

# The outer-connected domination number of a graph

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## Abstract

For a given graph  $G = (V, E)$ , a set  $D \subseteq V(G)$  is said to be an *outer-connected dominating set* if  $D$  is dominating and the graph  $G - D$  is connected. The *outer-connected domination number* of a graph  $G$ , denoted by  $\tilde{\gamma}_c(G)$ , is the cardinality of a minimum outer-connected dominating set of  $G$ . We study several properties of outer-connected dominating sets and give some bounds on the outer-connected domination number of a graph. We also show that the decision problem for the outer-connected domination number of a graph  $G$  is NP-complete even for bipartite graphs.

## 1 Introduction

Graph theory terminology not presented here can be found in [1, 5].

Let  $G = (V, E)$  be a simple graph. The *neighbourhood* of a vertex  $v$ , denoted by  $N_G(v)$ , is the set of all vertices adjacent to  $v$  in  $G$ . If  $v$  is a vertex of  $G$  then the integer  $\deg_G(v) = |N_G(v)|$  is said to be the *degree* of  $v$  in  $G$ . The *minimum* and *maximum degree* among all vertices of  $G$  are denoted by  $\delta(G)$  and  $\Delta(G)$ , respectively. A vertex of degree one in a graph is called an *end-vertex*. A *support* is the unique neighbour of an end-vertex. Let  $\Omega = \Omega(G)$  be the set of all end-vertices of  $G$ .

A set  $D \subseteq V(G)$  is a *dominating set* in  $G$  if  $N_G(v) \cap D \neq \emptyset$  for every vertex  $v \in V(G) - D$ . The *domination number* of a graph  $G$ , denoted  $\gamma(G)$ , is the cardinality of a minimum dominating set of  $G$ .

A set  $D \subseteq V(G)$  is said to be an *outer-connected dominating set* of  $G$  if  $D$  is dominating and either  $D = V(G)$  or  $G - D$  is connected. The cardinality of a minimum outer-connected dominating set in  $G$  is called the *outer-connected domination number* of  $G$  and is denoted by  $\tilde{\gamma}_c(G)$ . Observe that every graph  $G$  has an outer-connected dominating set, since the set of all vertices of  $G$  is an outer-connected dominating set in  $G$ .

## 2 Preliminary results

Let  $K_n$ ,  $C_n$  and  $P_n$  denote the complete graph, the cycle and the path of order  $n$ , respectively. For positive integers  $n_1, \dots, n_t$  let  $K_{n_1, \dots, n_t}$  be the complete multipartite graph with vertex set  $S_1 \cup S_2 \cup \dots \cup S_t$ , where  $|S_i| = n_i$  for  $1 \leq i \leq t$ . By a star of order  $n$  we mean the bipartite graph  $K_{1, n-1}$  for  $n \geq 2$ .

In our first observation we present the outer-connected domination number of complete graphs, cycles, paths and complete multipartite graphs.

### Observation 1

- (i)  $\tilde{\gamma}_c(K_n) = 1$  for  $n \geq 1$ ;
- (ii)  $\tilde{\gamma}_c(C_n) = n - 2$  for  $n \geq 3$ ;
- (iii)  $\tilde{\gamma}_c(P_n) = \begin{cases} n - 1, & n = 2, 3, \\ n - 2, & n \geq 4; \end{cases}$
- (iv) If  $t \geq 2$  and  $n_1 \leq n_2 \leq \dots \leq n_t$ , then

$$\tilde{\gamma}_c(K_{n_1, \dots, n_t}) = \begin{cases} n_2 & \text{if } t = 2 \text{ and } n_1 = 1, \\ 1 & \text{if } t \geq 3 \text{ and } n_1 = 1, \\ 2 & \text{if } t \geq 2 \text{ and } n_1 > 1. \end{cases}$$

It follows from the next theorem and from its proof that outer-connected dominating sets and outer-connected domination numbers of a disconnected graph are determined by outer-connected dominating sets and outer-connected domination numbers of its components.

**Theorem 1** *If  $G_1, \dots, G_r$  are the components of a graph  $G$ , then*

$$\tilde{\gamma}_c(G) = |V(G)| - \max\{|V(G_i)| - \tilde{\gamma}_c(G_i) : i = 1, \dots, r\}.$$

**Proof.** Let  $D_1, \dots, D_r$  be minimum outer-connected dominating sets of  $G_1, \dots, G_r$  respectively. Then  $V(G) - (V(G_1) - D_1), \dots, V(G) - (V(G_r) - D_r)$  are outer-connected dominating sets of  $G$  and therefore

$$\begin{aligned} \tilde{\gamma}_c(G) &\leq \min\{|V(G) - (V(G_i) - D_i)| : i = 1, \dots, r\} \\ &= |V(G)| - \max\{|V(G_i) - D_i| : i = 1, \dots, r\} \\ &= |V(G)| - \max\{|V(G_i)| - \tilde{\gamma}_c(G_i) : i = 1, \dots, r\}. \end{aligned}$$

