Postprint of: Baryshnikov Y., Duda J., Szpankowski W., Types of Markov Fields and Tilings, IEEE TRANSACTIONS ON INFORMATION THEORY, Vol. 62, Iss. 8 (2016), pp. 4361-4375, DOI: 10.1109/TIT.2016.2573834

Types of Markov Fields and Tilings*

July 22, 2014

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Abstract

The method of types is one of the most popular techniques in information theory and combinatorics. However, thus far the method has been mostly applied to one-dimensional Markov processes, and it has not been really thoroughly studied for Markov fields. Markov fields can be viewed as models for multi-dimensional systems involving a large number of variables with local mutual dependencies and interactions. These local dependencies can be represented by a *shape* $\mathcal{S}$ of interactions which are further marked by symbols from a finite alphabet of size m. Such a marked shape is called a *tile*. Two assignments in a Markov field have the same type if they have the same empirical distribution or they can be tiled by the same number of *tile types*. Our goal is to study the growth of the number of Markov field types or the number of tile types in a d-dimensional box of lengths $n_1, \ldots, n_d$ or its cyclic counterpart known as a d dimensional torus in which edges are glued together. We reduce this question to the enumeration of nonnegative integer solutions of a large system of Diophantine linear equations called the *conservation laws*. We view a Markov type as a vector in a $D = m^{|\mathcal{S}|}$ dimensional space and count the number of such vectors satisfying the conservation laws which turns out to be the number of integer points in a certain polytope. For the torus this polytope is of dimension $\mu = D - 1 - \text{rk}(C)$ where rk(C) is the number of linearly independent conservation laws. This provides an upper bound on the number of types. We also construct a matching lower bound leading to the conclusion that the number of types in the torus Markov field is $\Theta(N^\mu)$ where $N = n_1 \cdots n_d$. These results are derived by analytic geometry tools including ideas of discrete and convex multidimensional geometry.

Index Terms: Markov fields, Markov types, conservation laws, linear Diophantine equations, enumerative combinatorics, analytic and discrete geometry.

*A preliminary version of this paper was presented at ISIT, Honolulu, 2014.
This work was supported by NSF Center for Science of Information (CSoI) Grant CCF-0939370, and in addition by Grants from ARPA HR0011-07-1-0002 and ONR N000140810668, NSA Grant H98230-11-1-0141, and NSF Grants DMS-0800568, and CCF-0830140.

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1 Introduction

The *method of types* is one of the most popular and useful techniques in information theory and combinatorics. Two sequences of equal length are of the same type if they have identical empirical distributions. Furthermore, sequences of the same type are assigned the same probability by all distributions in a given class. The method of types was known for some time in probability and statistical physics. But only in the 1970's Csiszár and his group developed a general method and made it a basic tool of information theory of discrete memoryless sources [5, 6]; see also [4, 7, 10, 15, 23, 28]. The method of types is used in a myriad of applications [6], from the minimax redundancy [10] to simulation of information sources [16]. However, thus far this method has been studied only for *one-dimensional* processes, mostly Markov [10, 11, 28] but also general stationary ergodic processes [23]. Here we investigate types of *Markov fields* (Bayesian networks, Gibbs fields and/or factor graphs) [22] that find applications ranging from sensor networks [20] to images to information retrieval [17], and so forth [27]. We develop a novel approach based on multidimensional discrete and analytic geometry to study this important and intricate problem that has been left open for too long.

In order to gently introduce Markov fields and their types, we start with a one dimensional Markov chain over a finite alphabet $\mathcal{A} = \{1, 2, \ldots, m\}$. Let us write $\mathbf{x}^n = x_1 \ldots x_n \in \mathcal{A}^n$ for a sequence of length n generated by a Markov source. For Markov sources of order $r = 1$ we have two equivalent representations for the probability $\mathbf{P}(x^n)$ of x^n:

$$\mathbf{P}(\mathbf{x}^n) = \mathbf{P}(x_1) \prod_{i=2}^{n} \mathbf{P}(x_i | x_{i-1}) = \mathbf{P}(x_1) \prod_{(i,j) \in \mathcal{A}^2} p_{ij}^{\mathbf{T}(ij)}, \quad \sum_{x_1} \mathbf{P}(x_1) = 1,$$

where p_{ij} is the transition probability from $i \in \mathcal{A}$ to $j \in \mathcal{A}$, and the frequency count $\mathbf{T}(ij)$ is the number of pairs (ij) in the sequence $\mathbf{x}^n$. If we define $\mathfrak{t} = (ij)$ and $\mathfrak{T} = \mathcal{A}^2$, then the previous equation can be written as

$$\mathbf{P}(\mathbf{x}^n) = \mathbf{P}(x_1) \prod_{\mathfrak{t} \in \mathfrak{T}} p_{\mathfrak{t}}^{\mathbf{T}(\mathfrak{t})}. \tag{1}$$

Similarly, for one-dimensional Markov sources of order r we have

$$\mathbf{P}(\mathbf{x}^n) = \mathbf{P}(x_1^r) \prod_{i=r+1}^{n} \mathbf{P}(x_i | x_{i-1} \ldots x_{i-r}) = \mathbf{P}(x_1^r) \prod_{\mathfrak{t} \in \mathfrak{T}} p_t^{\mathbf{T}(\mathfrak{t})}, \quad \sum_{x_1^r} \mathbf{P}(x_1^r) = 1, \tag{2}$$

where $\mathfrak{t} = (i_1, \ldots, i_{r+1}) \in \mathfrak{T} := \mathcal{A}^{r+1}$ and $\mathbf{T}(\mathfrak{t})$ counts the appearances of $\mathfrak{t}$ in $\mathbf{x}^n$.

To motivate questions we ask in this paper, we remark first that the probability measures above are in fact *Gibbs distributions*, associated with *local interactions*, parameterized by some collections of neighboring sites (sequential $(r+1)$-tuples for Markov sources of order r). We will be calling these collections *plaques*, in view of the higher-dimensional generalizations that follow. All these collections in the case of Markov processes are in fact *shifts*, or *displacements* of the same *shape* (in 1D situation, the shape is just the interval $\mathcal{S} = \{0, 1, \ldots, r\}$; in higher dimensions there is a larger variety of shapes). *Plaques* are $\mathfrak{p} = \mathcal{S} + \mathbf{s}$ where $\mathbf{s}$ is a shift, however, we often simply write $\mathcal{S} \equiv \mathcal{S} + \mathbf{s}$. A marking of a shape with symbols of the alphabet $\mathcal{A}$ is called a tile $\mathfrak{t}$, that is, $\mathfrak{t} : \mathcal{S} \to \mathcal{A}$.

In this paper, we study a high-dimensional generalization of Markov processes called Markov fields that take values in the finite alphabet $\mathcal{A}$ and are supported by a subset of the integer lattice domain $\mathcal{D} \subset \mathbb{Z}^d$. We only consider *box-shaped* domains, with either free boundary conditions (called a *box* and denoted as $\mathcal{I}$) or with periodic (in all d dimensions) boundary conditions (referred to as a *torus* and denoted as $\mathcal{O}$). The set of all possible configurations on the domain $\mathcal{D}$ (i.e., functions $\mathcal{D} \to \mathcal{A}$) is represented as $\texttt{Conf}(\mathcal{D})$. We denote Markov fields as $\mathbf{x} \equiv \mathbf{x}^{\mathbf{n}} \in \texttt{Conf}(\mathcal{D})$, where $\mathbf{n} = (n_1, \ldots, n_d)$ is the size of a torus or a box. Finally, the set of all tiles is $\mathfrak{T} \equiv \texttt{Conf}(\mathcal{S})$.

