J. Fixed Point Theory Appl. 10 (2011) 233–252
DOI 10.1007/s11784-011-0065-9
Published online November 9, 2011

Journal of Fixed Point Theory and Applications

The Conley index, cup-length and bifurcation

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Abstract. A module structure of the cohomology Conley index is used to define a relative cup-length. This invariant is applied then to prove a multiplicity theorem for periodic solutions to Hamiltonian systems.

Mathematics Subject Classification (2010). 37B30, 37J20, 37J45, 34C25, 58E05.

Keywords. Conley index, cup product, bifurcation, Hamiltonian system, periodic solution.

1. Introduction

In this paper, we consider a module structure of the Conley index of smooth flows in $\mathbb{R}^n$. If $\overline{\Omega}$ is an isolating neighbourhood and (P_1, P_2) is a regular index pair in Ω, then the cohomology $H^*(P_1, P_2)$ is a module over $H^*(\overline{\Omega})$. We define a notion of relative cup-length of $H^*(P_1, P_2)$ over $H^*(\overline{\Omega})$. This notion can be used to derive several results on nontrivial structure of invariant sets. As an example we prove a theorem on a minimal number of periodic solutions to Hamiltonian systems. A natural action of the group S^1 on the space of periodic functions is being used. Some other applications of this tool to bifurcation theory are presented in the PhD thesis of the last author [17].

It is worth mentioning that the concept is not completely new. One can find a cup-length applied to Conley theory in [2] and [4]. A variant of a relative version appeared in [16]. We believe that our approach should also be useful for other problems considered in the bifurcation theory.

The paper is organized as follows. Section 2 contains an abstract algebraic definition of the relative cup-length and simple properties. In Section 3 we recall basic concepts from Conley index theory (main source is [13]) and specify the abstract notion to it. In Section 4 we prove an abstract result on a number of critical points for gradient-like flows. Section 5 contains a reduction procedure for bifurcation problems. In the latter sections this procedure is applied to Hamiltonian systems.

2. Relative cup-length

Throughout this section we assume that $A \subset X \subset Y$ are compact metric spaces and we denote by H^* the Alexander–Spanier cohomology with the coefficients in the fixed abelian group G. The cup product (see [15, Sec. 5.6])

$$\smile : H^k(X) \times H^l(X, A) \to H^{k+l}(X, A)$$

endows $H^*(X, A)$ with a structure of an $H^*(X)$-module. If $k : X \to Y$ denotes the inclusion map, then the formula

$$\beta \cdot \alpha := k^*(\beta) \smile \alpha$$

defines on $H^*(X, A)$ a structure of $H^*(Y)$-module. The following remark is a simple consequence of the naturality property of the cup product (see [6, Prop. 3.10]).

Remark 2.1. If $B \subset A$ is compact, then

$$H^*(X, A) \to H^*(X, B) \to H^*(A, B)$$

is an exact sequence of $H^*(Y)$-modules, where the maps are induced by inclusions.

Definition 2.1. Let $\beta \in H^p(Y)$, $p > 0$, $\beta \neq 0$, and let $A \subset X \subset Y$ be CW-complexes. The *relative cup-length* of β with respect to (X, A) is the number $\chi(\beta; X, A) \in \mathbb{N}$ defined as follows:

- $\chi(\beta; X, A) = 0$ if $H^*(X, A) = 0$;
- $\chi(\beta; X, A) = 1$ if $H^*(X, A) \neq 0$ and $\beta \cdot \alpha = 0$ for every $\alpha \in H^*(X, A)$;
- $\chi(\beta; X, A) = k \geq 2$ if there exists $\alpha_0 \in H^*(X, A)$ such that $\beta^{k-1} \cdot \alpha_0 \neq 0$ and $\beta^k \cdot \alpha = 0$ for every $\alpha \in H^*(X, A)$.

Definition 2.2. The *relative cup-length* of the $H^*(Y)$-module $H^*(X, A)$ is the number given by

$$\Upsilon(X, A; Y) := \max\{\chi(\beta; X, A);\ 0 \neq \beta \in H^k(Y),\ k > 0\}.$$

If $H^k(Y) = \{0\}$ for $k > 0$ but $H^*(X, A)$ is nonzero, we set $\Upsilon(X, A; Y) = 1$, and if $H^l(X, A)$ are trivial for all $l \geq 0$, then $\Upsilon(X, A; Y) := 0$.

Lemma 2.2. *If $B \subset A \subset X \subset Y$, then*

$$\Upsilon(X, B; Y) \leq \Upsilon(X, A; Y) + \Upsilon(A, B; Y).$$

Proof. Let $k_1 = \Upsilon(X, A; Y)$, $k_2 = \Upsilon(A, B; Y)$, $0 \neq \alpha \in H^p(X, B)$, $p \geq 0$, $0 \neq \beta \in H^q(Y)$, $q > 0$. Consider the following inclusions:

$$i : (X, B) \to (X, A), \qquad j : (A, B) \to (X, B).$$

Since $k_2 = \Upsilon(A, B; Y)$, $j^*(\beta^{k_2} \cdot \alpha) = 0$. By Remark 2.1, there exists $\gamma \in H^*(X, A)$ such that $\beta^{k_2} \cdot \alpha = i^*(\gamma)$. Therefore,

$$\beta^{k_1+k_2} \cdot \alpha = i^*(\beta^{k_1} \cdot \gamma).$$

But $\beta^{k_1} \cdot \gamma = 0$ by definition of k_1, and thus $\beta^{k_1+k_2} \cdot \alpha = 0$. This means that

$$\Upsilon(X, B; Y) \leq k_1 + k_2,$$

which ends the proof. □

Lemma 2.3. *If $A \subset X \subset Y_1 \subset Y_2$, then*

$$\Upsilon(X, A; Y_2) \leq \Upsilon(X, A; Y_1).$$

Proof. Consider the following inclusions:

$$s : X \hookrightarrow Y, \qquad k : A \hookrightarrow X, \qquad t : A \hookrightarrow Y.$$

If $\beta \in H^{>0}(Y_2)$, $\alpha \in H^*(X, A)$, then $\beta\alpha = t^*(\beta) \smile \alpha = k^*(s^*(\beta)) \smile \alpha$. Hence $\chi(X, A; \beta) = \chi(X, A; s^*(\beta))$ for all $\beta \in H^{>0}(Y_2)$. Since $t = k \circ s$, the condition $t^*(\beta) \smile \alpha \neq 0$ implies $s^*(\beta) \smile \alpha \neq 0$, and our inequality follows. □

Recall that the *cross product* is defined by the formula

$$a \times b := p_1^*(a) \smile p_2^*(b),$$

where p_1, p_2 denote projections $(X, A) \times (Y, B)$ onto (X, A) and (Y, B). For algebraic properties of the maps

$$\times : H^k(X; R) \times H^l(Y; R) \to H^{k+l}(X \times Y; R),$$
$$\times : H^k(X, A; R) \times H^l(Y, B; R) \to H^{k+l}(X \times Y, X \times B \cup A \times Y; R)$$

see, e.g., [6] or [1, pp. 240–242].

Let $\sigma :=$ generator $H^1(I, \partial I)$, $I := [-1, 1]$.

The formula

$$\mathfrak{S}(a) := a \times \sigma$$

defines a mapping

$$\mathfrak{S} : H^k(X, A) \to H^{k+1}((X, A) \times (I, \partial I)) = H^{k+1}(X \times I, X \times \partial I \cup A \times I).$$

The following lemma holds (cf. [6, Thm. 3.21] for more general version).