Now let  $D$  be a minimum outer-connected dominating set of  $G$ . Then  $|D| = \tilde{\gamma}_c(G)$  and in addition  $G - D$  is connected. Hence  $V(G) - D \subseteq V(G_l)$  for some  $l \in \{1, \dots, r\}$  and from the minimality of  $D$  it follows that  $D \cap V(G_l)$  is a minimum outer-connected dominating set of  $G_l$ . Thus  $D \cap V(G_l) = \tilde{\gamma}_c(G_l)$  and  $|V(G)| - \tilde{\gamma}_c(G) = |V(G) - D| = |V(G_l) - (D \cap V(G_l))| = |V(G_l)| - \tilde{\gamma}_c(G_l) \leq \max\{|V(G_i)| - \tilde{\gamma}_c(G_i) : i = 1, \dots, r\}$  and therefore  $\tilde{\gamma}_c(G) \geq |V(G)| - \max\{|V(G_i)| - \tilde{\gamma}_c(G_i) : i = 1, \dots, r\}$  which completes the proof.  $\square$



### 3 Bounds

It is obvious that if  $G$  is a graph of order  $n$ , then  $1 \leq \tilde{\gamma}_c(G) \leq n$ . In addition,  $\tilde{\gamma}_c(G) = 1$  if and only if  $G = K_1 + H$ , where  $H$  is a connected graph of order  $n - 1$ , while  $\tilde{\gamma}_c(G) = n$  if and only if  $G = \overline{K_n}$ . Hence  $\tilde{\gamma}_c(G) \leq n - 1$  if  $G$  has at least one edge. Moreover,  $\tilde{\gamma}_c(G) \leq n - 2$  if and only if  $G$  has at least one edge which is not an end-edge. In general,  $\tilde{\gamma}_c(G) \leq n - k$  if and only if there exists a proper connected subgraph  $H$  of  $G$  such that  $|V(H)| = k$  and every vertex of  $H$  has a neighbour which belongs to  $V(G) - V(H)$ .

A characterization of graphs  $G$  of order  $n$  for which  $\tilde{\gamma}_c(G) = n - 1$  is given in the following observation.

**Observation 2** *If  $G$  is a connected graph on  $n \geq 2$  vertices, then  $\tilde{\gamma}_c(G) = n - 1$  if and only if  $G$  is a star.*

Sampathkumar and Walikar [7] have proved that  $\frac{n(G)}{\Delta(G)+1} \leq \gamma_c(G) \leq 2m(G) - n(G)$  for a connected graph  $G$ . Now we present similar inequalities for the outer-connected domination number.

Let  $\mathcal{A}$  be the family of graphs defined as follows: a graph  $G$  belongs to  $\mathcal{A}$  if and only if there exists an outer-connected dominating set  $A$  of  $G$  such that  $|PN_G[v, A]| = \Delta(G) + 1$  for every vertex  $v$  belonging to  $A$ , where  $PN_G[v, A]$  is the private neighbourhood of  $v$  with respect to  $A$ , i.e.  $PN_G[v, A] = N_G[v] - N_G[A - \{v\}]$ .

**Theorem 2** *If  $G$  is a connected graph with  $n(G) \geq 2$ , then*

$$\frac{n(G)}{\Delta(G)+1} \leq \tilde{\gamma}_c(G) \leq 2m(G) - n(G) + 1.$$

*In addition,  $\tilde{\gamma}_c(G) = \frac{n(G)}{\Delta(G)+1}$  if and only if  $G$  belongs to the family  $\mathcal{A}$ , while  $\tilde{\gamma}_c(G) = 2m(G) - n(G) + 1$  if and only if  $G$  is a star.*

**Proof.** Since  $\frac{n(G)}{\Delta(G)+1} \leq \gamma(G)$  (see [8]) and  $\gamma(G) \leq \tilde{\gamma}_c(G)$ , we certainly have  $\frac{n(G)}{\Delta(G)+1} \leq \tilde{\gamma}_c(G)$ . Moreover, since  $G$  is connected and has at least two vertices, we have  $m(G) \geq n(G) - 1$  and  $\tilde{\gamma}_c(G) \leq n(G) - 1$ . Consequently,  $\tilde{\gamma}_c(G) \leq 2m(G) - n(G) + 1$ .

If  $G$  belongs to the family  $\mathcal{A}$ , then there exists an outer-connected dominating set  $A$  of  $G$  for which  $\tilde{\gamma}_c(G) \leq |A| = \frac{n(G)}{\Delta(G)+1} \leq \tilde{\gamma}_c(G)$ . Now assume that  $\tilde{\gamma}_c(G) = \frac{n(G)}{\Delta(G)+1}$  and let  $D$  be a minimum outer-connected dominating set in  $G$ . Then  $|D| = \frac{n(G)}{\Delta(G)+1}$  and this forces each vertex of  $D$  to dominate exactly  $\Delta(G) + 1$  vertices and moreover  $|PN_G[v, D]| = \Delta(G) + 1$ . Consequently  $G \in \mathcal{A}$ .

If  $G$  is a star, then  $\tilde{\gamma}_c(G) = n(G) - 1 = 2m(G) - n(G) + 1$ . If  $\tilde{\gamma}_c(G) = 2m(G) - n(G) + 1$ , then, since  $G$  has at least one edge,  $2m(G) - n(G) + 1 = \tilde{\gamma}_c(G) \leq n(G) - 1$ . Thus  $m(G) \leq n(G) - 1$  and by the connectivity of  $G$ ,  $m(G) = n(G) - 1$ . Consequently,  $\tilde{\gamma}_c(G) = n(G) - 1$  and, according to Observation 2,  $G$  is a star.  $\square$

Before stating the next theorem, we describe a family  $\mathcal{S}$  of graphs, which are the extremal graphs of the theorem.