It is well known that the distribution $\mathbf{P}(\mathbf{x})$ of a Markov field $\mathbf{x}$ can be rewritten as a Gibbs distribution

$$\mathbf{P}(\mathbf{x}) = Z^{-1} \prod_{\mathfrak{p}} \mathbf{w}(\mathbf{x}|_{\mathfrak{p}}), \tag{3}$$

where $\mathbf{w}$ is a *weight* – a (nonnegative) function (or equivalently vector) on configurations of plaques (shape $\mathcal{S}$ in some position), where $\mathbf{x}|_{\mathfrak{p}}$ is a restriction of $\mathbf{x}$ to $\mathfrak{p}$, and Z is the so called partition function

$$Z = \sum_{\mathbf{x}} \prod_{\mathfrak{p}} \mathbf{w}(\mathbf{x}|_{\mathfrak{p}}). \tag{4}$$

Observe that (3) is an instance of the Hammersley–Clifford Theorem [18].

In the case when all the plaques have the same shape and the weights are *translation invariant* (i.e. the weight $\mathbf{w}(\mathbf{x}|_{\mathfrak{p}})$ depends only on the restriction $\mathfrak{t} = \mathbf{x}|_{\mathfrak{p}}$, but not on the position of the plaque), the product in (4) can be rewritten as

$$\prod_{\mathfrak{p}} \mathbf{w}(\mathbf{x}|_{\mathfrak{p}}) = \mathbf{w}^{\mathbf{T}}, \tag{5}$$

where $\mathbf{w} = (w_{\mathbf{x}_1}, \ldots, w_{\mathbf{x}_K})$ for some K is the vector of weights, and the vector $\mathbf{T} = \{\mathbf{T}(\mathfrak{t})\}_{\mathfrak{t}\in\mathfrak{T}}$ is the *type* of the configuration $\mathbf{x}$, i.e. the number of the plaques $\mathfrak{p}$ (obtained by shifting the shape $\mathcal{S}$) for which the restriction $\mathbf{x}|_{\mathfrak{p}}$ is the same as $\mathfrak{t}$. Here, we use the convention that for two vectors $\mathbf{a} = (a_1, \ldots, a_K), \mathbf{b} = (b_1, \ldots, b_K)$, we denote $\mathbf{a}^{\mathbf{b}} = \prod_k a_k^{b_k}$.

Observe that the partition function (4) can be rewritten as

$$Z = \sum_{\mathbf{T}} M(\mathbf{T}) \mathbf{w}^{\mathbf{T}}, \tag{6}$$

where the summation is over all *types* $\mathbf{T}$, that is, the vector of numbers $\mathbf{T}(\mathfrak{t})$ of plaques $\mathfrak{p} = \mathbf{s} + \mathcal{S}$ such that the restriction of the configuration $\mathbf{x}$ to $\mathfrak{p}$ has the labeling $\mathfrak{t}$, and $M(\mathbf{T})$ is the number of configurations $\mathbf{x} \in \texttt{Conf}(\mathcal{O})$ (or in $\texttt{Conf}(\mathcal{I})$) having the type $\mathbf{T}$. This reformulation (6) allows one to decouple the effects of the weights $\mathbf{w}$ and of the combinatorics of the model encoded by the shape of the domain $\mathcal{O}$ or $\mathcal{I}$, and the shape of the plaques $\mathcal{S}$. Then two questions naturally arise:

1. for a given type, i.e. the counts $\{\mathbf{T}(\mathfrak{t})\}_{\mathfrak{t}\in\mathfrak{T}}$, how many sequences $\mathbf{x}^n$ realize it (i.e., what is $M(\mathbf{T})$ for a given type $\mathbf{T}$)?

2. how many distinct types are there, that is, what is the size of the set of types $\mathbf{T}$ for which $M(\mathbf{T}) > 0$?

In the language of Markov sources, the types $\{\mathbf{T}(\mathfrak{t})\}_{\mathfrak{t}\in\mathfrak{T}}$, as we define them, specialize to the familiar *Markov types*: for a one-dimensional Markov source, the type just encodes the number of times the transition $i \to j$ $(i, j \in \mathcal{A})$ is observed in a sequence.

In passing we should point out that the condition that a given type can be observed in a Markov trajectory is equivalent to a multigraph representing the type to be *Eulerian*, as discussed in some depth in [11]. The question, for a given Eulerian type $\mathbf{T}$ to determine the number of trajectories having this type is in our notation the question of finding $M(\mathbf{T})$.

These two problems (i.e., number of sequences of a given type and number of types) were studied quite extensively in the past for one dimensional Markov sources. The number of sequences of a given Markov type was first addressed by Whittle [28] and then re-established by analytic method

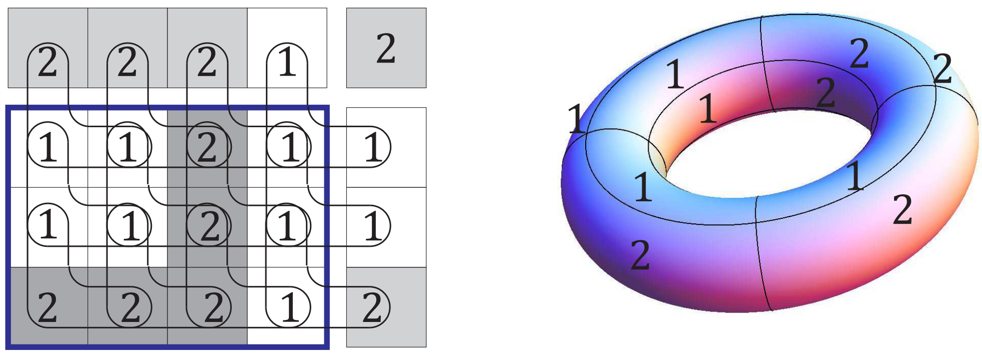


Figure 1: Illustration of cyclic fields: here $\mathbf{x}$ is defined on the torus $\mathcal{O}_{(4,3)}$ that is constructed from the box on the right by gluing the left and the right as we all as the top and the bottom edges.

in [10]. A precise evaluation of the number of Markov types was recently discussed in [11] (see also [15] for tree models).

In this paper, we address a more general, and much harder problem: the *enumeration of Markov field types* (i.e., the number of distinct empirical distributions of tile counts), that can be realized by a "trajectory" $\mathbf{x} \in \mathtt{Conf}(\mathcal{O})$. Let us be more precise. We need to introduce some notation. For dimension $d \in \mathbb{N}$, let $\mathbf{n} = (n_1, n_2, .., n_d)$ and $N := n_1 \cdot n_2 \cdot \ldots \cdot n_d$. Define the box $\mathcal{I}_\mathbf{n} = I_{n_1} \times I_{n_2} \times .. \times I_{n_d} \subset \mathbb{Z}^d$ with $I_n := \{0, 1, .., n-1\}$ on which the underlying Markov field is defined. We mostly work here with the *rectangular torus*, i.e. the fields on $\mathbb{Z}^d$ subject to $\mathbf{n}$-periodic boundary conditions (see Figure 1). Our results remain valid for general case, where the periodicity lattice is not necessarily rectangular. In the 1-dimensional case analyzed in [11] these periodic conditions translate into the *cyclic sequences* of symbols in $\mathcal{A}$, that is, sequences $\mathbf{x}^n$ in which $x_{k+n} = x_k$.

Now, the *tile count*, or *type* of the field $\mathbf{x}$ is a function $\mathbf{T} : \mathfrak{T} \to \mathbb{N}$ counting how often each tile occurs in the field $\mathbf{x}$, that is,

$$\mathbf{T}(\mathfrak{t}) = |\{\mathbf{s} \in \mathcal{D} \ : \ \mathcal{S} + \mathbf{s} \subset \mathcal{D}, \text{and } \ \mathbf{x}|_{\mathcal{S}+\mathbf{s}} = \mathfrak{t}\}| \tag{7}$$

where $f|_B$ denotes the restriction of a function f to a smaller domain B.