Lemma 2.4. *If $X \subset Y$, then $\mathfrak{S}$ is an isomorphism of $H^*(Y)$-modules. More exactly,*

$$\mathfrak{S}(b \cdot a) = p^*(b) \cdot \mathfrak{S}(a),$$

where p denotes the projection $Y \times I$ onto Y.

Proof. Let $b \in H^*(Y)$, $a \in H^*(X, A)$. Consider the following projections:

$$p_1 : (X \times I, A \times I) \to (X, A),$$
$$p_2 : (X \times I, X \times \partial I) \to (I, \partial I),$$
$$\overline{p}_1 : X \times I \to X.$$

The following diagram is commutative, where $i_1(x,t) = (i(x),t)$:

$$\begin{tikzcd} X \times I \arrow[r, "i_1"] \arrow[d, "\overline{p}_1"] & Y \times I \arrow[d, "p"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

Using this diagram and the naturality and associativity properties of the cup product (see [1, p. 239]), we obtain

$$\begin{aligned} \mathfrak{S}(b \cdot a) &= (b \cdot a) \times \sigma = p_1^*(i^*(b) \smile a) \smile p_2^*(\sigma) = \overline{p}_1^*(i^*(b)) \smile p_1^*(a) \smile p_2^*(\sigma) \\ &= \overline{p}_1^*(i^*(b)) \smile \mathfrak{S}(a) = i_1^*(p^*(b)) \smile \mathfrak{S}(a) = p^*(b) \cdot \mathfrak{S}(a). \end{aligned}$$ □

Theorem 2.5. *The following formula holds:*

$$\Upsilon((X,A) \times (I, \partial I); Y) = \Upsilon(X, A; Y).$$

Proof. Let us notice that formally $X \times I \subset Y \times I$ and thus $H^*(X \times I, X \times \partial I \cup A \times I)$ is an $H^*(Y \times I)$-module, but $p^* : H^*(Y) \to H^*(Y \times I)$ is an isomorphism which gives the naturally isomorphic $H^*(Y)$-module structure: $b \odot a := p^*(b) \cdot a$ for $b \in H^*(Y)$ and $a \in H^*(X \times I, X \times \partial I \cup A \times I)$. Taking this into account, the desired equality follows directly from Lemma 2.4. □

3. Conley index and the relative cup-length

In this section, we recall the basic notions of the Conley index theory; the reader can refer to [9] and [13] for details. Let X be a locally compact metric space. A continuous map $\eta : X \times \mathbb{R} \to X$ is a *flow* if it satisfies the conditions

$$\eta(x,0) = x,$$
$$\eta(x, t+s) = \eta(\eta(x,t), s).$$

A set $S \subset X$ is an *invariant set* for the flow η if

$$\eta(S, \mathbb{R}) := \bigcup_{t \in \mathbb{R}} \eta(S,t) = S.$$

For an arbitrary set $N \subset X$ one can define its invariant part

$$\mathrm{Inv}(N, \eta) := \{x \in N \mid \eta(x, \mathbb{R}) \subset N\}.$$

A compact set $N \subset X$ is an *isolating neighbourhood* if $\mathrm{Inv}(N,\eta) \subset \mathrm{int}\, N$. A set S is called an *isolated invariant set* if there is an isolating neighbourhood N such that $S = \mathrm{Inv}(N, \varphi)$. A flow $\eta : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is *generated* by a smooth vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ if $\eta(x,t)$ is the solution of the Cauchy problem $\dot{u} = -F(u), u(0) = x$ evaluated at time t. Such a flow is a *gradient flow* if $F = \nabla f$ for some smooth function $f : \mathbb{R}^n \to \mathbb{R}$.

Let S be an isolated invariant set for the flow η. A compact pair $N_0 \subset N_1$ of subsets of X is called an *index pair* for S if the following hold:

(a) $\overline{\mathrm{int}(N_1 \setminus N_0)}$ is an isolating neighbourhood for S;
(b) N_0 is *positively invariant* relative to N_1; i.e., if $x \in N_0$ and $\eta(x, [0,t]) \subset N_1$, then $\eta(x, [0,t]) \subset N_0$;

(c) N_0 is an *exit set* for N_1; i.e., if $x \in N_1$ and $t_1 > 0$ such that $\eta(x, t_1) \notin N_1$, then there exists $t_0 \in [0, t_1]$ for which $\eta([0, t_0], x) \subset N_1$ and $\eta(x, t_0) \in N_0$.

The following result implies the correctness of the definition of the *homotopy Conley index* (cf. [9, Thms. 2.2.1 and 2.2.2] or [13, Thms. 23.7 and 23.12]).

Theorem 3.1. *Let S be an isolated invariant set for the flow η. Then there exists an index pair for S. Moreover, if (N_1, N_0) and (N_1', N_0') are index pairs for S, then the pointed topological spaces*

$$(N_1/N_0, [N_0]) \qquad \text{and} \qquad (N_1'/N_0', [N_0'])$$

are homotopically equivalent.

Definition 3.1. Let S be an isolated invariant set for the flow η. The *homotopy Conley index* of S is the homotopy type of the pointed space

$$h(S) = h(S, \eta) := [N_1/N_0, [N_0]],$$

where (N_1, N_0) is an index pair for S.

It is useful to consider the *cohomology Conley index* defined by

$$CH^*(S) := H^*(N, L) = H^*(N/L),$$

where H^* denotes the Alexander–Spanier cohomology and (N, L) is an index pair for S. The last equality means that we identify $H^*(N, L)$ and $H^*(N/L)$ via the isomorphism induced by the quotient map.

It is convenient to extend the index to an index of isolating neighbourhood: if N is an isolating neighbourhood for η, then the *homotopy* (resp., *cohomology*) *Conley index* of N is defined as

$$h(N) = h(N, \eta) := h(\mathrm{Inv}(N, \eta)),$$
$$\big(\text{resp., } CH^*(N) = CH^*(N, \eta) := CH^*(\mathrm{Inv}(N, \eta))\big).$$

Before giving the definition of the relative cup-length of Conley index, we need some useful lemmas. If (N_0, N_1) is an index pair and $t \geq 0$, then, following [13], we set

$$\begin{aligned} N_1^t &:= \{x \in N_1;\ \eta(x, [-t, 0]) \subset N_1\}, \\ N_0^{-t} &:= \big\{x \in N_1;\ \text{there are } x' \in N_0 \text{ and } t' \in [0, t] \\ &\qquad \text{with } \eta(x', [-t', 0]) \subset N_1 \text{ and } \eta(x', t) = x\big\}. \end{aligned}$$

For $t \geq 0$, define a map

$$g : N_1/N_0^{-t} \to N_1^t/(N_0 \cap N_1^t)$$

by

$$g([x]) := \begin{cases} [\eta(x, t)] & \text{if } \eta(x, [0, t]) \subset N_1 \setminus N_0; \\ * & \text{otherwise.} \end{cases}$$

It is known (see [13, Lem. 23.14]) that g is a homeomorphism. Therefore, it induces an isomorphism

$$g^* : H^*(N_1^t, N_0 \cap N_1^t) \to H^*(N_1, N_0^{-t}).$$

Lemma 3.2. *Assume that N is an isolating neighbourhood for η and (N_1, N_0) is an index pair for $S \subset N$. If $N_1 \subset N$, then the inclusion $i : (N_1, N_0 \cap N_1^t) \to (N_1, N_0^{-t})$ induces an isomorphism*

$$i^* = (g^*)^{-1} : H^*(N_1, N_0^{-t}) \to H^*(N_1, N_0 \cap N_1^t).$$

Proof. Consider the following diagram, where the vertical arrows denote the quotient maps.

$$\begin{tikzcd}
(N_1, N_0^{-t}) \arrow[d] & (N_1, N_0 \cap N_1^t) \arrow[l, "i"'] \arrow[d] \\
N_1/N_0^{-t} \arrow[r, "g"] & N_1/(N_0 \cap N_1^t)
\end{tikzcd}$$

From the definition of g, it is obvious that the diagram is homotopy commutative and the conclusion follows. $\square$

Definition 3.2. Let N be an isolating neighbourhood for the flow η. We define the *relative cup-length* of η with respect to N as

$$\Upsilon(\eta, N) := \Upsilon(N_1, N_0; N),$$

where (N_1, N_0) is an index pair for S.