Let  $\mathcal{S}$  be the family of graphs, where a graph  $G$  belongs to  $\mathcal{S}$  if and only if there exists an independent set  $I$  in  $G$  such that  $G - I$  is a tree and every vertex of  $G - I$  is adjacent to exactly one vertex of  $I$ .

**Theorem 3** *If  $G$  is a graph, then*

$$\tilde{\gamma}_c(G) \geq n(G) - \frac{m(G) + 1}{2}. \tag{1}$$

*In addition,  $\tilde{\gamma}_c(G) = n(G) - \frac{m(G)+1}{2}$  if and only if  $G$  belongs to the family  $\mathcal{S}$ .*

**Proof.** Let  $D$  be a minimum outer-connected dominating set in  $G$  and let  $m_G(D)$  denote the number of edges joining  $D$  and  $V(G) - D$  in  $G$ . Then

$$m(G - D) \geq n(G) - \tilde{\gamma}_c(G) - 1, \tag{2}$$

and

$$m_G(D) \geq n(G) - \tilde{\gamma}_c(G). \tag{3}$$

Hence

$$m(G) \geq m(G - D) + m_G(D) \geq 2n(G) - 2\tilde{\gamma}_c(G) - 1 \tag{4}$$

and thus  $\tilde{\gamma}_c(G) \geq n(G) - \frac{m(G)+1}{2}$ .

We now prove that  $\tilde{\gamma}_c(G) = n(G) - \frac{m(G)+1}{2}$  if and only if  $G$  belongs to the family  $\mathcal{S}$ .

Assume first that  $G$  belongs to  $\mathcal{S}$ . Then there exists an independent set  $I$  in  $G$  such that  $G - I$  is a tree and every vertex of  $G - I$  is adjacent to exactly one vertex of  $I$ . The set  $I$  is an outer-connected dominating set in  $G$ . Thus  $\tilde{\gamma}_c(G) \leq |I| = n(G) - n(G - I)$  and because  $n(G - I) = m_G(I) = m(G) - m(G - I) = m(G) - n(G - I) + 1$  (and therefore  $n(G - I) = \frac{m(G)+1}{2}$ ),

$$\tilde{\gamma}_c(G) \leq n(G) - n(G - I) = n(G) - \frac{m(G) + 1}{2}.$$

Consequently, by (1),  $\tilde{\gamma}_c(G) = n(G) - \frac{m(G)+1}{2}$ .

Assume now that  $\tilde{\gamma}_c(G) = n(G) - \frac{m(G)+1}{2}$ . Let  $D$  be a minimum outer-connected dominating set of  $G$ . Then by (4) we have

$$\begin{aligned} m(G) &\geq m(G - D) + m_G(D) \\ &\geq 2n(G) - 2\tilde{\gamma}_c(G) - 1 \\ &= 2n(G) - 2 \left( n(G) - \frac{m(G)+1}{2} \right) - 1 = m(G). \end{aligned} \tag{5}$$

Hence, and by (2) and (3), it follows that

$$m(G - D) = n(G) - \tilde{\gamma}_c(G) - 1 \tag{6}$$

and

$$m_G(D) = n(G) - \tilde{\gamma}_c(G). \tag{7}$$



Since  $G - D$  is connected (by the choice of  $D$ ) and has  $n(G - D) - 1$  edges,  $G - D$  is a tree. Moreover, since  $m(G) = m(G - D) + m_G(D) + m(G[D])$  and  $m(G) = m(G - D) + m_G(D)$  by (5),  $m(G[D]) = 0$  and  $D$  is an independent set. Now, since  $D$  is dominating, it follows from (7) that each vertex of  $G - D$  has exactly one neighbour in  $D$ . This completes the proof of the fact that  $G$  belongs to the family  $S$ .  $\square$

Probably, the statement of the next lemma is well-known, but since we have not seen such a result anywhere, we state it here with a short proof.

**Lemma 4** *In a graph  $G$  with  $\delta(G) \geq 2$  there is a cycle of length at least  $\delta(G) + 1$ .*

**Proof.** Let  $(v_0, v_1, \dots, v_l)$  be a longest path in  $G$ . Then  $N_G(v_0) \subseteq \{v_1, v_2, \dots, v_l\}$  and therefore  $v_k \in N_G(v_0) \cap \{v_1, v_2, \dots, v_l\}$  for some  $k \geq \deg_G(v_0) \geq \delta(G)$ . Consequently  $(v_0, v_1, \dots, v_k, v_0)$  is a required cycle.  $\square$

**Theorem 5** *If  $G$  is a connected graph of order  $n$ , then*

$$\tilde{\gamma}_c(G) \leq n - \delta(G).$$

**Proof.** The result is obvious if  $\delta(G) \leq 2$ . Now assume that  $\delta(G) \geq 3$ . By Lemma 4 there exists a cycle of length at least  $\delta(G) + 1$  in  $G$ . Let  $C = (v_0, v_1, \dots, v_l, v_0)$  be a shortest cycle in  $G$  of length at least  $\delta(G)$ . We claim that the set  $D = V(G) - V(C)$  is an outer-connected dominating set of  $G$ . Certainly  $G - D = G[V(C)]$  is connected. Suppose  $D$  is not dominating. Then  $N_G(v) \cap D = \emptyset$  for some vertex  $v \in V(C)$ . We may assume, without loss of generality, that  $v = v_0$  and  $\deg_G(v_0) = r$ . Then  $N_G(v_0) = \{v_1, v_{i_1}, v_{i_2}, \dots, v_{i_{r-2}}, v_l\}$  where  $1 < i_1 < i_2 < \dots < i_{r-2} < l$ . Now  $(v_0, v_{i_1}, v_{i_1+1}, \dots, v_l, v_0)$  is a cycle of length at least  $\delta(G)$  which is shorter than  $C$ , a contradiction.  $\square$

In the next observation we describe the main properties of minimum outer-connected dominating sets of a graph.