While tilings and asymptotically counting them are discussed in many references [1, 12, 13, 19], our problem is distinctly different: these references are concerned with (asymptotic) evaluation of what we call $M(\mathbf{T})$. Here we address the issue of the *support* of the function M, especially of its size:

$$\mathcal{P}_\mathbf{n} = \mathcal{P}_\mathbf{n}(\mathcal{A}, \mathcal{S}) = \{\mathbf{T} \in \mathbb{Z}^{\mathfrak{T}} : \text{exists } \mathbf{x} \in \mathtt{Conf}(\mathcal{D}), \text{ such that } \mathbf{x} \text{ is of type } \ \mathbf{T}\}. \tag{8}$$

The cardinality of $\mathcal{P}_\mathbf{n}$, i.e. the number of realizable *types*, is our main concern in this paper. While the question of understanding the structure of the set of types for the multi-dimensional fields is very natural and important, we could not find any relevant literature, beyond the 1D situation.

We shall view the types $\{\mathbf{T}(\mathfrak{t})\}_{\mathfrak{t} \in \mathfrak{T}}$ as a $D := |\mathfrak{T}| = m^{|\mathcal{S}|}$–dimensional vector indexed by the tiles $\mathfrak{t} \in \mathfrak{T}$. Clearly, $\mathbf{T}(\mathfrak{t}) \geq 0$ for all $\mathfrak{t} \in \mathfrak{T}$. However, this vector satisfies a number of *equality* constraints that have a major impact on the cardinality of $\mathcal{P}_\mathbf{n}$. First of all, one has the normalization condition

$$\sum_{\mathfrak{t}} \mathbf{T}(\mathfrak{t}) = \mathtt{I}(\mathcal{D}), \tag{9}$$

where $\mathtt{I}(\mathcal{D})$ is the number of different plaques $\mathfrak{p} = \mathbf{s} + \mathcal{S}$ in $\mathcal{D}$. It is quite obvious for the torus that $\mathtt{I} = N = n_1 \cdot \ldots \cdot n_d$. Further, in order to tile a torus the number of tiles "ending" with a subtile $\mathfrak{t}' : \mathcal{S}' \to \mathcal{A}$ for some subshape $\mathcal{S}' \subset \mathcal{S}$ must be equal to the number of tiles that "begin" with $\mathfrak{t}'$. This leads, as in the 1D case, to what we call the *conservation laws* (discussed in depth in Section 2):

$$\sum_{\mathfrak{t}:\mathfrak{t}|_{\mathcal{S}'_1}=\mathfrak{t}'} \mathbf{T}(\mathfrak{t}) = \sum_{\mathfrak{t}:\mathfrak{t}|_{\mathcal{S}'_2}=\mathfrak{t}'} \mathbf{T}(\mathfrak{t}) \tag{10}$$

for all pairs of subshapes $\mathcal{S}'_1, \mathcal{S}'_2 \subset \mathcal{S}$ such that $\mathcal{S}'_2 = \mathcal{S}'_1 + \mathbf{s}$ for some $\mathbf{s}$, and some tile $\mathfrak{t}' : \mathcal{S}' \to \mathcal{A}$.

The system of equations (9)–(10) constitutes a *linear system of Diophantine equations* in $\mathbb{Z}^D$. We denote by $\mathcal{F}_\mathbf{n} := \mathcal{F}_\mathbf{n}(\mathcal{A}, \mathcal{S})$ the set of nonnegative integer solutions to (9)–(10). Clearly, $|\mathcal{P}_\mathbf{n}| \leq |\mathcal{F}_\mathbf{n}|$ since all types in $\mathcal{P}_\mathbf{n}$ satisfy the conservation laws, and thus lie in $\mathcal{F}_\mathbf{n}$. However, we will see that unlike the 1D situation, these sets are very different.

As we said, little is known about the set of realizable types in higher dimensions. Let us briefly survey the available 1D results. In [11] an analytic approach was used to enumerate precisely $\mathcal{F}_n$ for $d = 1$. Another analytic approach is suggested in Stanley [25], however, it allows only to find the order of growth. We remark here that extending analytic techniques of [11] to estimate asymptotically $|\mathcal{F}_\mathbf{n}|$ is in general quite complicated, however, in Section 2 we discuss it in some details. Furthermore, for the $d = 1$ case $|\mathcal{P}_n| \sim |\mathcal{F}_n|$ as $n \to \infty$. This does not hold any longer for the multidimensional case where the set of types $\mathcal{P}_\mathbf{n}$ is a *proper* subset of $\mathcal{F}_\mathbf{n}$. Thus we can only establish an upper bound on the size of the set of types through $\mathcal{F}_\mathbf{n}$, and we propose another approach to find a lower bound.

To analyze the cardinality of $\mathcal{F}_\mathbf{n}$ and, ultimately, $\mathcal{P}_\mathbf{n}$ we need to understand the geometry of D-dimensional count vectors $\mathbf{T}$. In particular, we must estimate the dimension of the affine subspace spanned by $\mathcal{F}_\mathbf{n}$. To accomplish it we shall write the conservation law (10) as $\mathbf{C} \cdot \mathbf{T} = \mathbf{0}$ where $\mathbf{C}$ is a matrix describing a system of conservation laws (10), or, perhaps, its submatrix of the same rank. This allows us to define the cone $\mathcal{C}$ (recall that a set $\mathcal{C}$ is a cone if $\mathbf{T} \in \mathcal{C}$ implies $\lambda\mathbf{T} \in \mathcal{C}$ for all $\lambda > 0$):

$$\mathcal{C} \equiv \mathcal{C}(\mathcal{A}, \mathcal{S}) = \{\mathbf{T} \in \mathbb{R}^D_{\geq 0} \; : \; \mathbf{C} \cdot \mathbf{T} = \mathbf{0}\}$$

and the corresponding commutative monoid (a "lattice analogue of a cone"),

$$\mathcal{C}_\mathbb{Z} := \mathcal{C} \cap \mathbb{Z}^D.$$

Then

$$\mathcal{F}_\mathbf{n} \equiv \mathcal{F}_\mathbf{n}(\mathcal{A}, \mathcal{S}) = \{\mathbf{T} \in \mathcal{C}_\mathbb{Z} : \sum_{\mathfrak{t}} \mathbf{T}(\mathfrak{t}) = N\}. \tag{11}$$

The dimensionality (of the affine span of) $\mathcal{F}_\mathbf{n}$ depends on D and the set of constraints represented by the matrix $\mathbf{C}$. We shall show that $\mathcal{F}_\mathbf{n}$ lies in an affine subspace of dimension $\mu = D - 1 - \mathrm{rk}(\mathbf{C})$ where $\mathrm{rk}(\mathbf{C})$ is the rank of $\mathbf{C}$. This is illustrated in Figure 2(a). In our first main result Theorem 3 we present a precise characterization of $\mathrm{rk}(\mathbf{C})$.

