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The quick measure of a NURBS surface curvature for accurate triangular meshing

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ABSTRACT

NURBS surfaces are the most widely used surfaces for three-dimensional models in CAD/CAE programs. When a model for FEM calculation is prepared with a CAD program it is inevitable to mesh it finally. There are many algorithms for meshing planar regions. Some of them may be used for meshing surfaces but it is necessary to take the curvature of the surface under consideration to avoid poor quality mesh. The mesh must be denser in the curved regions of the surface. In this paper, instead of analysing a surface curvature, the method to assess how close is a mesh triangle to the surface to which its vertices belong, is presented. The distance between a mesh triangle and a parallel tangent plane through a point on a surface is the measure of the triangle quality. Finding the surface point whose projection is located inside the mesh triangle and which is the tangency point to the plane parallel to this triangle is an optimization problem. Mathematical description of the problem and the algorithm to find its solution are also presented in the paper.

Keywords: triangulation; meshing; curvature; NURBS surface; optimization

INTRODUCTION

Triangular meshing is achieved with algorithms which use either Delaunay Triangulation (DT) presented in [2] and [6] or Constrained Delaunay Triangulation (CDT) presented in [3]. Empty circumcircle condition of Delaunay Triangulation ensures best shaped triangles on a given set of vertices. Also mesh refinement algorithms as in [10], which improve mesh by inserting new vertices to break triangles having excessive angles hold empty circumcircle condition. Checking in two-dimensional space if there is any other vertex within circle circumscribed on any mesh triangle, is not difficult. However it requires precise calculations due to round-off vulnerability of floating math – see [11]. Provided that points A, B, C are in the counter-clockwise order point D lies within ABC circumcircle when condition presented in formula (1.1) is satisfied.

$$\det\begin{vmatrix} x_A - x_D & y_A - y_D & (x_A - x_D)^2 + (y_A - y_D)^2 \\ x_B - x_D & y_B - y_D & (x_B - x_D)^2 + (y_B - y_D)^2 \\ x_C - x_D & y_C - y_D & (x_C - x_D)^2 + (y_C - y_D)^2 \end{vmatrix} < 0 \tag{1.1}$$

In our considerations we assume the surface to be described as NURBS (Non-Uniform Rational B-Spline) surface. A NURBS surface of degree p in u direction and degree q in v direction is a bivariate vector-valued function described by the formula (1.2) as in [5] and [9].

$$\mathbf{S}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} R_{i,j}(u,v)\mathbf{P}_{i,j} \tag{1.2}$$

The $\{P_{i,j}\}$ is a bidirectional net of control points and $R_{i,j}(u, v)$ is a piecewise rational basis function as in formula (1.3).

$$R_{i,j}(u,v) = \frac{N_{i,p}(u)N_{j,q}(v)\varpi_{i,j}}{\sum_{k=0}^{n}\sum_{l=0}^{m} N_{k,p}(u)N_{l,q}(v)\varpi_{k,l}} \tag{1.3}$$

The $\{\varpi_{i,j}\}$ are the weights while $\{N_{i,p}(u)\}$ and $\{N_{j,q}(v)\}$ are non-rational B-spline basis functions defined on the knot vectors:

$$U = \{0,..., 0, u_{p+1},..., u_{r-p-1}, 1,..., 1\}$$

$$V = \{0,..., 0, v_{q+1},..., u_{s-q-1}, 1,..., 1\}$$

Formula (1.2) and (1.3) is a parametric form of the NURBS surface. Parameters u and v define parametric space for the surface which is a plane. Vector-valued function returns a vector of co-ordinates (x, y, z) – point on a surface, for given (u, v) pair of parameters. If a surface was developable it should be enough to triangulate it in parameter space (on a plane). Then the result triangles in 3D space should have the same properties as in the parameter space. However the surface may be curved in such a way that transformation from parametric space do 3D space causes distortion which changes length and angles. This is the reason why it is not possible to mesh only in the parametric space. To properly mesh the surface the triangulation must be denser in the regions where the distortion is more significant.