**Observation 3** *Let  $G$  be a connected graph on at least 3 vertices. If  $D$  is a minimum outer-connected dominating set in  $G$  and  $\Omega$  is the set of end-vertices of  $G$ , then*

- (i)  $\Omega \subseteq D$  if  $G$  is not a star;
- (ii)  $D = \Omega$  or  $|\Omega \cap D| = |\Omega| - 1$  if  $G$  is a star;
- (iii)  $\tilde{\gamma}_c(G) \geq |\Omega|$ ;
- (iv)  $\tilde{\gamma}_c(G) = |\Omega|$  if and only if every vertex of  $G$  is either a support or an end-vertex.

**Proof.** Since (ii) is obvious and (iii) easily follows from (i) and Observation 2, we only prove (i) and (iv).

(i) Assume  $G$  is not a star and suppose to the contrary that  $\Omega - D \neq \emptyset$ . Then, since  $G - D$  is connected,  $\tilde{\gamma}_c(G) = |V(G)| - 1$  and therefore, by Observation 2,  $G$  is a star, a contradiction.

(iv) The statement is trivial for stars. Thus assume  $G$  is not a star and  $\tilde{\gamma}_c(G) = |\Omega|$ .



Then, by (i),  $\Omega$  is a minimum outer-connected dominating set of  $G$  and this implies that every vertex belonging to  $V(G) - \Omega$  is a support. Conversely, if every vertex of the graph  $G$  is a support or an end-vertex, then  $\Omega$  is an outer-connected dominating set of  $G$  and, by (i), it is a minimum outer-connected dominating set of  $G$  and therefore  $\tilde{\gamma}_c(G) = |\Omega|$ .  $\square$

A subdivision of an edge  $uv$  is obtained by inserting a new vertex  $w$  and replacing the edge  $uv$  with the edges  $uw$  and  $wv$ . A spider is the tree obtained from a star by subdividing all of its edges. A wounded spider is a tree obtained from a spider by removing at least one end-vertex. Certainly, a star is also a wounded spider.

The next theorem provides a lower bound for the outer-connected domination number of a tree.

**Theorem 6** *If  $T$  is a tree of order  $n \geq 3$ , then*

$$\tilde{\gamma}_c(T) \geq \Delta(T).$$

*Furthermore,  $\tilde{\gamma}_c(T) = \Delta(T)$  if and only if  $T$  is a wounded spider.*

**Proof.** The result is obvious if  $T$  is a star. Now let  $T$  be a tree of order  $n \geq 3$  and assume  $T$  is not a star. Since  $T$  has at least  $\Delta(T)$  end-vertices and since all end-vertices belong to every outer-connected dominating set of  $T$  we certainly have  $\tilde{\gamma}_c(T) \geq \Delta(T)$ .

Clearly, if  $T$  is a wounded spider, then  $\tilde{\gamma}_c(T) = \Delta(T)$ . Now assume  $T$  is a tree for which  $\tilde{\gamma}_c(T) = \Delta(T)$ . Then since  $\Delta(T) \leq |\Omega(T)| \leq \tilde{\gamma}_c(T)$  we have  $\Delta(T) = |\Omega(T)|$  (and  $\tilde{\gamma}_c(T) = |\Omega(T)|$ ). From the equality  $\Delta(T) = |\Omega(T)|$ , it follows that there exists a unique vertex, say  $u$ , of maximum degree, and  $\Omega(T)$  is a minimum outer-connected dominating set of  $T$ . In addition, every inner vertex of a path joining  $u$  to an end-vertex of  $T$  (if any) is of degree 2. This and the fact that  $\Omega(T)$  is dominating implies that every such a path is of length at most two and at least one of them is of length one. This proves that  $T$  is a wounded spider.  $\square$

Let  $\mathcal{R}, \mathcal{R}', \mathcal{R}''$  be families of trees on at least 3 vertices defined as follows: a tree  $T$  belongs to  $\mathcal{R}$  if  $T$  is the corona of another tree, while a tree  $T$  belongs to  $\mathcal{R}'$  or  $\mathcal{R}''$ , respectively, if  $T$  is obtained from a tree  $S$  belonging to  $\mathcal{R}$  by adding a new vertex and joining it to an end-vertex of  $S$  or to an inner vertex of  $S$ , respectively.