Our ultimate goal, however, is to estimate the cardinality of the number of types $\mathcal{P}_\mathbf{n}$, that is, the number of *realizable* tiling types, or the number of distinct count vectors $\mathbf{T}$. We shall see that the Hausdorff distance between the *normalized* set $\hat{\mathcal{P}}_\mathbf{n} := \mathcal{P}_\mathbf{n}/N$ is close to $\hat{\mathcal{F}}_\mathbf{n} := \mathcal{F}_\mathbf{n}/N$ leading to our main Theorem 7 in which we establish that $|\mathcal{P}_\mathbf{n}| = \Theta(N^\mu)$ where as before $\mu = D - 1 - \mathrm{rk}(\mathbf{C})$. However, unlike $d = 1$ where we proved $|\mathcal{P}_n| \sim |\mathcal{F}_n|$, in the multidimensional case $|\mathcal{F}_\mathbf{n}|$ is not asymptotically equivalent to $|\mathcal{P}_\mathbf{n}|$ even if the growth of both is the same. Finally, we briefly discuss the non-cyclic Markov field types and provide an upper bound on the number of types in a box. In this case lack of cyclic boundary conditions introduces some imbalance in the conservation laws replacing $\mathbf{C} \cdot \mathbf{T} = \mathbf{0}$ by $\mathbf{C} \cdot \mathbf{T} = \mathbf{b}$ for some vector $\mathbf{b}$ as illustrated in Figure 2(b). This leads to an upper bound $O(N^{D-1}/(n)^{\mathrm{rk}(\mathbf{C})})$ on the number of types in the box $\mathcal{I}_\mathbf{n}$. However, whether this is the right growth for the number of types in the box case, remains an open question.

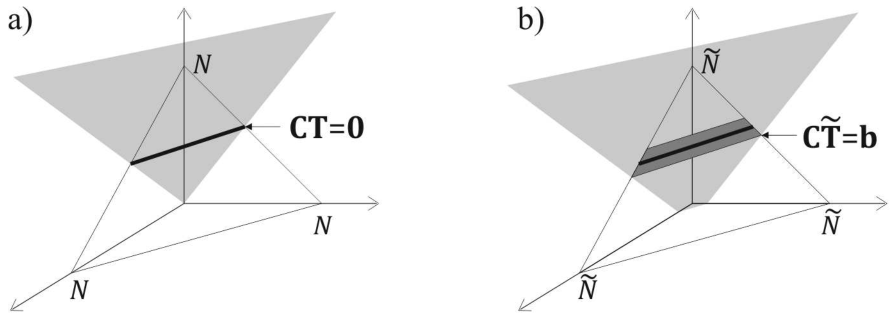


Figure 2: Geometry of type vectors $\{\mathbf{T}(\mathfrak{t})\}$ for (a) torus and (b) box. Here the gray area denotes the cone $\mathcal{C}$ of non-negative type vectors satisfying the conservation laws while the intersection with the simplex $\{\sum_{\mathfrak{t}} \mathbf{T}(\mathfrak{t}) = N\}$ representing a polyhedron is displayed in bold.

Regarding methodology used to establish our results. As mentioned before, in [11] for $d = 1$ we applied an analytic approach through multidimensional Cauchy's integral. We still can use this approach for some simple shapes in the multidimensional case, as discussed in Section 2. However, in the general case we have to switch to tools of discrete, convex, and analytic multidimensional geometry that somewhat resembles the method discussed in [25]. In particular, we need to understand how to count the number of lattice points in a polytope [2, 8]. This will allow us to find the number of nonnegative integer solutions of a linear system of Diophantine equations (i.e., conservation laws) that leads to the enumeration of the Markov field types.

The paper is organized as follows. In the next section we present our main results and some consequences. Most of the proofs are delayed till the last Section 3.

2 Main Results

In this main section we first illustrate the notation and definitions, formulate conservation laws essential to our discussion, and finally present our main results regarding the enumeration of types for Markov fields and tilings. We sprinkle this section with many examples to illustrate our definitions and results.

2.1 Basic Definitions and Examples

We start with some examples illustrating our definitions discussed in the introduction.

Example 1: *1D Markov chain.*
The first example we consider is the 1D case. For $d = 1$ the torus becomes a length $N = n_1$ cycle, fields are length N sequences with cyclic boundary condition: $x_{N+i} = x_i$. We are usually interested in the distribution of pairs, so the shape is $\mathcal{S} = \{0, 1\}$; for r order Markov the shape is $\mathcal{S} = \{0, \ldots, r\}$. For $m = 2$ and $\mathcal{O}_{10} = \{0, \ldots, 9\}$ cycle (1D torus), we consider a 1D (cyclic) sequence $\mathbf{x} = (1122111212)$. Now $\mathbf{T}(21) = 3$ because this "21" pattern (i.e., $\mathfrak{t}(0) = 2, \mathfrak{t}(1) = 1$) appears in $\mathbf{x}$ for 3 different shifts: $\mathbf{s} \in \{3, 7, 9\}$. Similarly, $\mathbf{T}(11) = 3$, $\mathbf{T}(12) = 3$, $\mathbf{T}(22) = 1$. Clearly, $\sum_{(ij)\in\{1,2\}} T(ij) = 10$. Note that we can view $\mathbf{T}$ as a $D = m^2 = 4$ dimensional vector $\mathbf{T}$.

Example 2: *2D Markov Field with the* `L` *Shape.*
Let $d = 2$. The torus is an $n_1 \times n_2$ rectangle with cyclic boundary conditions: $x_{i,j} = x_{i+n_1,j} = x_{i,j+n_2}$. Let us take 3x4 torus $\mathcal{O}_{\{3,4\}} = \{0, 1, 2\} \times \{0, 1, 2, 3\}$. Fields assign an element from the

alphabet $\mathcal{A} = \{1, 2\}$ to each point of this torus. For example, for the field

$$\mathbf{x} = \begin{array}{c} 1121 \\ 1121 \\ 2221 \end{array}$$

we have $\mathbf{x}(0,0) = 2$, $\mathbf{x}(0,1) = 1$, $\mathbf{x}(1,0) = 2$, but also conditions $\mathbf{x}(4,0) = \mathbf{x}(0,3) = \mathbf{x}(4,3) = \mathbf{x}(0,0)$, where we use north-east coordinates with $x(0,0)$ in the lower left corner. The first shape we consider here is the simplest nontrivial L-shape: $\mathcal{S} = \{(0,0),(0,1),(1,0)\}$. We find

$$\mathbf{T}\left(\begin{array}{c} 1 \\ 12 \end{array}\right) = 2$$

because this pattern appears in $\mathbf{s} \in \{(3,0),(1,1)\}$ positions.

Example 3: *2D Markov Field with the Square $\square$ Shape.*
The second 2D shape we consider is a 2×2 square shape $\mathcal{S} = \{0,1\} \times \{0,1\}$.For the same torus $\mathcal{O}_{\{3,4\}}$ and field $\mathbf{x}^{\mathbf{n}}$ as in the previous example, we find

$$\mathbf{T}\left(\begin{array}{c} 11 \\ 11 \end{array}\right) = 2$$

because this pattern appears in $\mathbf{s} \in \{(0,1),(3,1)\}$ positions. $\square$

2.2 Conservation Laws

Conservation laws are associated with the different ways we can embed a smaller shape $\mathcal{S}'$ into a larger shape $\mathcal{S}$. Recall that shapes are subsets of $\mathbb{Z}^d$, and thus our embeddings are just displacements by a vector in $\mathbb{Z}^d$. For example, the subshape $\mathcal{S}' = \{0,1\} \times \{0\}$ has six embeddings into $\mathcal{S} = \{0,1,2\} \times \{0,1,2\}$, that can be identified with $\mathbf{s} \in \{0,1\} \times \{0,1,2\}$ shifts: $\mathcal{S}' + \mathbf{s} \subset \mathcal{S}$.