NURBS CURVATURE

To be able to identify regions where the triangular mesh is to be denser it is necessary to define some measure of distortion in mapping between parametric space and the surface. The simplest measure is curvature. The curvature is defined for parametric curve C as its second derivative with respect to the curve length - see formula (2.1) as in [5].

$$k = k(s) = \frac{d\mathbf{C}'(s)}{ds} = \mathbf{C}''(s) \tag{2.1}$$

In Euclidean three-dimensional space there are two curvatures defined for a parametric surface S(u, v): Gaussian curvature and mean curvature. Both requires finding two principal directions on the surface - the surface curves with maximum and minimum curvatures. Hence to compute a surface curvature at (u, v) it is necessary to select two surface curves from all surface curves crossing this point: one with maximum curvature k1 and one with minimum curvature k2. The Gaussian curvature is the product of k1 and k2 – see formula (2.2) as in [1].

$$K = k1 \cdot k2 \tag{2.2}$$

The mean curvature is the mean value of k1 and k2 – see formula (2.3) as in [1].

$$H = \frac{k1 + k2}{2} \tag{2.3}$$

In case of general parametric surface S(u, v) finding the principal directions might be computationally expensive.

TANGENT PLANE AND DISTANCE

Triangular meshing of a surface is a method of surface approximation. The maximum distance between a point on the surface and its approximating mesh seems to be a good measure of mesh quality. It seems even better than finding the maximum curvature point – see Fig. 1.

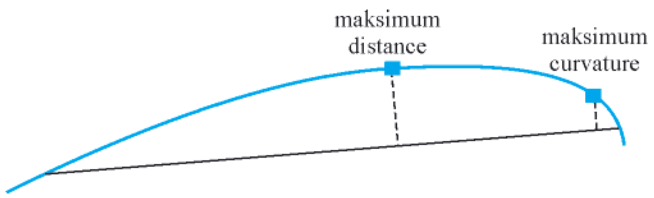

Fig. 1. *Maximum distance and maximum curvature points*

In [8] Laug proposed two measures: distance deviation A0 and angle deviation (difference in a normal vector angle) A1. However he did not give any suggestions how to compute the measures and how expensive the calculations are. He focused mainly on discussing when to apply each of the measures to obtain pertinent results.

Finding maximum distance for the entire mesh requires checking every mesh triangle. To assess the quality of a single mesh triangle it is necessary to find a plane tangent to the surface, which is parallel to the triangle. When a point of tangency lies between vertices of the triangle in the parametric space then the distance between the parallel plane and the triangle is the maximum distance between a point on the surface and the triangle – see Fig. 2.

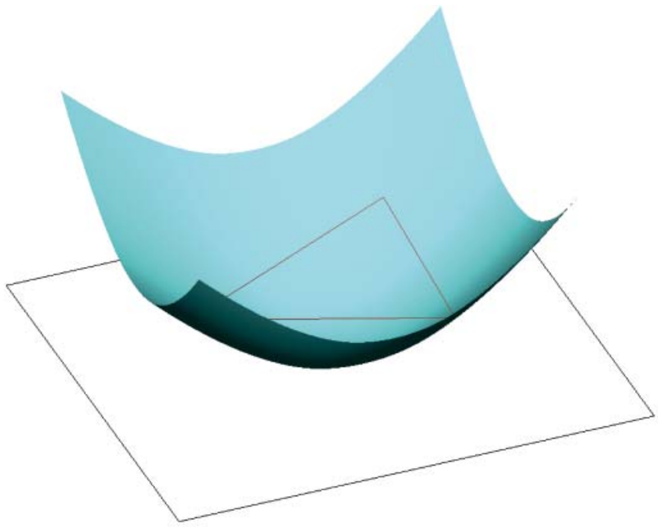

Fig. 2. *A mesh triangle and the parallel plane tangent to the surface*

OPTIMIZATION PROBLEM

Finding the parallel tangent plane requires solving a system of two equations presented in formula (4.1). Vector N is known as we know three vertices P_1, P_2, P_3 of the mesh triangle, which also belong to the surface S.

$$\begin{gathered} \mathbf{N} \bullet \mathbf{S}'_u(u, v) \\ \mathbf{N} \bullet \mathbf{S}'_v(u, v) \end{gathered} \tag{4.1}$$

where:
$\mathbf{N} = \mathbf{P_1P_2} \times \mathbf{P_1P_3}$
$\mathbf{S}'_u(u, v)$ – the first derivative with respect to u
$\mathbf{S}'_v(u, v)$ – the first derivative with respect to v

Geometric interpretation of the formula (4.1) is as follows: the plane tangent to the surface S is defined by two partial derivatives: S'_u with respect to parameter u, and S'_v with respect to parameter v. Each partial derivative is a vector tangent to the surface S in the same point but in different direction. If vector N is perpendicular to the plane tangent to the surface S, then its dot product with each partial derivative S'_u and S'_v must be zero.