The following lemma states that $\Upsilon(\eta, N)$ is well defined.

Lemma 3.3. *Let N be an isolating neighbourhood for η and let $S \subset N$ be an isolated invariant set. If (N_1, N_0) and $(\overline{N}_1, \overline{N}_0)$ are index pairs for S such that $N_1, \overline{N}_1 \subset N$, then*

$$\Upsilon(\overline{N}_1, \overline{N}_0; N) = \Upsilon(N_1, N_0; N).$$

Proof. As in the proof of [13, Lem. 23.17], we consider the following sequence of maps, where j, $\hat{i}$, $\hat{i}_1$ are defined by inclusion maps of pairs of spaces and g, $\hat{g}$ are as above. All of them are homotopy equivalences of pointed spaces, as it is proved in detail in [13]:

$$\begin{tikzcd}
N_1/N_0 \arrow[r, "j"] & N_1/N_0^{-t} \arrow[r, "g"] & N_1^t/\big(N_0 \cap N_1^{-t}\big) \arrow[d, "\hat{i_1}"] \\
\overline{N}_1/\overline{N}_0 & \overline{N}_1^t/\left(\overline{N}_0 \cap \overline{N}_1^t\right) \arrow[l, "\hat{i}"'] & \overline{N}_1/\overline{N}_0^{-t} \arrow[l, "\hat{g}"']
\end{tikzcd}$$

By Lemma 3.2 and definition of j, it follows that the following sequence of isomorphisms

$$\begin{tikzcd}
H^*(N_1, N_0) & H^*\left(N_1, N_0^{-t}\right) \arrow[l, "\approx"'] & H^*\left(N_1^t, N_0 \cap N_1^{-t}\right) \arrow[l, "\approx"'] \\
H^*\left(\overline{N}_1, \overline{N}_0\right) \arrow[r, "\approx"] & H^*\left(\overline{N}_1^t, \overline{N}_0 \cap \overline{N}_1^{\,t}\right) \arrow[r, "\approx"] & H^*\left(\overline{N}_1, \overline{N}_0^{\,-t}\right) \arrow[u, "\approx"]
\end{tikzcd}$$

are all induced by inclusions. Therefore, they all are isomorphisms of $H^*(N)$-modules and the conclusion follows. □

One of the main properties of the Conley index is the continuation. The same holds true for the relative cup-length.

Lemma 3.4. *Consider a continuous family of flows $\eta_\lambda : X \times \mathbb{R} \to X$; $\lambda \in [0,1]$. Let $N \subset X$ be an isolating neighbourhood for all flows η_λ. Then*

$$\Upsilon(\eta_0, N) = \Upsilon(\eta_1, N).$$

Proof. Similarly as in the proof of Lemma 3.3 we shall use parts of the proof of [13, Thm. 23.31]. Given $\mu \in [0,1]$, there exists a neighbourhood W of μ in $[0,1]$ with the property that for all $\lambda \in W$, we can find pairs $(N_1, N_0) \subset (P_1^\lambda, P_0^\lambda) \subset (\overline{N_1}, \overline{N_0})$ such that (N_1, N_0), $(\overline{N_1}, \overline{N_0})$ are index pairs for η_μ in N, and $(P_1^\lambda, P_0^\lambda)$ is an index pair for η_λ in N (see [13, Lem. 23.28]). Then it is shown in the proof of [13, Thm. 23.31] that the inclusion $i : (N_1, N_0) \to (P_1^\lambda, P_0^\lambda)$ induces a homotopy equivalence of pointed spaces N_1/N_0 and P_1^λ/P_0^λ. The same argument applies to show that $i^* : H^*(P_1^\lambda, P_0^\lambda) \approx H^*(N_1, N_0)$ is an isomorphism of $H^*(N)$-modules. Therefore, $\Upsilon(\eta_\lambda, N) = \Upsilon(\eta_\mu, N)$. Since $[0,1]$ is compact and connected, this completes the proof. □

One easily sees that the continuation holds for more general parameter space Λ as in [13].

4. Gradient-like flows

Throughout this section, as before, η denotes a flow on a locally compact metric space X.

Let N be an isolating neighbourhood for η and let $\varphi : \operatorname{int} N \to \mathbb{R}$ be continuous. The flow η is called *gradient-like* with respect to φ if $\eta(x, [0,t]) \subset \operatorname{int} N$ and $\eta(x,t) \neq x$ imply $\varphi(\eta(x,t)) > \varphi(x)$. We define the *critical level set* of φ with respect to η as

$$\operatorname{Crit}(\varphi, \eta) := \varphi(\{x \in U; \eta(x,t) = x \text{ for all } t \in \mathbb{R}\}).$$

In other words, $c \in \operatorname{Crit}(\varphi, \eta)$ if and only if there is $x \in N$ which is a rest point of the flow and $\varphi(x) = c$.

The aim of this section is to give a proof of the following theorem.

Theorem 4.1. *Assume that X is locally contractible and N is an isolating neighbourhood for η. If η is gradient-like with respect to $\varphi : \operatorname{int} N \to \mathbb{R}$ and $\operatorname{Crit}(\varphi, \eta)$ is finite, then*

$$\#\operatorname{Crit}(\varphi, \eta) \geq \Upsilon(\eta, N).$$

Before giving the proof of the theorem we shall recall some definitions and results concerning Morse decompositions.

Recall that the *omega limit set* of $x \in X$ is given by

$$\omega(x) := \bigcap_{t>0} cl\big(\eta(x, [t, \infty))\big)$$

and the *alpha limit set* is

$$\alpha(x) := \bigcap_{t<0} cl\big(\eta(x, (-\infty, t])\big).$$

Assume that S is an isolated invariant set for η. A *Morse decomposition* of S is a finite collection, $\{M_i : 1 \leq i \leq n\}$, of disjoint compact invariant subsets of S which can be ordered $(M_1, M_2, \ldots, M_n)$ in such a way that if $x \in S \setminus \bigcup\{M_i : 1 \leq i \leq n\}$, then there are indices $i < j$ such that $\omega(x) \subset M_i$ and $\alpha(x) \subset M_j$. Such an ordering will be called admissible. The elements M_i of the Morse decomposition of S will be called *Morse sets* of S. For an admissible ordering $(M_1, \ldots, M_n)$ of a Morse decomposition S, define subsets M_{ij}, $i < j$, by

$$M_{ij} := \{x \in S : \omega(x) \cup \alpha(x) \subset M_i \cup M_{i+1} \cup \cdots \cup M_j\}.$$

The proof of the following existence theorem can be found in [13, Thm. 23.7] or in [12, Cor. 4.4].