**Theorem 7** *If  $T$  is a tree of order  $n \geq 3$ , then*

$$\tilde{\gamma}_c(T) \geq \left\lceil \frac{n}{2} \right\rceil$$

*with equality  $\tilde{\gamma}_c(T) = \lceil \frac{n}{2} \rceil$  if and only if  $T$  belongs to  $\mathcal{R} \cup \mathcal{R}' \cup \mathcal{R}''$ .*

**Proof.** Let  $T = (V, E)$  be a tree and let  $D$  be a minimum outer-connected dominating set of  $T$ . Suppose, on the contrary, that  $\tilde{\gamma}_c(T) < \lceil \frac{n}{2} \rceil$ . Then  $\tilde{\gamma}_c(T) < \frac{n}{2}$  and by the pigeon hole principle  $|N_T(v) \cap (V - D)| \geq 2$  for some  $v \in D$ . But then any path joining two vertices of  $N_T(v) \cap (V - D)$  in the connected graph  $T - D$  together with  $v$  form a cycle in  $T$ , which is impossible.



Now we prove that  $\tilde{\gamma}_c(T) = \lceil \frac{n}{2} \rceil$  if and only if  $T$  belongs to  $\mathcal{R} \cup \mathcal{R}' \cup \mathcal{R}''$ .

If  $T \in \mathcal{R}$ , then  $T$  is a corona and  $\Omega(T)$  is a minimum outer-connected dominating set of  $T$  and  $\tilde{\gamma}_c(T) = |\Omega(T)| = \frac{n}{2} = \lceil \frac{n}{2} \rceil$ .

Assume  $T \in \mathcal{R}' \cup \mathcal{R}''$ . Then there exists an end-vertex  $v$  such that  $T - v$  is a corona. If  $T \in \mathcal{R}'$  and  $u$  is a neighbour of  $v$ , then  $\Omega(T) \cup \{u\}$  is a minimum outer-connected dominating set of  $T$  and  $\tilde{\gamma}_c(T) = |\Omega(T)| + 1 = \frac{n-1}{2} + 1 = \lceil \frac{n}{2} \rceil$ . Finally, if  $T \in \mathcal{R}''$ , then  $\Omega(T)$  is a minimum outer-connected dominating set of  $T$  and  $\tilde{\gamma}_c(T) = |\Omega(T)| = \lceil \frac{n}{2} \rceil$ .

Let  $T$  be a tree of order at least 3 such that  $\tilde{\gamma}_c(T) = \lceil \frac{n}{2} \rceil$ . If  $n = 3$ , then certainly  $T = P_3 \in \mathcal{R} \cup \mathcal{R}' \cup \mathcal{R}''$ . Thus assume  $T$  has at least 4 vertices. Then  $\lceil \frac{n}{2} \rceil < n - 1$  which implies that  $\tilde{\gamma}_c(T) < n - 1$ , so  $T$  is not a star. Consequently, by Observation 3,  $\Omega(T) \subseteq D$ . If  $D = \Omega(T)$ , then  $|\Omega(T)| = \lceil \frac{n}{2} \rceil$  and therefore every vertex belonging to  $V - \Omega(T)$  is adjacent to exactly one vertex in  $\Omega(T)$  or one of them is adjacent to two end-vertices and each of the other vertices is adjacent to exactly one end-vertex. This implies  $T$  belongs to  $\mathcal{R}$  or  $\mathcal{R}''$ .

Finally assume  $\Omega(T) \subsetneq D$ . Then there exists a vertex  $v \in D$  such that  $\deg_T(v) \geq 2$ . We shall prove that  $\deg_T(v) = 2$  and  $v$  is the only such vertex. From the connectivity of  $T - D$  it follows that  $|N_T(v) \cap (V - D)| \leq 1$ . We claim that  $|N_T(v) \cap D| \leq 1$ . Suppose, to the contrary, that two vertices  $x$  and  $y$  belong to  $N_T(v) \cap D$ . Since  $T$  is a tree we have  $|N_T(\{x, y, v\}) \cap (V - D)| \leq 1$ . This, and the fact that no two vertices in  $V - D$  share common neighbour in  $D$ , imply that  $|V - D| = |N_T(\{x, y, v\}) \cap (V - D)| + |N_T(D - \{x, y, v\}) \cap (V - D)| \leq 1 + |D| - 3 = |D| - 2$ . Hence,  $n - |D| \leq |D| - 2$  and  $|D| \geq \frac{n}{2} + 1 > \lceil \frac{n}{2} \rceil$ . Thus  $v$  has exactly one neighbour in  $D$  and exactly one neighbour in  $V - D$ . Suppose now that  $|D - \Omega(T)| \geq 2$ . Then there exist  $u, v \in D$  such that  $\deg_T(u) = \deg_T(v) = 2$ . Denote by  $u_1$  and  $v_1$  the neighbour of  $u$  and  $v$  in  $D$ , respectively. Then  $|V - D| = |N_T(\{u, u_1, v, v_1\}) \cap (V - D)| + |N_T(D - \{u, u_1, v, v_1\}) \cap (V - D)| \leq 2 + |D| - 4 = |D| - 2$ . Hence,  $n - |D| \leq |D| - 2$  and  $\lceil \frac{n}{2} \rceil = |D| \geq \frac{n}{2} + 1 > \lceil \frac{n}{2} \rceil$ , a contradiction. We obtain that  $v$  is the unique vertex of degree two in  $D$  and the vertex  $x \in N_T(v) \cap D$  is an end-vertex of  $T$ . Thus, we conclude that  $T$  belongs to the family  $\mathcal{R}'$ .  $\square$

As an immediate consequence of this theorem and of Ore's theorem [6] we have the following corollary.