Solution of the system of equations (4.1) is a pair of parameters (u, v). With these parameters we obtain a point P = S(u, v), which is a point of tangency. The distance between the point P and the triangle P_1 P_2 P_3 shows how close is the mesh triangle to the surface. The closer the point the better the quality of the triangle and vice versa.

There might be more than one solution for the system of two equations presented in formula (4.1). It is due to the fact that the surface is a free-form surface of any shape. Finding more than one solution is a natural indication for making triangular mesh denser.

OPTIMIZATION ALGORITHM

To solve the system of equations presented in formula (4.1) we use Newton iteration. The method is similar to that used to solve the point inversion problem presented in [9] and [4]. First we change each equation into the function as in formula (5.1)

$$\begin{gathered} f(u, v) = \mathbf{N} \bullet \mathbf{S}'_u(u, v) = 0 \\ g(u, v) = \mathbf{N} \bullet \mathbf{S}'_v(u, v) = 0 \end{gathered} \tag{5.1}$$

Iterations require to change the values of parameters u and v.

$$u_{i+1} = u_i - \Delta u \qquad v_{i+1} = v_i - \Delta v \tag{5.2}$$

where:

$$\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} = \begin{bmatrix} f'_u(u_i, v_i) & f'_v(u_i, v_i) \\ g'_u(u_i, v_i) & g'_v(u_i, v_i) \end{bmatrix}^{-1} \cdot \begin{bmatrix} f(u_i, v_i) \\ g(u_i, v_i) \end{bmatrix} \tag{5.3}$$

$$f'_u(u, v) = \mathbf{N} \bullet \mathbf{S}''_{uu}(u, v) \tag{5.4}$$

$$f'_v(u, v) = \mathbf{N} \bullet \mathbf{S}''_{uv}(u, v) \tag{5.5}$$

$$g'_u(u, v) = \mathbf{N} \bullet \mathbf{S}''_{vu}(u, v) \tag{5.6}$$

$$g'_v(u, v) = \mathbf{N} \bullet \mathbf{S}''_{vv}(u, v) \tag{5.7}$$

Iterations are finished when the convergence criterion (5.8) has been met or maximum number of iterations has been reached.

$$\left| \mathbf{S}(u_i, v_i) - \mathbf{S}(u_{i-1}, v_{i-1}) \right| < \varepsilon \tag{5.8}$$

The found solution is the local extreme. As it should not take many iterations it is possible to start searching for the solution from each of the vertices of the mesh triangle. If the result is the same from each vertex it is possible that the solution found is the only solution.

The Newton iteration has been chosen to find solution because it involves relatively not expensive calculations: i.e. the first and second partial derivatives and matrix multiplication. It is also recognized capable of to bringing most reliable results in other fundamental calculations performed on NURBS e.g. point inversion. To find the maximum distance between a mesh triangle and the surface one could also use random search methods. It would simplify calculations, however it would require more iterations as the process of choosing subsequent solutions is of a stochastic nature.

EXPERIMENT RESULTS

The NURBS surface of second degree (quadratic) has been taken to experiment. Tab. 1 lists control points of the surface.

The shape of the NURBS surface is presented with green color in Fig. 3. The rectangular net of control points is shown above the surface.

Six triangles with vertices on the surface have been chosen to check the tangent plane search algorithm presented in the previous chapter. For each triangle a parallel plane tangent to the surface has been found and a distance between a plane and a triangle has been calculated. The results are presented in Tab. 2.

Each triangle vertex in Tab. 2 is described both in three-dimensional space with the co-ordinates (x, y, z) and in parametric space of the NURBS surface with the pair of parameters (u, v).