Theorem 4.2. *Let S be an isolated invariant set for η and $(M_1, M_2, \ldots, M_n)$ an admissible ordering of a Morse decomposition of S. Then there exists an increasing sequence of compact sets (a (Morse) filtration of S),*

$$N_0 \subset N_1 \subset \cdots \subset N_n$$

such that for any $i < j$, the pair (N_j, N_{i-1}) is an index pair for M_{ij}. In particular, (N_n, N_0) is an index pair for S, and (N_j, N_{j-1}) is an index pair for M_j.

Furthermore, given any isolating neighbourhood N of S, and any neighbourhood U of S, the sets N_j can be chosen so that $cl(N_n \setminus N_0) \subset U$ and each N_j is positively invariant relative to N.

Proof of Theorem 4.1. Let

- $S := \operatorname{Inv} N$;
- $\operatorname{Crit}(\varphi, \eta) = \{c_1 < c_2 < \cdots < c_k\}$;
- $M_i := \operatorname{Crit}(\varphi, \eta) \cap \varphi^{-1}(c_i)$.

Choose

$$N_0 \subset N_1 \subset \cdots \subset N_n$$

satisfying the conditions of Theorem 4.2. Lemma 2.2 implies

$$\Upsilon(N_i, N_0; N) \leq \Upsilon(N_{i-1}, N_0; N) + \Upsilon(N_i, N_{i-1}; N) \tag{1}$$

for $i = 1, 2, \dots, k$. Since M_i is finite and X is locally contractible, we can find a neighbourhood $U \subset N$ of M_i consisting of pairwise disjoint contractible sets. Then we find an index pair (N_i', N_{i-1}') in U. Therefore, $H^*(N_i', N_{i-1}')$ has a trivial structure as an $H^*(N)$-module. Thus by Lemma 3.3, we obtain

$$\Upsilon(N_i, N_{i-1}; N) \leq 1.$$

Therefore,

$$\Upsilon(\eta, N) = \Upsilon(N_k, N_0; N) \leq k. \qquad \square$$

5. Bifurcation

Throughout this section we let E_1, E_0 be Banach spaces, H a Hilbert space and we assume that $E_1 \subset E_0 \subset H$, where the embeddings are continuous.

We assume also that a compact Lie group G acts orthogonally on H, and the action on E_1, E_0 is by isometries (i.e., the norms on E_1, E_0 are G-invariant).

Definition 5.1. Given an open $\Omega \subset E$ and a continuous $f : \Omega \to E_0$, we say that f is a *generalized gradient map* if there is an open $\Omega_0 \subset E_0$, with $\Omega \subset \Omega_0$, and a C^1-function $\varphi : \Omega_0 \to \mathbb{R}$ such that

$$D\varphi(x)(y) = \langle f(x), y\rangle \quad \text{for all } x \in \Omega,\ y \in E_0.$$

Here $\langle \cdot, \cdot \rangle$ denotes the scalar product in H. Similarly, in the case of an open $\Omega \subset E \times \mathbb{R}$ and a continuous $f : \Omega \to E_0$, we say that f is a *generalized gradient map* if $f_\lambda : \Omega_\lambda \to E_0$ is a generalized gradient map. Here $\Omega_\lambda = \{x \in E; (x, \lambda) \in \Omega\}$.

If X is a Banach space, we denote the open ϵ-ball in X by $B_X(\epsilon) := \{x \in X; \|x\| < \epsilon\}$ and $B_X(x_0, \epsilon) := \{x \in X; \|x - x_0\| < \epsilon\}$.

If $V \subset E$ is a finite-dimensional linear subspace, then there is the *orthogonal decomposition determined by V*

$$E_1 = W_1 \oplus V, \qquad E_0 = W_0 \oplus V, \tag{2}$$

where $W_0 := \{x \in E_0; \langle x, y\rangle = 0 \text{ for all } y \in V\}$, $W_1 := E_1 \cap W_0$.

Definition 5.2. Let $[\lambda_1, \lambda_2] \subset \mathbb{R}$. We say that a C^1-gradient equivariant map

$$f : \Omega_f \to E_0,$$

where $\Omega_f \subset E \oplus \mathbb{R}$ is open G-invariant, $\{0\} \times [\lambda_1, \lambda_2] \subset \Omega_f$, defines a *bifurcation problem* on $[\lambda_1, \lambda_2]$ if

$$f(0, \lambda) = 0 \quad \text{for } (0, \lambda) \in \Omega_f$$

and

$$D_x f(0, \lambda_i) : E \approx E_0, \quad i = 1, 2.$$

We shall also simply say that f is a *bifurcation problem.*

Definition 5.3. Let $f_i : \Omega_i \to E_0$, $i = 1, 2$, be two bifurcation problems on $[\lambda_1, \lambda_2]$. We say that f_1 and f_2 are *equivalent* if there exists an equivariant diffeomorphism

$$\Psi : \Omega_1 \to \Omega_2$$

such that

$$f_2 = f_1 \circ \Psi.$$

Theorem 5.1. *Let $f : \Omega_f \to E_0$ be a bifurcation problem on $[\lambda_1, \lambda_2]$. If there exist decompositions*

$$E_1 = V \oplus W_1, \quad E_0 = V \oplus W_0, \quad f(x, y, \lambda) = (f_1(x, y, \lambda), f_2(x, y, \lambda)),$$

such that

$$Df_2(0, \lambda)_{|W_1} : W_1 \approx W_0 \quad \textit{for } \lambda \in [\lambda_1, \lambda_2],$$

then there exist

(1) *an open invariant $\Omega \subset \Omega_f$, $\{0\} \times [\lambda_1, \lambda_2] \subset \Omega$;*
(2) *$g : \Omega_g \to E_0$—a bifurcation problem on $[\lambda_1, \lambda_2]$;*

such that

(a) *$f_{|\Omega}$ is a bifurcation problem on $[\lambda_1, \lambda_2]$ equivalent to g;*
(b) *$g(V \cap \Omega_g) \subset V$ and $g^{-1}(0) \subset V$;*
(c) *if $D_1 f_2(0, 0, \lambda) = 0$, then $D_1 g(0, 0, \lambda) = D_1 f_1(0, 0, \lambda)$.*

The proof is based on the following two theorems.

Theorem 5.2 (Equivariant implicit function theorem). *Let V_1, V_2, W be Banach G-spaces, $\Omega \subset V_1 \times V_2$ a G-invariant open set, $(x_0, 0) \in \Omega$ and $F : \Omega \to W$ be continuously differentiable G-map. Assume that $F(x_0, 0) = 0$ and*

$$D_2F(x_0, 0) : V_2 \to W$$

is a G-equivariant Banach space isomorphism. Then there exist $\epsilon_1, \epsilon_2 > 0$, $B_{V_1}(x_0, \epsilon_1) \times B_{V_2}(\epsilon_2) \subset \Omega$, and a continuously differentiable G-equivariant map $\psi : B_{V_1}(x_0, \epsilon_1) \to B_{V_2}(\epsilon_2)$ such that

$$F(x, \psi(x)) = 0 \tag{3}$$

and

$$D\psi(x) = -\big(D_2F(x, \psi(x))\big)^{-1} D_1F(x, \psi(x)) \tag{4}$$

for all $x \in B_{V_1}(x_0, \epsilon_1)$. Furthermore, for every $x \in B_{V_1}(x_0, \epsilon_1)$, $\psi(x)$ is the only solution of (3) *in $B_{V_2}(\epsilon_2)$.*

Proof. The theorem is an equivariant reformulation of [7, Thm. 10.1]. Since the mapping

$$G(x, y) := y - L_0^{-1} F(x, y), \quad L_0 := D_2F(x_0, 0),$$

defined in [7, p. 134], is in our case equivariant, the proof carries over directly. □