**Corollary 1** *For a tree  $T \neq K_1$  we have  $\tilde{\gamma}_c(T) = \gamma(T)$  if and only if  $T$  is a corona.*

### 4 Edge subdivision and vertex removing

Now we examine the effects on  $\tilde{\gamma}_c(G)$  when  $G$  is modified by an edge subdivision. We start with some notation. If  $uv$  is an edge of  $G$  then by  $G \oplus w_{uv}$  we denote the graph obtained from  $G$  by the subdivision of  $uv$ .

**Theorem 8** *For every integer  $k$  there exist a graph  $G$  and an edge  $uv$  of  $G$  such that  $\tilde{\gamma}_c(G \oplus w_{uv}) - \tilde{\gamma}_c(G) = k$ .*



**Proof.** We consider three cases.

Case 1. If  $k \leq -2$  then we construct graphs  $G$  and  $G \oplus w_{uv}$  as follows. We begin with four spiders  $S_i$  with  $|V(S_i)| = 2|k| - 1$  and denote its centers by  $x_i, i = 1, 2, 3, 4$ . Next we add four end-vertices  $y_i$  and four edges  $x_i y_i$ . Finally, to obtain the graph  $G$ , we add vertices  $u, v$  and edges  $uv, ux_1, ux_2, vx_3, vx_4, x_1 x_3$  (see Fig. 1). It is easy to observe that  $D = N_G[x_2] \cup \Omega(G)$  is a minimum outer-connected dominating set of  $G$  and thus  $\tilde{\gamma}_c(G) = 5|k| + 1$ .

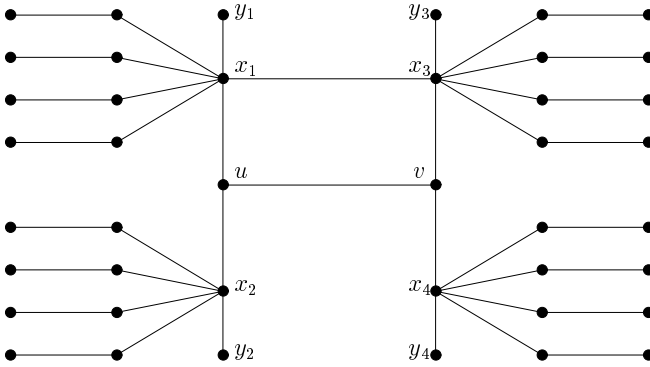


Figure 1: Graph  $G$  for  $k = -5$

Let  $G \oplus w_{uv}$  be a graph which results if the edge  $e = uv$  is subdivided (see Fig. 1). Notice that  $D = \{w\} \cup \Omega(G)$  is the minimum outer-connected dominating set of  $G \oplus w_{uv}$  of cardinality  $4|k| + 1$ . Thus  $\tilde{\gamma}_c(G \oplus w_{uv}) = 4|k| + 1$  and  $\tilde{\gamma}_c(G \oplus w_{uv}) - \tilde{\gamma}_c(G) = -|k| = k$ .

Case 2. Define  $A = \{u, v, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$ . Then for  $k = -1, 0, 1$  let  $H$  be the subgraph of  $G$  induced by  $A, A - \{y_4\}, A - \{y_3, y_4\}$ , respectively. The difference  $\tilde{\gamma}_c(H \oplus w_{uv}) - \tilde{\gamma}_c(H) = k$  is easy to verify.

Case 3. If  $k \geq 2$ , then let  $G$  be the join  $P_{3k} + K_1$ , where  $x_1, x_2, \dots, x_{3k}$  are consecutive vertices of  $P_{3k}$  and  $x$  is the universal vertex of  $G$ . Then obviously  $\{x\}$  is a minimum outer-connected dominating set of  $G$ , so  $\tilde{\gamma}_c(G) = 1$ . It is also easy to see that  $D = \{x, x_1, \dots, x_k\}$  is a minimum outer-connected dominating set of  $G \oplus w_{x_k x_{k+1}}$ . Hence  $\tilde{\gamma}_c(G \oplus w_{x_k x_{k+1}}) = k + 1$  and the proof is complete.  $\square$

Now we investigate how removing a vertex influences an outer-connected domination number. We have the following two propositions.

**Proposition 9** For every connected graph  $G$  and a vertex  $v \in V(G)$  such that  $G - v$  is connected we have

$$\tilde{\gamma}_c(G) \leq \tilde{\gamma}_c(G - v) + 1.$$





**Proof.** If  $D$  is a minimum outer-connected dominating set of  $G - v$ , then clearly  $D \cup \{v\}$  is a minimum outer-connected dominating set in  $G$  and therefore  $\tilde{\gamma}_c(G) \leq \tilde{\gamma}_c(G - v) + 1$ .  $\square$

**Proposition 10** *For every integer  $k \geq -1$ , there exists a graph  $G$  such that  $\tilde{\gamma}_c(G - v) - \tilde{\gamma}_c(G) = k$ .*

**Proof.** If  $k \geq 1$ , then let  $G$  be the graph which results if we add to a path  $P_{k+3}$  a vertex  $v$  and edges joining  $v$  to all vertices from the path. The vertex  $v$  is a universal non-cut vertex of  $G$  and thus we have  $\tilde{\gamma}_c(G) = 1$ . Next we remove  $v$  with all edges incident to  $v$ . Notice that  $G - v$  is a path on at least four vertices, so by Observation 1,  $\tilde{\gamma}_c(G - v) = k + 3 - 2$ . Thus  $\tilde{\gamma}_c(G - v) - \tilde{\gamma}_c(G) = k$ . For  $k = 0$  and  $k = -1$  let  $G$  be a path  $P_2$  and  $P_3$ , respectively, and let  $v$  be an end-vertex of  $G$ . It is easy to verify that  $\tilde{\gamma}_c(G - v) - \tilde{\gamma}_c(G) = k$ .  $\square$

## 5 Comparing $\tilde{\gamma}_c$ to other types of domination numbers

In this section we investigate relations between the outer-connected domination number and other types of domination numbers. We begin with some definitions.