Let $\boldsymbol{\varepsilon} : \mathcal{S}' \to \mathcal{S}$ be an embedding. A tile on $\mathcal{S}$ is a mapping $\mathfrak{t} : \mathcal{S} \to \mathcal{A}$, and composing it with $\boldsymbol{\varepsilon}$ we obtain a (sub)tile $\mathfrak{t}'$ on the smaller shape: restriction of $\mathfrak{t}$ to $\boldsymbol{\varepsilon}(\mathcal{S})$, that we denote as $\mathfrak{t}' = \boldsymbol{\varepsilon}^*(\mathfrak{t}) : \mathcal{S}' \to \mathcal{A}$. Further, recall that a type $\mathbf{T} : \mathcal{A}^{\mathcal{S}} \to \mathbb{N}$ is a vector with components indexed by tiles on $\mathcal{S}$. The mapping $\mathfrak{t} \mapsto \boldsymbol{\varepsilon}^*(\mathfrak{t})$ defines a mapping $\hat{\boldsymbol{\varepsilon}} : \mathcal{T}_{\mathcal{S}} \to \mathcal{T}_{\mathcal{S}'}$, where $\mathcal{T}_{\mathcal{S}}$ is set of types for shape $\mathcal{S}$, taking a type $\mathbf{T}$ (on shape $\mathcal{S}$) into a type $\mathbf{T}' = \hat{\boldsymbol{\varepsilon}}\mathbf{T}$ defined on $\mathcal{S}'$. Clearly,

$$(\hat{\boldsymbol{\varepsilon}}\mathbf{T})(\mathfrak{t}') = \mathbf{T}'(\mathfrak{t}') := \sum_{\mathfrak{t}:\boldsymbol{\varepsilon}^*(\mathfrak{t})=\mathfrak{t}'} \mathbf{T}(\mathfrak{t}) \tag{12}$$

is just the sum of the counts $\mathbf{T}(\mathfrak{t})$ over all tiles $\mathfrak{t}$ such that their restriction to $\boldsymbol{\varepsilon}(\mathcal{S}')$ coincides with $\mathfrak{t}'$. Now, if there are two different embeddings $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 : \mathcal{S}' \to \mathcal{S}$, one obtains two types on $\mathcal{S}'$, namely $\hat{\boldsymbol{\varepsilon}}_1\mathbf{T}$ and $\hat{\boldsymbol{\varepsilon}}_2\mathbf{T}$ having the same subtile $\mathfrak{t}'$. The next lemma introduces a conservation law.

Lemma 1. *If the type $\mathbf{T}$ is the count vector for a configuration $\mathbf{x}$ on a torus $\mathcal{O}_{\mathbf{n}}$, then*

$$\hat{\boldsymbol{\varepsilon}}_1\mathbf{T}(\mathfrak{t}') - \hat{\boldsymbol{\varepsilon}}_2\mathbf{T}(\mathfrak{t}') = 0 \tag{13}$$

where $\hat{\boldsymbol{\varepsilon}}_1$ and $\hat{\boldsymbol{\varepsilon}}_2$ are mappings with the same corresponding subtile $\mathfrak{t}'$ (i.e., $\boldsymbol{\varepsilon}_1^*(\mathfrak{t}) = \boldsymbol{\varepsilon}_2^*(\mathfrak{t}) = \mathfrak{t}'$) satisfying (12).

This obvious lemma, again generalizes the Eulerian condition (that every vertex has the same number of incoming and outgoing edges) on the multigraphs describing the types in 1D situation, and is the starting point of our study. It is also equivalent to (10) discussed in the introduction.

Example 4: *Continuation of Example 1.*
Returning to the 1D case of Example 1, we have $\mathcal{S} = \{0, 1\}$, $\mathcal{S}' = \{0\}$, with $\boldsymbol{\varepsilon}_1$ placing the node (subshape) 0 at 0, and $\boldsymbol{\varepsilon}_2$ at 1. In this case,

$$(\hat{\boldsymbol{\varepsilon}}_1\mathbf{T})(1) = \mathbf{T}(11) + \mathbf{T}(12) =: \mathbf{T}(1*) \qquad (\hat{\boldsymbol{\varepsilon}}_1\mathbf{T})(2) = \mathbf{T}(21) + \mathbf{T}(22) =: \mathbf{T}(2*)$$

and

$$(\hat{\boldsymbol{\varepsilon}}_2\mathbf{T})(1) = \mathbf{T}(11) + \mathbf{T}(21) =: \mathbf{T}(*1) \qquad (\hat{\boldsymbol{\varepsilon}}_2\mathbf{T})(2) = \mathbf{T}(12) + \mathbf{T}(22) =: \mathbf{T}(*2).$$

(We use the mnemonic $\mathbf{T}(a*)$ and like to denote the summation over the don't-care variable.) Observe also that the conservation law $(\hat{\boldsymbol{\varepsilon}}_1\mathbf{T})(1) = (\hat{\boldsymbol{\varepsilon}}_2\mathbf{T})(1)$ simply means that the number of edges in the type $\mathbf{T}$ *entering* 1 is the same as the number of edges *leaving* 1, that is, the Euler condition. Furthermore, using the vector count $(\mathbf{T}(11), \mathbf{T}(12), \mathbf{T}(21), \mathbf{T}(22))$ in the space of types $\mathbb{R}^4$, we can re-write this conservation law as

$$\begin{aligned} 0 &= (\hat{\boldsymbol{\varepsilon}}_1\mathbf{T})(1) - (\hat{\boldsymbol{\varepsilon}}_2\mathbf{T})(1) = \mathbf{T}(12) - \mathbf{T}(21) \\ &= (0, 1, -1, 0) \cdot (\mathbf{T}(11), \mathbf{T}(12), \mathbf{T}(21), \mathbf{T}(22)) \end{aligned}$$

where in the last line we use the matrix $\mathbf{C} = (0, 1, -1, 0)$. In fact, this suggests that we can rephrase our discussion in terms of linear functionals and dual spaces as discussed in details below. In particular, in this example, we can use the following linear function (functional):

$$\boldsymbol{v}_{(\mathcal{S}', \mathfrak{t}'=\text{"}1\text{"}, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2)}(\mathbf{T}) = (0, 1, -1, 0) \cdot \mathbf{T},$$

which is formally a covector in the dual space (space of all covectors). □

We now re-formulate our conservation laws in the language of *linear functionals* and *dual spaces* [14]. This formalism allows us to rigorously prove our statements.

Definition 2. *Consider the* vector space of types, $\mathscr{V} := \mathbb{R}^D$. *Let* $\mathcal{S}'$ *be a shape,* $\boldsymbol{\varepsilon}_i : \mathcal{S}' \to \mathcal{S}$ *be different embeddings of the* $\mathcal{S}'$ *into* $\mathcal{S}$*, and* $\mathfrak{t}'$ *be a tile on* $\mathcal{S}'$*. The linear function (*covector*)* $\boldsymbol{v}_{(\mathcal{S}', \mathfrak{t}', \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2)} \in \mathscr{V}^*$ *defined by*

$$\boldsymbol{v}_{(\mathcal{S}', \mathfrak{t}', \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2)}\mathbf{T} \mapsto (\hat{\boldsymbol{\varepsilon}}_1\mathbf{T})(\mathfrak{t}') - (\hat{\boldsymbol{\varepsilon}}_2\mathbf{T})(\mathfrak{t}')$$

is called the conservation law *corresponding to the tuple*

$$(\mathcal{S}', \mathfrak{t}', \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2).$$

In the standard basis of $\mathscr{V}$, the mapping $\hat{\boldsymbol{\varepsilon}}$ is a linear combination of $\mathbf{T}$ coordinates with 0 or 1 coefficients, $\boldsymbol{v}$ is subtraction of two of them, so all nonzero coefficients of $\boldsymbol{v}_{(\mathcal{S}', \mathfrak{t}', \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2)}$ are ± 1. In fact, all conservation laws for all possible $\mathcal{S}', \boldsymbol{\varepsilon}, \mathfrak{t}'$ form a (huge) matrix $\mathbf{C}$ with coefficients in $\{-1, 0, 1\}$. In Example 4, $\mathfrak{t}' = \text{"}2\text{"}$ leads to $(0, 1, -1, 0)$ vector, forming the matrix

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}$$

with linearly dependent rows (redundant conservation laws). Generally the number of independent rows (i.e., rank of $\mathbf{C}$) is much smaller.