Theorem 5.3 (Parametrized equivariant implicit function theorem). *Let V_1, V_2, W be Banach G-spaces, $\Omega \subset V_1 \times V_2 \times \mathbb{R}$ a G-invariant open set, $(0,0,\lambda) \in \Omega$ for $\lambda \in [\lambda_1, \lambda_2]$. Assume that $F : \Omega \to W$ is a continuously differentiable G-map, $F(0,0,\lambda) = 0$ if $(0,0,\lambda) \in \Omega$ and*

$$D_2F(0,0,\lambda) : V_2 \to W$$

is a G-equivariant Banach space isomorphism if $(0,0,\lambda) \in \Omega$. Then there exist $\epsilon_1, \epsilon_2 > 0$, $B_{V_1}(\epsilon_1) \times B_{V_2}(\epsilon_2) \times (\lambda_1 - \epsilon_1, \lambda_2 + \epsilon_1) \subset \Omega$, and a continuously differentiable G-equivariant map $\psi : B_{V_1}(\epsilon_1) \times (\lambda_1 - \epsilon_1, \lambda_2 + \epsilon_1) \to B_{V_2}(\epsilon_2)$ such that

$$F(x, \psi(x,\lambda), \lambda) = 0 \tag{5}$$

and

$$D\psi(x,\lambda) = -\big(D_2F(x,\psi(x,\lambda))\big)^{-1} D_1F(x,\psi(x,\lambda)) \tag{6}$$

for all $x \in B_{V_1}(\epsilon_1) \times [\lambda_1, \lambda_2]$. Furthermore, for every $(x,\lambda) \in B_{V_1}(\epsilon_1) \times [\lambda_1, \lambda_2]$, $\psi(x,\lambda)$ is the only solution of (5) in $B_{V_2}(\epsilon_2)$.

Proof. The theorem follows from Theorem 5.2. One should consider $V_1 \oplus R$ instead of V_1 and then use the compactness of $[\lambda_1, \lambda_2]$. □

Proof of Theorem 5.1. We apply Theorem 5.3 to the map f_2 and obtain a G-equivariant mapping

$$\psi : B_V(\epsilon_1) \times (\lambda_1 - \epsilon_1, \lambda_2 + \epsilon_1) \to B_{W_0}(\epsilon_2).$$

Observe that for each $\lambda \in [\lambda_1, \lambda_2]$, the following holds true:

$$\text{if } x \in B_V(\epsilon_1),\ y \in B_{W_0}(\epsilon_2) \quad \text{then} \quad f_2(x,y,\lambda) = 0 \iff y = \psi(x,\lambda).$$

Taking ϵ_2 smaller if necessary, we define a G-equivariant diffeomorphism

$$\Psi : B_V(\epsilon_1) \times (\lambda_1 - \epsilon_1, \lambda_2 + \epsilon_1) \to \Omega_f$$

by the following formula:

$$\Psi(x,y,\lambda) := (x, y + \psi(x,\lambda), \lambda).$$

The desired map g is given by

$$g := f \circ \Psi.$$

Since V is finite dimensional, for $\epsilon > 0$ small enough, we have

$$g^{-1}(0) \cap (B_V(\epsilon) \times B_{W_1}(\epsilon) \times [\lambda_1, \lambda_2]) \subset B_V(\epsilon) \times [\lambda_1, \lambda_2].$$

Considering $Dg(0,0)$ in a block form, we obtain the last assertion. □

6. Bifurcation in $\mathbb{R}^n$

In this section to simplify the notation, we consider a finite-dimensional bifurcation problem on $I = [-1, 1]$ defined by a map f. More precisely, we assume that $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a C^1-map, $f(0, \lambda) = 0$ for $\lambda \in \mathbb{R}$ and $Df(0, \pm 1) : \mathbb{R}^n \approx \mathbb{R}^n$.

Let $A_\lambda := D_x f(0, \lambda)$. Then $f(x, \lambda) = A_\lambda(x) + f_0(x, \lambda)$. For $\tau \in [0, 1]$, we set

$$f_\tau(x, \lambda) := A_\lambda(x) + \tau f_0(x, \lambda).$$

Assume further that there exist $\rho, C > 0$ such that

$$\langle f_\tau(x, 1), x\rangle \geq C|x|^2 \quad \text{for } |x| \leq 2\rho \tag{7}$$

and

$$\langle f_\tau(x, -1), x\rangle \leq -C|x|^2 \quad \text{for } |x| \leq 2\rho. \tag{8}$$

For $\alpha > 0$ and $0 < \epsilon < \rho$, define

$$F_\tau : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

by $F_\tau(x, \lambda) := (f_\tau(x, \lambda), \alpha(|x| - \epsilon))$. Let

$$\Omega = \{x \in \mathbb{R}^n; |x| \leq 2\rho\} \times [-1, 1]$$

and $M := \sup\{|f_\tau(x, \lambda)|; (x, \lambda) \in \Omega,\ \tau \in [0, 1]\}$.

Lemma 6.1. *If*

$$\alpha \geq \frac{2M}{\rho(\rho - \epsilon)},$$

then there exists $\delta > 0$ such that $\delta < \epsilon$ and for all $\tau \in [0, 1]$, the set $N := \{(x, \lambda) \in \Omega;\ |x| \geq \delta\}$ is an isolating neighbourhood for the flow generated by F_τ.

Proof. First we prove that Ω is an isolating neighbourhood. We fix τ and let $\eta(x, \lambda, t) = (\eta_1(x, \lambda, t), \eta_2(x, \lambda, t)) \in \mathbb{R}^n \times \mathbb{R}$ denote the flow generated by F_τ. It is enough to show that for all $(x, \lambda) \in \partial\overline{\Omega}$,

(a) there exists $T > 0$ such that either $\eta(x, \lambda, T) \notin \overline{\Omega}$ or $\eta(x, \lambda, -T) \notin \overline{\Omega}$.

Let $K := \{(x, \lambda) \in \overline{\Omega}; |x| = 2\rho, \lambda \in [-1, 1]\}$. If $(x, \lambda) \in \partial\Omega \setminus K$, then (a) follows immediately from the definition of F_τ.

To complete the proof of our first claim we start from the following simple observations:

(b) if $\eta(x, \lambda, t) \in \overline{\Omega}$ for all $t \in [0, T]$, then

$$|\eta_1(x, \lambda, t) - \eta_1(x, \lambda, 0)| \leq TM \quad \text{for } t \in [0, T];$$

(c) if $\eta(x, \lambda, t) \in A := \{(x, \lambda) \in \overline{\Omega}; |x| \geq \rho\}$ for all $t \in [0, T]$, then

$$\eta_2(x, \lambda, t) - \eta_2(x, \lambda, 0) \geq (\rho - \epsilon)\alpha t \quad \text{for } t \in [0, T].$$

Let $(x_0, \lambda_0) \in K$ and

$$T_1 := \inf\{t \in (0, \infty); \eta(x_0, \lambda_0, t) \notin A\}.$$

Since every point of A leaves A in a finite time, $T_1 < \infty$. (One can call T_1 the *exit time* of (x_0, λ_0) from A.) Let $(x_1, \lambda_1) := \eta(x_0, \lambda_0, T_1)$. If $(x_1, \lambda_1) \in \partial\Omega$, then (a) holds. Suppose that $(x_1, \lambda_1) \in \Omega$. Then $|x_1| = \rho$, $\lambda_1 \in (-1, 1)$ and (b) implies $\rho \leq MT_1$. Applying (c), one obtains

$$\lambda_1 \geq \lambda_0 + \alpha(\rho - \epsilon)\frac{\rho}{M} > \lambda_0 + 2 > 1.$$

We have obtained a contradiction. Therefore, Ω is an isolating neighbourhood for all η_τ and thus the invariant part

$$\mathrm{Inv}(\Omega, \eta) = \bigcup_{\tau\in[0,1]} \mathrm{Inv}(\Omega, \eta_\tau) \subset \mathrm{int}(\Omega)$$

is compact. Moreover, one easily checks that it is disjoint with $\{0\} \times [-1, 1]$. Thus there exists $\epsilon > \delta \geq 0$ such that $\mathrm{Inv}(\Omega) \in \mathrm{int}(N)$. This proves that N is an isolating neighbourhood for all η_τ. □

Assume now that $V = (\mathbb{R}^n, \varphi)$ is an orthogonal representation of a compact Lie group G; i.e., $\varphi : G \to O(n)$ is a group homomorphism. Let $S(V) := \{x \in V; |x| = 1\}$. The use of V instead of $\mathbb{R}^n$ is a bit of notation abuse—we try to emphasize that $S(V)$ is a G-space.