A set  $D \subseteq V(G)$  is a *connected dominating set* of  $G$  if it is dominating and the induced subgraph  $G[D]$  is connected. The cardinality of a minimum connected dominating set of  $G$  is the *connected domination number* and is denoted by  $\gamma_c(G)$ .

We say that a set  $D \subseteq V(G)$  is a *doubly connected dominating set* of  $G$  if it is dominating and the induced subgraphs  $G[D]$  and  $G[V(G) - D]$  are connected. The cardinality of a minimum doubly connected dominating set in  $G$  is a *doubly connected domination number* and is denoted by  $\gamma_{cc}(G)$ . Properties of the doubly connected domination number of a graph are studied in [2].

A set  $D \subseteq V(G)$  is a *restrained dominating set* if every vertex in  $V(G) - D$  is adjacent to a vertex in  $D$  and to another vertex in  $V(G) - D$ . By  $\gamma_r(G)$  we denote the size of a smallest restrained dominating set of  $G$ . This type of domination was studied for example in [3].

Since for an arbitrary graph  $G$  every connected dominating set is a dominating set and every doubly connected dominating set is a connected dominating set, we have the following inequality chain

$$\gamma(G) \leq \tilde{\gamma}_c(G) \leq \gamma_{cc}(G).$$

However, each of the differences  $\tilde{\gamma}_c(G) - \gamma(G)$  and  $\gamma_{cc}(G) - \tilde{\gamma}_c(G)$  may be arbitrarily large.

**Proposition 11** *For any non-negative integers  $r$  and  $t$ , there exists a graph  $G$  such that  $\tilde{\gamma}_c(G) - \gamma(G) = r$  and  $\gamma_{cc}(G) - \tilde{\gamma}_c(G) = t$ .*

**Proof.** Let  $G$  be the graph obtained from the star  $K_{1,r+t+1}$  by subdividing  $t$  of its edges. It is easy to verify that  $\gamma(G) = 1+t$ ,  $\tilde{\gamma}_c(G) = r+t+1$  and  $\gamma_{cc}(G) = |V(G)| - 1 = r + 2t + 1$ .  $\square$



In the next proposition we prove that the numbers  $\tilde{\gamma}_c(G)$  and  $\gamma_c(G)$  are incomparable.

**Proposition 12** *For every positive integer  $r$  there exist graphs  $G_1$  and  $G_2$  such that  $\tilde{\gamma}_c(G_1) - \gamma_c(G_1) = r$  and  $\gamma_c(G_2) - \tilde{\gamma}_c(G_2) = r$ .*

**Proof.** Let  $G_1$  be a star of order  $r + 2$  and let  $G_2$  be a graph pictured in Figure 2. It is straightforward to verify that  $\tilde{\gamma}_c(G_1) = r + 1$ ,  $\gamma_c(G_1) = 1$  and  $\tilde{\gamma}_c(G_2) = r + 2$ ,  $\gamma_c(G_2) = 2r + 2$ . Therefore,  $\tilde{\gamma}_c(G_1) - \gamma_c(G_1) = r$  and  $\gamma_c(G_2) - \tilde{\gamma}_c(G_2) = r$ .  $\square$

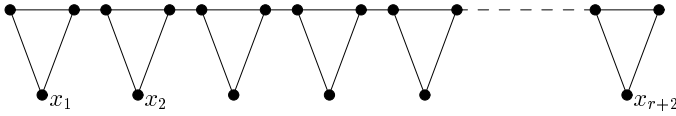


Figure 2: Graph  $G_2$

**Theorem 13** *For any connected graph  $G$  with  $n(G) > 1$ ,*

- (i)  $\gamma_r(G) \leq \tilde{\gamma}_c(G) + 1$ ;
- (ii)  $\gamma_r(G) = \tilde{\gamma}_c(G) + 1$  if and only if  $G$  is a star;
- (iii) *For any non-negative integer  $k$  there exists a graph  $G$  such that  $\tilde{\gamma}_c(G) - \gamma_r(G) = k$ .*

**Proof.**

- (i) If  $\tilde{\gamma}_c(G) \leq n(G) - 2$ , then every outer-connected dominating set of  $G$  is a restrained dominating set of  $G$  and therefore  $\gamma_r(G) \leq \tilde{\gamma}_c(G) \leq \tilde{\gamma}_c(G) + 1$ . Otherwise  $\tilde{\gamma}_c(G) = n(G) - 1$  and, by Observation 2,  $G$  is a star, so  $\gamma_r(G) = n(G) = \tilde{\gamma}_c(G) + 1$ .
- (ii) The result follows immediately from (i).
- (iii) Let  $G = (k + 1)K_2 + K_1$ . It is an easy exercise to verify that  $\tilde{\gamma}_c(G) - \gamma_r(G) = k$ .  $\square$

## 6 Complexity issues for $\tilde{\gamma}_c$

In this section we consider the decision problem of the OUTER-CONNECTED DOMINATING SET as follows

### OUTER-CONNECTED DOMINATING SET (OCDS)

**INSTANCE:** A graph  $G = (V, E)$  and a positive integer  $k \leq |V|$ .

**QUESTION:** Does  $G$  have an outer-connected dominating set of cardinality at most  $k$ ?



The decision problem of OCDS stays  $NP$ -complete even when restricted to connected bipartite graphs.