We aim at finding a matrix $\mathbf{C}_m$ with independent rows. There are several sources of such dependencies among the rows of $\mathbf{C}$:

1. The normalization equation $\sum_{\mathfrak{t}'\in\mathtt{Conf}(\mathcal{S}')}(\hat{\boldsymbol{\varepsilon}}\mathbf{T})(\mathfrak{t}') = \mathbf{T}(*) = N$ implies that

$$\sum_{\mathfrak{t}'\in\mathtt{Conf}(\mathcal{S}')} \boldsymbol{v}_{(\mathcal{S}',\mathfrak{t}',\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2)} = 0$$

for any two embeddings $\boldsymbol{\varepsilon}_i : \mathcal{S}' \to \mathcal{S}$. This eliminates for every pair of embeddings $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 : \mathcal{S}' \to \mathcal{S}$ one equation (from $m^{|\mathcal{S}'|}$ to $m^{|\mathcal{S}'|} - 1$ equations), since summing over all $\mathfrak{t}'$ we obtain the trivial equation $N = N$.

2. Clearly, the functional $\boldsymbol{v}_{(\mathcal{S}',\mathfrak{t}',\boldsymbol{\varepsilon}_2,\boldsymbol{\varepsilon}_3)}$ can be represented as

$$\boldsymbol{v}_{(\mathcal{S}',\mathfrak{t}',\boldsymbol{\varepsilon}_2,\boldsymbol{\varepsilon}_3)} = \boldsymbol{v}_{(\mathcal{S}',\mathfrak{t}',\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_3)} - \boldsymbol{v}_{(\mathcal{S}',\mathfrak{t}',\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2)}.$$

Hence, for any $\mathcal{S}'$ (admitting at least two different embeddings into $\mathcal{S}$), we can fix one of the embeddings

$$\boldsymbol{\varepsilon}_{\mathcal{S}'} : \mathcal{S}' \to \mathcal{S}$$

as the canonical one, and restrict our attention only to the conservation laws

$$\boldsymbol{v}_{(\mathcal{S}',\mathfrak{t}',\boldsymbol{\varepsilon}_{\mathcal{S}'},\boldsymbol{\varepsilon})},$$

where $\boldsymbol{\varepsilon}$ runs over all embeddings $\boldsymbol{\varepsilon} : \mathcal{S}' \to \mathcal{S}$ different from $\boldsymbol{\varepsilon}_{\mathcal{S}'}$: there are $(|\{\boldsymbol{\varepsilon} : \mathcal{S}' \to \mathcal{S}\}| - 1)$ such choices. (We will discuss a way to produce such a choice consistently later on.) We remark that the number of conservation laws after these two restrictions is:

$$(m^{|\mathcal{S}'|} - 1)(|\{\boldsymbol{\varepsilon} : \mathcal{S}' \to \mathcal{S}\}| - 1).$$

3. These two reductions are sufficient for small shapes S. However, for larger shapes such as the 2×2 squares, there are further relations. Specifically, let us choose one symbol $m \in \mathcal{A}$. Observe that counts using this symbol can be expressed without it, for example $T(m1) = T(*1) - \sum_{i=1..m-1} T(i1)$. In this case, we can express conservation laws over the whole alphabet $\mathcal{A}$ without using one symbol, say the m-th one. It is sufficient to focus on laws for the reduced alphabet $\mathcal{A}' = \mathcal{A} \setminus \{m\}$: there are $(m-1)^{|\mathcal{S}'|}$ of them for given embedding.

We are now in the position to formulate our main result regarding the rank of matrix $\mathbf{C}$. We shall do it in the formalism we have just established. In particular, we use the notion of the *kernel* or *null space* of the underlying linear functionals defined in our case as $\{\mathbf{T} : \ \mathbf{C}\cdot\mathbf{T} = \mathbf{0}\}$. To pick up further information on linear algebra and linear functionals the reader is refer to [14]. In Section 3 we establish the following result.

Theorem 3. *The submatrix $\mathbf{C}_m$ of $\mathbf{C}$ formed by the rows corresponding to the functionals*

$$\boldsymbol{v}_{(\mathcal{S}',\mathfrak{t}',\boldsymbol{\varepsilon}_{\mathcal{S}'},\boldsymbol{\varepsilon})} \tag{14}$$

with $\mathfrak{t}'$ over $\mathcal{A}' = \mathcal{A} \setminus \{m\}$ has the same rank as the full matrix $\mathbf{C}$, and therefore defines the same kernel (i.e., $\{\mathbf{T} : \mathbf{C}_m \cdot \mathbf{T} = \mathbf{0}\} = \{\mathbf{T} : \mathbf{C}\cdot\mathbf{T} = \mathbf{0}\}$). Here $\boldsymbol{\varepsilon}_{\mathcal{S}'}$ is a canonical *embedding of a shape $\mathcal{S}'$ embeddable into $\mathcal{S}$.*

In particular, the matrix $\mathbf{C}$ has the corank (the dimension of its kernel: $\{\mathbf{T} : \mathbf{C}\cdot\mathbf{T} = \mathbf{0}\}$) equal to the number of tilings of the reduced alphabet $\mathcal{A}' = \mathcal{A} - \{m\}$ of all subshapes $\mathcal{S}'$ (including the empty one) embeddable into $\mathcal{S}$, i.e.

$$\mu + 1 = \sum_{\mathcal{S}' : |\{\boldsymbol{\varepsilon}:\mathcal{S}'\to\mathcal{S}\}|\geq 1} (m-1)^{|\mathcal{S}'|} \tag{15}$$

The rank of the matrix $\mathbf{C}$ *is given by*

$$\mathrm{rk}(\mathbf{C}) = D - \mu - 1 = \sum_{\mathcal{S}':|\{\boldsymbol{\varepsilon}:\mathcal{S}'\to\mathcal{S}\}|\geq 1} (|\{\boldsymbol{\varepsilon}:\mathcal{S}'\to\mathcal{S}\}|-1)(m-1)^{|\mathcal{S}'|} \tag{16}$$

where the summation is again over all shapes $\mathcal{S}'$ *embeddable into* $\mathcal{S}$.

For the box shapes formula (16) (requiring an enumeration of all subshapes fitting into $\mathcal{S}$) can be significantly simplified:

Corollary 4. *If* $\mathcal{S} = I_{l_1} \times I_{l_2} \times \ldots \times I_{l_d}$, *one has*

$$\mu = D - 1 - \mathrm{rk}(\mathbf{C}) = \sum_{\mathbf{s}\in\{0,1\}^d} m^{\prod_i (l_i - s_i)} \cdot (-1)^{\sum_i s_i} \tag{17}$$

where $\mathbf{l} = (l_1, \ldots, l_d) \in \mathbb{N}^d$.

2.3 More Examples

We now discuss a few examples illustrating the dimensionality reductions associated with the conservation laws and Theorem 3. We already observed in Example 1 that there is a single independent conservation law corresponding to $(0, 1, -1, 0) \cdot \mathbf{T} = \mathbf{0}$ in the $D = 4$ dimensional space of types leading to $\mu = 4 - 1 - 1 = 2$.