Lemma 6.2. *Let $f : \Omega_f \to \mathbb{R}^n$ be a gradient equivariant bifurcation problem on $[-1, 1]$ and $A_\lambda := D_x f(0, \lambda)$, $\lambda \in [-1, 1]$. Assume that there is $C > 0$ such that*

$$\langle A_1(x), x\rangle \geq C|x|^2 \quad \textit{for } x \in \mathbb{R}^n \tag{9}$$

and

$$\langle A_{-1}(x), x\rangle \leq -C|x|^2 \quad \textit{for } x \in \mathbb{R}^n. \tag{10}$$

Then for sufficiently small ϵ, the number of zero G-orbits of f in $S(\mathbb{R}^n, \epsilon) \times (-1, 1)$ is not less than the cup-length of $S(V)/G$.

Proof. We keep the notation from the beginning of this section. From (9) and (10), it follows that there exists $\rho > 0$ such that assumptions (7) and (9) are satisfied. Now for $\epsilon < \rho$, we find α and δ as in Lemma 6.1 and obtain an isolating neighbourhood $N = \{(x, \lambda); \delta \leq |x| \leq 2\rho, -1 \leq \lambda \leq 1\}$ which is clearly an invariant set with respect to the action of G (trivial on the parameter space). By Lemma 3.4, it is enough to calculate the equivariant Conley index (and the relative cup-length) for the flow generated by the map $g(x, \lambda) := (D_x f(0, \lambda)(x), \alpha(|x| - \epsilon)) = (A_\lambda x, \alpha(|x| - \epsilon))$.

Now we can make another simplification. Consider a map $B : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ given by $B(x, \lambda) = \lambda x$ and a family of flows generated by vector fields $F_\tau : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $F_\tau(x, \lambda) = (\tau A_\lambda x + (1 - \tau)B(x, \lambda), \alpha(|x| - \epsilon))$ with $\tau \in [0, 1]$. It is easy to verify that N is an isolating neighbourhood for this family of flows. Thus we can do all the calculations for $\tau = 0$. We can easily find an index pair: $N_1 := N$ and

$$\begin{aligned} N_0 := \{(x,1) : 1 \geq |x| \geq \epsilon\} &\cup \{(x,\lambda); |x| = 2\rho, \lambda \in [0,1]\} \\ &\cup \{(x,\lambda); |x| = \delta, \lambda \in [-1,0]\} \\ &\cup \{(x,-1); \delta \leq |x| \leq \epsilon\}. \end{aligned}$$

Since all the sets are G-invariant, their quotient sets constitute an index pair for the flow generated on the orbit space. N is equivariantly homotopy equivalent to $S(V) \times [-1,1] \times [-1,1]$ and $N_0 \approx S(V) \times L$, where $L : \{(t,s) \in \partial([-1,1] \times [-1,1]); ts \geq 0\}$. Therefore, $\overline{N_1} := N_1/G \approx S(V)/G \times [-1,1] \times [-1,1]$, $\overline{N_0} := N_0/G \approx S(V)/G \times L$. Their quotient $\overline{N_1}/\overline{N_0} \approx S(V)/G \wedge S^1$. Thus, by Theorem 2.5, $\Upsilon(\overline{N_1}, \overline{N_0}; \overline{N_1})$ is equal to the cup-length of $S(V)/G$.

Now we can apply Theorem 4.1, since the gradient flow generated by f gives rise to a gradient-like flow on the orbit space and the critical points of this flow are images of the zero G-orbits of f. $\square$

7. Bifurcations of periodic solutions to Hamiltonian systems

By

$$J : \mathbb{R}^{2N} = \mathbb{R}^N \oplus \mathbb{R}^N \to \mathbb{R}^N \oplus \mathbb{R}^N = \mathbb{R}^{2N}$$

we denote a linear automorphism given by the matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Throughout this section we assume that $H : \mathbb{R}^{2N} \to \mathbb{R}$ is a C^2-function (Hamiltonian) such that

(H1) $H(0) = 0$, $\nabla H(0) = 0$;

(H2) the Hessian $\nabla^2 H(0)$ is nondegenerate.

The main object of our investigation is periodic solutions to the following equation:

$$\dot{u}(t) = J\nabla H(u(t)). \tag{11}$$

We shall use the following Banach spaces:

(1) $\mathcal{E}_0 := C(S^1, \mathbb{R}^{2N})$. The elements of $\mathcal{E}_0$ are identified with continuous functions

$$u : \mathbb{R} \to \mathbb{R}^{2N}, \quad u(t + 2\pi) = u(t), \quad \|u\| := \sup\{|u(t)|; t \in \mathbb{R}\}.$$

(2) $\mathcal{E} := C^1(S^1, \mathbb{R}^{2N})$. As a linear space $\mathcal{E}$ is a subspace of $\mathcal{E}_0$. The norm in $\mathcal{E}$ is defined by a formula

$$\|u\|_1 := \|u\| + \|\dot{u}\|.$$

The above automorphism J defines also automorphisms of our Banach spaces

$$J : \mathcal{E} \to \mathcal{E}, \qquad J : \mathcal{E}_0 \to \mathcal{E}_0.$$

More precisely,

$$J\left(\sum_{i=1}^{2N} u_i \mathbf{e}_i\right) := \sum_{i=1}^{2N} u_i J(\mathbf{e}_i),$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{2N}\}$ is the standard basis of $\mathbb{R}^{2N}$.

In the space $\mathcal{E}_0$ we have a continuous inner product

$$\langle u, v \rangle := \sum_{j=1}^{2N} \int_0^{2\pi} u_j(t) v_j(t) dt, \tag{12}$$

where

$$u = \sum_{j=1}^{2N} u_j \mathbf{e}_j, \qquad v = \sum_{j=1}^{2N} v_j \mathbf{e}_j.$$

(In other words, we consider $\mathcal{E}_0$ as a subspace of $\mathcal{L}^2(S^1, \mathbb{R}^{2N})$.)

The formula

$$\mathcal{L}(u) := J(\dot{u})$$

defines a bounded linear operator

$$\mathcal{L} : \mathcal{E} \to \mathcal{E}_0.$$

Denote by

$$\mathcal{H} : \mathcal{E} \to \mathcal{E}_0$$

a map (nonlinear in general) given by a formula

$$(\mathcal{H}(u))(t) := \nabla H(u(t)).$$

Our further considerations are based on the following well-known remark.