Tab. 1. *Control points $\{P_{i,j}\}$ of the NURBS surface*

$P_{i,j}$	**u direction**			
v direction	(0.0, 0.0, 0.0)	(10.0, 0.0, 0.0)	(20.0, 0.0, 0.0)	(30.0, 0.0, 0.0)
	(0.0, 10.0, 0.0)	(10.0, 10.0, 10.0)	(20.0, 10.0, 10.0)	(30.0, 10.0, 0.0)
	(0.0, 20.0, 2.0)	(10.0, 20.0, 12.0)	(20.0, 20.0, 12.0)	(30.0, 20.0, 2.0)
	(0.0, 30.0, 0.0)	(10.0, 30.0, 10.0)	(20.0, 30.0, 10.0)	(30.0, 30.0, 0.0)
	(0.0, 40.0, 0.0)	(10.0, 40.0, 0.0)	(20.0, 40.0, 0.0)	30.0, 40.0, 0.0)

The surface knot vector in u direction : U = {0.0, 0.0, 0.0, 0.5, 1.0, 1.0, 1.0}
The surface knot vector in v direction : V = {0.0, 0.0, 0.0, 0.4, 0.6, 1.0, 1.0, 1.0}

Tab. 2. *Tangent plane search results for the six triangles on the NURBS surface*

No	1-st vertex	2-nd vertex	3-rd vertex	Normal vector	Tangency point	Distance	Iterations
1	(11.02, 14.23, 9.48) (0.33, 0.33)	(11.02, 25.41, 9.61) (0.33, 0.66)	(18.71, 20.00, 10.64) (0.66, 0.50)	(12.25, 0.98, -85.952)	(14.20, 19.81, 11.61)	1.593	2
2	(13.95, 18.33, 11.48) (0.45, 0.45)	(13.95, 21.67, 11.48) 0.45, 0.55)	(16.05, 20.00, 11.57) (0.55, 0.50)	(0.28, 0.00, -7.00)	(14.79, 20.00, 11.66)	0.146	2
3	(3.80, 13.13, 4.13) (0.10, 0.30)	(3.80, 16.67, 4.93) (0.10, 0.40)	(12.80, 14.95, 10.47) (0.40, 0.35)	(21.00, 7.26, -31.88)	(8.87, 16.21, 8.81)	0.530	3
4	(7.20, 13.13, 6.75) (0.20, 0.30)	(7.20, 16.67, 7.73) (0.20, 0.40)	(12.80, 14.95, 10.47) (0.40, 0.35)	(11.39, 5.51, -19.83)	(10.11, 15.51, 9.41)	0.271	4
5	(7.20, 9.17, 5.13) (0.20, 0.20)	(12.80, 9.17, 7.53) (0.40 0.20)	(10.20, 30.83, 6.63) (0.30 0.80)	(-52.00, -1.20, 121.33)	(11.90, 19.84, 10.92)	3.372	2
6	(7.20, 9.17, 5.13) (0.20 0.20)	(12.80, 9.17, 7.53) (0.40 0.20)	(10.20, 16.67, 9.73) (0.30 0.40)	(-18.00, -18.56, 42.00)	(11.58, 13.04, 9.26)	0.455	4

Fig. 3. *A view of the NURBS surface shape below the rectangular net of control points*

SUMMARY

The presented optimization method does not guarantee to always find a plane parallel to the mesh triangle and tangent to the NURBS surface. Especially, when the NURBS surface has inflection points and is partially convex and partially concave there might be many solutions. With Newton iteration only the solution closest to the point from which the search started, is found.

However the distance between a parallel tangent plane and a mesh triangle is a good measure of accurate meshing. The experiment results presented in Tab. 2 e.g. in line 1 and 2 show that the smaller triangle in line 2 is about ten times closer to the surface than the bigger triangle in line 1. Both tests required only two iterations – a very quick convergence. Next pairs of lines in Tab. 2, e.g. 3 and 4 or 5 and 6, also shows significant reduction of the distance between the NURBS surface and mesh triangles with reduction of the size of a triangle. Quick convergence of the search method results in the short execution time. The presented method seems to be promising and may be applied even if every triangle of the meshed surface was to be checked in this way.

Advantages of the presented method:

- distance between a mesh triangle and a parallel plane tangent to the NURBS surface is a good measure of mesh quality,
- quick convergence – several iterations to find solution,
- short execution time due to quick convergence and not complicated operations (NURBS surface derivatives)

An alternative way to assess mesh quality would be calculation of volume contained between the NURBS surface and a mesh triangle plane. However volume calculation requires calculating integrals of the NURBS surface, which is more complicated and time consuming than calculating derivatives. The comparison of the two measures could be interesting although it is out of the scope of this paper.

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