To prove that the decision problem for arbitrary graphs is  $NP$ -complete, we need to use a well-known  $NP$ -completeness result, called Exact Three Cover (X3C), which is defined as follows.

**EXACT COVER BY 3-SETS (X3C)**

**INSTANCE:** A finite set  $X$  with  $|X| = 3q$  and a collection  $\mathcal{C}$  of 3-element subsets of  $X$ .

**QUESTION:** Does  $\mathcal{C}$  contain an exact cover for  $X$ , that is, a subcollection  $\mathcal{C}' \subseteq \mathcal{C}$  such that every element of  $X$  occurs in exactly one member of  $\mathcal{C}'$ ?

Garey and Johnson in [4] proved that X3C is  $NP$ -complete.

**Theorem 14** *OCDS for bipartite graphs is  $NP$ -complete.*

**Proof.** We know that the OCDS problem for bipartite graphs is in the  $NP$  class of decision problems as it is easy to verify in polynomial time whether a given subset of vertices of  $G$  is an outer-connected dominating set of  $G$ . To show that OCDS is an  $NP$ -complete problem, we will establish a polynomial transformation from X3C. Let  $X = \{x_1, x_2, \dots, x_{3q}\}$  and  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$  be an arbitrary instance of X3C.

We will construct a bipartite graph  $G$  and a positive integer  $k$  such that this instance of X3C will have an exact three cover if and only if  $G$  has an outer-connected dominating set of cardinality at most  $k$ .

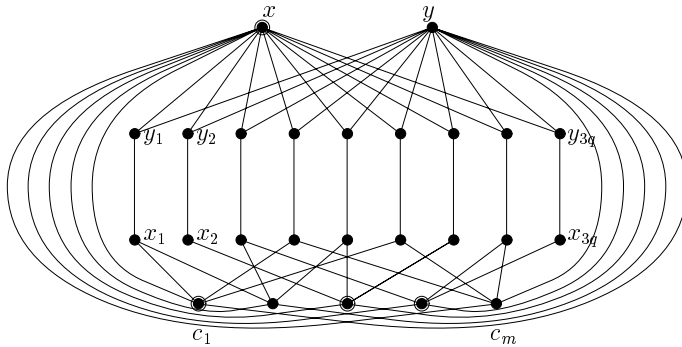


Figure 3: Reduction from X3C to OCDS

Now we describe the construction of  $G$  and  $k$  as follows:

$$\begin{aligned}
 V(G) &= \{x_1, x_2, \dots, x_{3q}\} \cup \{y_1, y_2, \dots, y_{3q}\} \cup \{c_1, c_2, \dots, c_m\} \cup \{x, y\}, \\
 E(G) &= \{x_i y_i : i \in \{1, 2, \dots, 3q\}\} \\
 &\quad \cup \{x_i c_j : x_i \in C_j, i \in \{1, 2, \dots, 3q\}, j \in \{1, 2, \dots, m\}\} \\
 &\quad \cup \{x y_i, y y_i : i \in \{1, 2, \dots, 3q\}\} \\
 &\quad \cup \{x c_j, y c_j : j \in \{1, 2, \dots, m\}\} \\
 k &= q + 1.
 \end{aligned}$$



The graph  $G$  so obtained is connected and bipartite.

Assume first that  $\mathcal{C}$  has an exact 3-cover, say  $\mathcal{C}'$ . Then  $\{c_j: C_j \in \mathcal{C}'\} \cup \{x\}$  is an outer-connected dominating set of cardinality  $q + 1$ .

Now assume that  $D$  is an outer-connected dominating set of cardinality at most  $q + 1$ . If  $x$  and  $y$  do not belong to  $D$ , then since  $D$  is dominating, at least  $3q$  vertices of  $G$  belong to  $D$  to dominate  $y_i$ ,  $i = 1, 2, \dots, 3q$ , so  $|D| \geq 3q$ , a contradiction. Hence, at least one of  $x$  and  $y$ , say  $x$ , belongs to  $D$ . Notice that  $N[x] \cap X = \emptyset$ . Moreover, for each vertex  $u$  belonging to  $\{x_1, \dots, x_{3q}, y_1, \dots, y_{3q}\}$ ,  $|N_G(u) \cap X| = 1$  and for every vertex  $v$  belonging to  $\{c_1, \dots, c_m\}$ ,  $|N_G(v) \cap X| = 3$ . Hence  $y \notin D$  and exactly  $q$  vertices of  $\{c_1, \dots, c_m\}$ , say vertices  $c_{j_1}, \dots, c_{j_q}$ , must belong to  $D$  in such a way that the corresponding set  $\{C_{j_1}, \dots, C_{j_q}\}$  is an exact cover of  $X$ .  $\square$

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(Received 13 July 2005; revised 27 Nov 2006)