Define a map

$$f : \mathcal{E} \times (0, \infty) \to \mathcal{E}_0, \quad f(u, \lambda) := \mathcal{L}(u) + \lambda \mathcal{H}(u). \tag{13}$$

Remark 7.1. A function $u \in \mathcal{E}$ is a periodic solution to equation (11) of period $\frac{2\pi}{\lambda}$ if and only if $f(u, \lambda) = 0$. The map f is (generalized) gradient in the sense introduced in Definition 5.1 with respect to the potential $\chi(u) := \int_0^{2\pi} u(t) dt$.

A change of variables $t \mapsto \lambda t$ gives the first part of the remark. The second part is well known.

Let $A := \nabla^2 H(0)$. The map JA is a *Hamiltonian* (i.e., $(JA)^T J + J(JA) = 0$). Observe that in [8] the notion *Hamiltonian matrix* is used.

Now we describe briefly the spectral decomposition of JA. We try to follow the notation of [8, Sec. 3.3], where further details can be found.

The eigenvalues of JA fall into three groups (because of (H2)):

(1) the pure imaginary $\pm i\omega_1, \ldots, \pm i\omega_r$;
(2) the real eigenvalues $\alpha_1, \ldots, \alpha_s$;
(3) the truly complex $\pm\beta_1 \pm i\gamma_1, \ldots, \pm\beta_t \pm i\gamma_t$.

This defines a direct sum decomposition

$$\mathbb{R}^{2N} = \mathbb{V} \oplus \mathbb{X} \oplus \mathbb{Y}, \tag{14}$$

where their complexifications are composed of generalized eigenspaces as follows:

$$\mathbb{V}^c = \bigoplus_{j=1}^{r} \left(\eta^\dagger(i\omega_j) \oplus \eta^\dagger(-i\omega_j)\right),$$

$$\mathbb{X}^c = \bigoplus_{j=1}^{s} \left(\eta^\dagger(\alpha_j) \oplus \eta^\dagger(-\alpha_j)\right),$$

$$\mathbb{Y}^c = \bigoplus_{j=1}^{t} \left(\eta^\dagger(\beta_j + i\gamma_j) \oplus \eta^\dagger(\beta_j - i\gamma_j) \oplus \eta^\dagger(-\beta_j + i\gamma_j) \oplus \eta^\dagger(-\beta_j - i\gamma_j)\right).$$

We are especially interested in part (1):

$$\begin{aligned}\sigma_0(JA) &= \sigma(JA) \cap \{i\mathbb{R}\} \\ &= \{\pm i\omega_1, \pm i\omega_2, \ldots, \pm i\omega_r\}, \quad 0 < \omega_1 < \omega_2 < \cdots < \omega_r.\end{aligned} \tag{15}$$

Denote by $\mathbb{V}_j$, $\mathbb{U}_j$ the subspaces of $\mathbb{R}^{2N}$ such that

$$\mathbb{V}_j^c := \eta^\dagger(i\omega_j) \oplus \eta^\dagger(-i\omega_j), \qquad \mathbb{U}_j^c := \eta(i\omega_j) \oplus \eta(-i\omega_j).$$

Obviously,

$$\mathbb{V} = \bigoplus_{j=1}^{r} \mathbb{V}_j \tag{16}$$

and each summand is A-invariant (and so are $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{U}_j$).

Denote by $A^c : \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ the complexification of A and let $\mathbb{U}_j \subset \mathbb{R}^{2N}$, $j = 1, \ldots, r$, denote the subspace such that

$$\mathbb{U}_j^c = \mathrm{Ker}(A + i\omega_j) \oplus \mathrm{Ker}(A - i\omega_j).$$

Let $d_j := \frac{1}{2} \dim \mathbb{U}_j$. Clearly d_j is an integer. Let

$$d = d(A) := d_1 + d_2 + \cdots + d_r. \tag{17}$$

In order to make our setup precise, we introduce the following terminology. If $u : \mathbb{R} \to \mathbb{R}^{2N}$ is a periodic C^1-solution to (11) and $\tau \in \mathbb{R}$, then we let u_τ denote the periodic solution to (11) defined by $u_\tau(t) := u(t + \tau)$, $t \in \mathbb{R}$. We say that two periodic solutions u, v to (11) are *geometrically distinct* if $u_\tau \neq v$ for all $\tau \in \mathbb{R}$.

Now we can formulate the main result of this section.

Theorem 7.2. *If H satisfies* (H1) *and* (H2)*, then there exists an $\epsilon_0 > 0$ such that $0 < \epsilon < \epsilon_0$ implies the existence of at least d geometrically distinct periodic solutions to* (11) *in* $\{u \in C(\mathbb{R}, \mathbb{R}^{2N}); \|u\| = \epsilon\}$.

8. Proof of Theorem 7.2

Define the operators $\mathcal{A}, \mathcal{D}_\lambda : \mathcal{E} \to \mathcal{E}_0$ by

$$\mathcal{A}(u)(t) := A(u(t)) = (\nabla^2 H(0))(u(t)), \qquad \mathcal{D}_\lambda u := J\dot{u} + \lambda A(u). \tag{18}$$

Note that $\mathcal{D}_\lambda = Df(0, \lambda)$. For any subspace $\mathbb{Z} \subset \mathbb{R}^{2N}$, we denote

$$\mathcal{E}(\mathbb{Z}) := C^1(S^1, \mathbb{Z}), \qquad \mathcal{E}_0(\mathbb{Z}) := C^0(S^1, \mathbb{Z}).$$

We consider the above function spaces together with the S^1-action defined by

$$(\gamma u)(t) := u(t - \theta) \quad \text{for } \gamma := e^{i\theta}.$$

Notation. Throughout this section we tacitly assume that the considered maps are S^1-equivariant and gradient (in the generalized sense, see Definition 5.1). The gradient structure should be clear from the context.

Define an equivalence relation in the set $\mathfrak{S} := \{\omega_1, \omega_2, \ldots, \omega_q\}$ by

$$\omega_j \sim \omega_k \iff n\omega_j = m\omega_k, \quad n, m \in \mathbb{N}.$$

This relation divides $\mathfrak{S}$ into pairwise disjoint classes

$$\mathfrak{S} = \bigcup_{k=1}^{p} \mathfrak{S}_k.$$

For $k \in \{1, 2, \ldots, q\}$, set $\mathcal{J}_k := \{j \in \{1, \ldots, r\}; \omega_j \in \mathfrak{S}_k\}$, $\mathfrak{D}_k := \mathcal{D}_{\lambda_k}$,

$$\mathbb{W}_k := \bigoplus_{j \in \mathcal{J}_k} \mathbb{V}_j, \quad b_k := \sum_{j \in \mathcal{J}_k} d_j,$$

$\mathcal{W}_k := \mathcal{E}(\mathbb{W}_k) \cap \operatorname{Ker} \mathfrak{D}_k$.

For each k, let ν_k denote the greatest real number such that for every $\omega \in \mathfrak{S}_k$ there is $n \in \mathbb{N}$ such that $\omega = n\nu_k$ and let $\lambda_k := \nu_k^{-1}$.

Suppose $\omega_j \in \mathfrak{S}_k$ and let $n_j := \frac{\omega_j}{\nu_k} \in \mathbb{N}$.

If $z = x + iy \in \mathbb{C}^{2N}$, $x, y \in \mathbb{R}^{2N}$, is an eigenvector corresponding to the eigenvalue $i\omega_j$ of $(JA)^c$, then $(JA)^c(x) = -\omega_j y$, $(JA)^c(y) = \omega_j x$ and thus $x - iy$ is an eigenvector corresponding to the eigenvalue $-i\omega_j$. Therefore, vectors x, y span a subspace of $\mathbb{R}^{2N}$ which is invariant for A. Let $\mathbf{z}_p = \mathbf{x}_p + i\mathbf{y}_p$, $p := 1, \ldots, d_j$, be a basis of $\operatorname{Ker}((JA)^c + i\omega_j I)$. Then the vectors

$$\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2, \ldots, \mathbf{x}_{d_j}, \mathbf{y}_{d_j} \tag{19}$$

form a basis of $\mathbb{U}_j$. Let $\mathbf{c}_j(t) := \cos(n_j t)$, $\mathbf{s}_j(t) := \sin(n_j t)$. Denote by $\mathcal{U}_j$ the $2d_j$-dimensional subspace of $C^1(S^1, \mathbb{U}_j) \subset \mathcal{E}$ spanned by

$$\mathbf{c}_j \mathbf{x}_p + \mathbf{s}_j \mathbf{y}_p, \quad \mathbf{s}_j \mathbf{x}_p - \mathbf{c}_j \mathbf{y}_p, \quad p = 1, \ldots, d_j.$$

Then

$$\mathcal{U}_j = \mathcal{E}(\mathbb{V}_j) \cap \operatorname{Ker} \mathfrak{D}_k \qquad \text{and} \qquad \mathcal{W}_k = \bigoplus_{j \in \mathcal{J}_k} \mathcal{U}_j.$$

Remark 8.1. The assignments

$$\mathbf{c}_j\mathbf{x}_p + \mathbf{s}_j\mathbf{y}_p \mapsto e_p, \quad \mathbf{s}_j\mathbf{x}_p - \mathbf{c}_j\mathbf{y}_p \mapsto ie_p, \quad p = 1, \ldots, d_j,$$

where $\{e_p\}$ denotes the standard basis of $\mathbb{C}^{d_j}$, define an isomorphism of real linear spaces

$$\mathcal{A}_j : \mathcal{U}_j \to \mathbb{C}^{d_j}.$$

Lemma 8.2. *The cup-length of $S(\mathcal{W}_k)$ equals b_k.*

Proof. Consider the complex linear space

$$\mathbf{V} := \bigoplus_{j\in\mathcal{J}_k} \mathbb{C}^{d_j}$$

whose points we write as $z = (z_1, \ldots, z_q), z_j \in \mathbb{C}^{d_j}$. Let $\mathbf{Y}$ and $\mathbf{Z}$ denote, respectively, the representations of S^1 on $\mathbf{V}$ determined by

$$\gamma(z_1, \ldots, z_q) := (\gamma^{d_1} z_1, \ldots, \gamma^{d_q} z_q),$$
$$\gamma(z_1, \ldots, z_q) := (\gamma^{b_k} z_1, \ldots, \gamma^{b_k} z_q).$$

To avoid misunderstandings we denote by $\mathbf{X}$ the standard representation of S^1 on $\mathbf{V}$. Let $\alpha : S(\mathbf{X}) \to S(\mathbf{Y})$, $\beta : S(\mathbf{Y}) \to S(\mathbf{Z})$ denote the S^1-equivariant maps between unit spheres in corresponding representations defined by

$$\alpha(z_1, \ldots, z_q) := (z_1^{d_1}, \ldots, z_q^{d_q}),$$
$$\beta(z_1, \ldots, z_q) := (z_1^{b_k - d_1}, \ldots, z_q^{b_k - d_q}).$$

Obviously $S(\mathbf{X})/S^1 = CP^{d-1}$. A slight modification of the arguments given in [6, Sec. 3.2] permits to prove that $S(\mathbf{Z})$ is diffeomorphic to CP^{a-1} and $\beta \circ \alpha$ induces a monomorphism of cohomology rings

$$(\beta \circ \alpha)^* : H^*(S(\mathbf{Z})/S^1) \to H^*(S(\mathbf{X})/S^1).$$

Therefore,

$$\alpha^* : H^*(S(\mathbf{Y})/S^1) \to H^*(S(\mathbf{X})/S^1)$$

is also a monomorphism. Thus the cup-length of $S(\mathbf{Y})$ equals b_k. Since

$$\bigoplus_{j\in\mathcal{J}_k} \mathcal{A}_j : \mathcal{W}_k \to \mathbf{Y}$$

is an isomorphism of real representations of S^1, the proof is completed. □

Let

$$\mathcal{W}_k^{\perp,0} := \{w \in \mathcal{E}_0(W_k); \langle w, v\rangle = 0 \text{ for } v \in \mathcal{W}_k\},$$
$$\mathcal{W}_k^{\perp} := \mathcal{E}(\mathbb{W}_k) \cap \mathcal{W}_k^{\perp,0}.$$

From (12) and (18), we have

$$\langle u, \mathfrak{D}_k(v)\rangle = \langle \mathfrak{D}_k(u), v\rangle = 0 \quad \text{for } u \in \mathcal{W}_k, v \in \mathcal{E}(\mathbb{W}_k).$$

Therefore, $\mathfrak{D}_k(\mathcal{W}_k^\perp) \subset \mathcal{W}_k^{\perp,0}$. Since $\mathfrak{D}_k$, as an operator from $\mathcal{E}(\mathbb{W}_k)$ into $\mathcal{E}_0(\mathbb{W}_k)$, is Fredholm of index 0, it maps isomorphically $\mathcal{W}_k^\perp$ onto $\mathcal{W}_k^{\perp,0}$. Applying Theorem 5.1, we obtain an $\epsilon > 0$ and a mapping

$$g : B(\mathcal{E}, \epsilon) \times [\lambda_k - \delta, \lambda_k + \delta] \to \mathcal{E}_0,$$

where $B(\mathcal{E}, \epsilon) := \{u \in \mathcal{E}; \|u\| < \epsilon\}$, such that

- f and g determine equivalent bifurcation problems on $[\lambda_k - \delta, \lambda_k + \delta]$;
- $g(B(\mathcal{W}_k, \epsilon) \times [\lambda_k - \delta, \lambda_k + \delta]) \subset \mathcal{W}_k$;
- $Dg(0, \lambda) = Df(0, \lambda)$ for $\lambda \in [\lambda_k - \delta, \lambda_k + \delta]$.

Setting $\varphi(w, \lambda) := g(w, \lambda)$, $w \in \mathcal{W}_k$, we obtain

$$\varphi : \mathcal{W}_k \times [\lambda_k - \delta, \lambda_k + \delta] \to \mathcal{W}_k,$$

which determines a finite-dimensional bifurcation problem on $[\lambda_k - \delta, \lambda_k + \delta]$ (one may call it a *reduction* of f to $\mathcal{W}_k$). Applying Lemmas 8.2 and 6.2, we obtain the following conclusion.

Conclusion 8.3. *For each $k \in \{1, \ldots, q\}$, there exist $\delta, \epsilon > 0$ such that*

(a) *the mapping φ defines a bifurcation problem on $[\lambda_k - \delta, \lambda_k + \delta]$;*
(b) *$f^{-1}(0) \cap (S(\mathcal{E}, \epsilon) \times [\lambda_k - \delta, \lambda_k + \delta])$ contains at least b_k different S^1-orbits.*

Now, to complete the proof of Theorem 7.2, it is enough to observe that, for sufficiently small δ and ϵ, different S^1 orbits in

$$f^{-1}(0) \cap \left(\bigcup_{k=1}^{q} S(\mathcal{E}, \epsilon) \times [\lambda_k - \delta, \lambda_k + \delta] \right)$$

correspond to geometrically distinct solutions to (11).

Acknowledgment

This research was supported by the Polish Ministry of Higher Education grant number N N 201 394037.

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