Example 5: *2D Markov Field with the L-Shape – Continuation.*
For the L-shape $\mathcal{S} = \{(0,0), (0,1), (1,0)\}$ in 2D and $m = 2$, the frequency vector $\mathbf{T}$ has $D = m^3 = 8$ coordinates

$$\left(\begin{pmatrix} 1 \\ 11 \end{pmatrix}, \begin{pmatrix} 1 \\ 21 \end{pmatrix}, \begin{pmatrix} 1 \\ 12 \end{pmatrix}, \begin{pmatrix} 1 \\ 22 \end{pmatrix}, \begin{pmatrix} 2 \\ 11 \end{pmatrix}, \begin{pmatrix} 2 \\ 21 \end{pmatrix}, \begin{pmatrix} 2 \\ 12 \end{pmatrix}, \begin{pmatrix} 2 \\ 22 \end{pmatrix} \right);$$

however, only five of them are independent. The only nontrivial subshape is a single point $\mathcal{S}' = \{(0,0)\}$, which can be embedded in all three positions: $\boldsymbol{\varepsilon}_1((0,0)) = (0,0)$, $\boldsymbol{\varepsilon}_2((0,0)) = (1,0)$, $\boldsymbol{\varepsilon}_3((0,0)) = (0,1)$ leading to the following conservation laws:

$$0 = \mathbf{T}\begin{pmatrix} * \\ 1* \end{pmatrix} - \mathbf{T}\begin{pmatrix} * \\ *1 \end{pmatrix} = \mathbf{T}\begin{pmatrix} 1 \\ 12 \end{pmatrix} + \mathbf{T}\begin{pmatrix} 2 \\ 12 \end{pmatrix} - \mathbf{T}\begin{pmatrix} 1 \\ 21 \end{pmatrix} - \mathbf{T}\begin{pmatrix} 2 \\ 21 \end{pmatrix},$$

$$0 = \mathbf{T}\begin{pmatrix} * \\ 1* \end{pmatrix} - \mathbf{T}\begin{pmatrix} 1 \\ ** \end{pmatrix} = \mathbf{T}\begin{pmatrix} 2 \\ 11 \end{pmatrix} + \mathbf{T}\begin{pmatrix} 2 \\ 12 \end{pmatrix} - \mathbf{T}\begin{pmatrix} 1 \\ 21 \end{pmatrix} - \mathbf{T}\begin{pmatrix} 1 \\ 22 \end{pmatrix},$$

$$0 = \mathbf{T}\begin{pmatrix} * \\ *1 \end{pmatrix} - \mathbf{T}\begin{pmatrix} 1 \\ ** \end{pmatrix} = \mathbf{T}\begin{pmatrix} 2 \\ 11 \end{pmatrix} + \mathbf{T}\begin{pmatrix} 2 \\ 21 \end{pmatrix} - \mathbf{T}\begin{pmatrix} 1 \\ 12 \end{pmatrix} - \mathbf{T}\begin{pmatrix} 1 \\ 22 \end{pmatrix}.$$

These equations define the functionals $\boldsymbol{v}_{(\mathcal{S}',\mathfrak{t}',\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2)}$, $\boldsymbol{v}_{(\mathcal{S}',\mathfrak{t}',\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_3)}$ and $\boldsymbol{v}_{(\mathcal{S}',\mathfrak{t}',\boldsymbol{\varepsilon}_2,\boldsymbol{\varepsilon}_3)}$ with $\mathfrak{t}' = 1$. Obviously one of these equations is redundant - choosing the lower left position as the canonical embedding $\boldsymbol{\varepsilon}_{\mathcal{S}'} := \boldsymbol{\varepsilon}_1$, there remain only the first two of the above equations - in the basis above, they can be written as:

$$\mathbf{CT} = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 \end{pmatrix} \cdot \mathbf{T} = \mathbf{0}.$$

These two independent conservation laws restrict the space of $\mathbf{T}$ to a $\mu + 1$ =6–dimensional cone, and the normalization equation further restricts it to a $\mu = 5$ dimensional polytope.

Example 6: *2D Markov Field with Square Shape* $\square$ – *Continuation.*
For the 2×2 square shape and $m = 2$ the frequency vector $\mathbf{T}$ is in $D = m^4 = 16$–dimensional space. As $\mathcal{A}' = \mathcal{A} \setminus \{m\} = \{1\}$, the ultimate set of independent conservation laws (16) are

$$\mathbf{T}\begin{pmatrix} ** \\ 1* \end{pmatrix} = \mathbf{T}\begin{pmatrix} 1* \\ ** \end{pmatrix} = \mathbf{T}\begin{pmatrix} *1 \\ ** \end{pmatrix} = \mathbf{T}\begin{pmatrix} ** \\ *1 \end{pmatrix},$$

$$\mathbf{T}\begin{pmatrix} ** \\ 11 \end{pmatrix} = \mathbf{T}\begin{pmatrix} 11 \\ ** \end{pmatrix} \qquad \mathbf{T}\begin{pmatrix} 1* \\ 1* \end{pmatrix} = \mathbf{T}\begin{pmatrix} *1 \\ *1 \end{pmatrix}.$$

The first line contains three equations for a single point shape. The second line contains the remaining two single conditions for $\mathcal{S}' = \{(0,0),(1,0)\}$ and $\mathcal{S}' = \{(0,0),(0,1)\}$, respectively, and both their embeddings. By combining these five equations we can obtain the remaining ones. For example,

$$\mathbf{T}\begin{pmatrix} 1* \\ ** \end{pmatrix} = \mathbf{T}\begin{pmatrix} 1* \\ 1* \end{pmatrix} + \mathbf{T}\begin{pmatrix} 1* \\ 2* \end{pmatrix} \qquad \text{implies}$$

$$\mathbf{T}\begin{pmatrix} 1* \\ 2* \end{pmatrix} = \mathbf{T}\begin{pmatrix} 1* \\ ** \end{pmatrix} - \mathbf{T}\begin{pmatrix} 1* \\ 1* \end{pmatrix} = \mathbf{T}\begin{pmatrix} *1 \\ ** \end{pmatrix} - \mathbf{T}\begin{pmatrix} *1 \\ *1 \end{pmatrix} = \mathbf{T}\begin{pmatrix} *1 \\ *2 \end{pmatrix}.$$

Thus, $\mathbf{T}$ in $D = m^4 = 16$-dimensional space has $\mu + 1 = 11$ components by the above five independent conservation laws. The normalization restricts it further to $\mu = 10$–dimensional polytope. In Figure 3 we show all 21 vertices of this polytope and the corresponding tiling. Observe that some vertices (vectors) are not realizable by a Markov field. $\square$

Finally, we illustrate (17) of Corollary 4 for the box shape.

Example 7. *The Box Shape*.
Let us now consider a general box shape $\mathcal{I}_{l_1} \times .. \times \mathcal{I}_{l_d}$. Observe that:

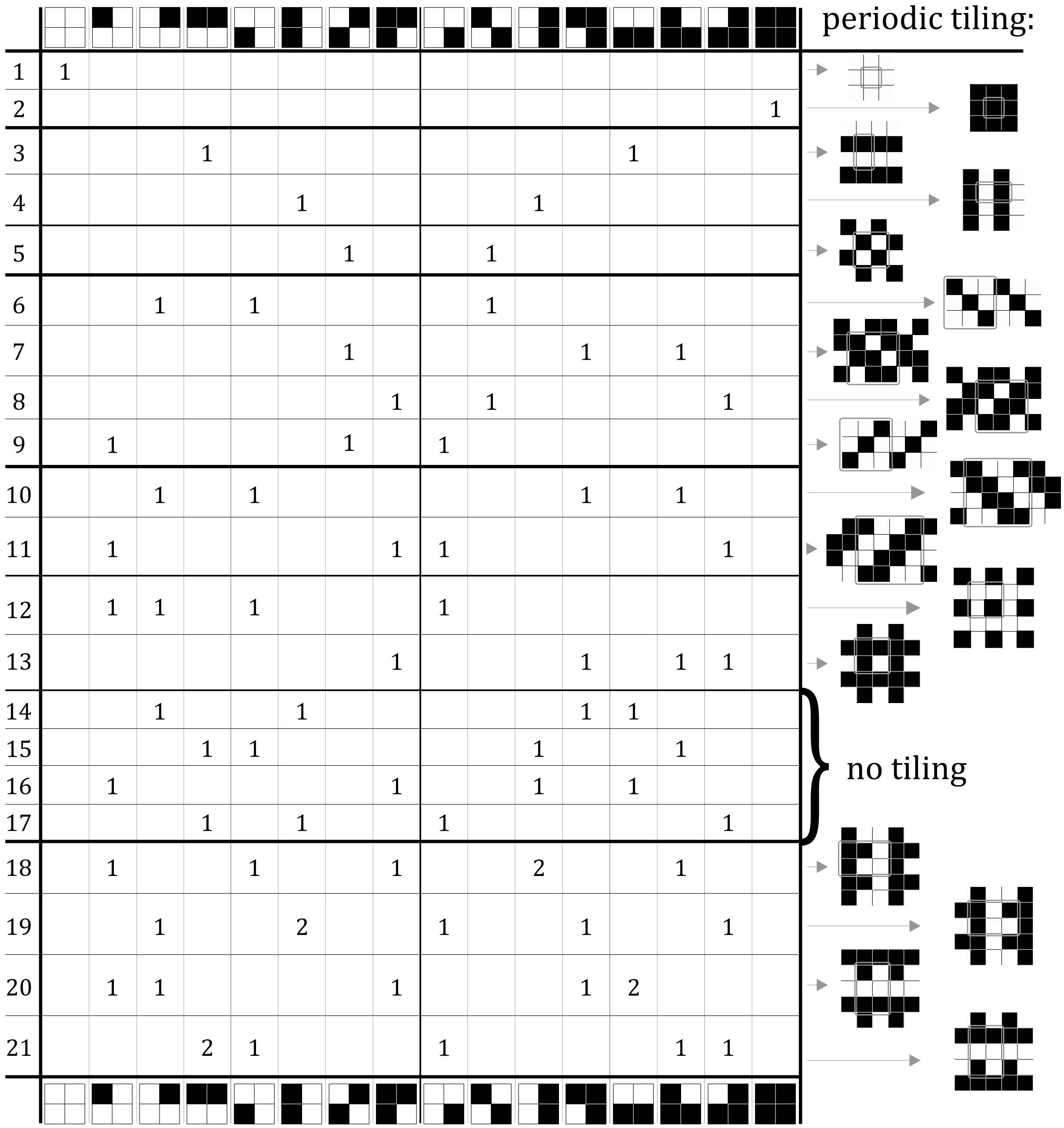
- For the $d = 1$ dimensional shape $\mathcal{S} = \{(0),(1)\}$ we have $\mu = m^2 - m$, as known already from [11]. For $\mathcal{S} = \{(0),(1),(2)\}$ we find $\mu = m^3 - m^2$, while for $\mathcal{S} = \{(0),(1),(2),(3)\}$ we have $\mu = m^4 - m^3$.
- For $d = 2$ the 2×2 square shape ($l_1 = l_2 = 2$) leads to $\mu = m^4 - 2m^2 + m$, the 3×2 rectangular shape gives $\mu = m^6 - m^4 - m^3 + m^2$ and the 3×3 square ends up with $\mu = m^9 - 2m^6 + m^4$.
- For $d = 3$ the $2 \times 2 \times 2$ cube leads to $\mu = m^8 - 3m^4 + 3m^2 - m$, while the $2 \times 3 \times 4$ box gives $\mu = m^{24} - m^{18} - m^{16} - m^{12} + m^{12} + m^9 + m^8 - m^6$.
- For $d = 4$ we have $\mu = m^{16} - 4m^8 + 6m^4 - 4m^2 + m$ for the $2 \times 2 \times 2 \times 2$ cube.
- Finally, in $d = 5$ space the $2\times2\times2\times2\times2$ cube leads to $\mu = m^{32} - 5m^{16} + 10m^8 - 10m^4 + 5m^2 - m$.

2.4 Geometry and Enumeration

We explore now the geometry of the vector counts $\mathbf{T} = \{\mathbf{T}(\mathfrak{t})\}_{\mathfrak{t}\in\mathfrak{T}}$ in the D-dimensional space. As discussed, the conservation laws (which we write as a linear system $\mathbf{CT} = \mathbf{0}$) restrict the vectors $\mathbf{T}$ to a $D - \text{rk}(\mathbf{C}) = \mu + 1$ dimensional cone $\mathcal{C}$ and the normalization equation $\sum_{\mathfrak{t}} \mathbf{T}(\mathfrak{t}) = N$ (for torus) further restricts $\mathbf{T}$ to the polytope $\mathcal{F}_{\mathbf{n}}$. Let us recall some definitions. Formally, let us define

$$\mathcal{C} \quad = \quad \{\mathbf{T} \in \mathbb{R}^D_{\geq 0} \ : \ \mathbf{C} \cdot \mathbf{T} = \mathbf{0}\}, \tag{18}$$

$$\mathcal{F}_{\mathbf{n}} = \mathcal{F}_N \equiv \mathcal{F}_{\mathbf{n}}(\mathcal{A}, \mathcal{S}) \quad = \quad \{\mathbf{T} \in \mathcal{C} \cap \mathbb{N}^D : \sum_{\mathfrak{t}} \mathbf{T}(\mathfrak{t}) = N\}. \tag{19}$$



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1															
2																1
3				1									1			
4						1					1					
5							1			1						
6			1		1					1						
7							1					1		1		
8								1		1					1	
9		1					1		1							
10			1		1							1		1		
11		1						1	1						1	
12		1	1		1				1							
13								1				1		1	1	
14			1			1						1	1			
15				1	1						1			1		
16		1						1			1		1			
17				1		1			1						1	
18		1			1			1			2			1		
19			1			2			1			1			1	
20		1	1					1				1	2			
21				2	1				1					1	1	

Figure 3: For $m = 2$ and the 2×2 square shape $\square$, we show all 21 vertices (only nonzero coordinates are displayed) of the $\mu = 10$ dimensional polytope in a $D = 16$ dimensional space. On the right-hand side, we also show the corresponding periodic tilings: Four of them (14-17) cannot be realized. Periodic tilings 1-7 correspond to all 7 vertices for the L like shape.

We also define the normalized polytope $\hat{\mathcal{F}}$ of frequency vectors $\hat{\mathbf{T}}$ as

$$\hat{\mathcal{F}} \equiv \hat{\mathcal{F}}(\mathcal{A}, \mathcal{S}) = \{\hat{\mathbf{T}} \in \mathbb{R}^D_{\geq 0} \; : \mathbf{C} \cdot \hat{\mathbf{T}} = \mathbf{0}; \; \sum_{\mathfrak{t}} \mathbf{T}(\mathfrak{t}) = 1\} \tag{20}$$

and we write

$$\mathcal{F}_N = N\hat{\mathcal{F}} \cap \mathbb{N}^D. \tag{21}$$

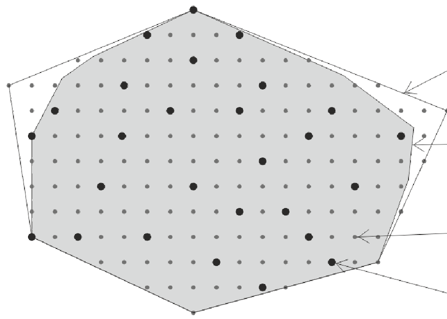


Figure 4: An illustration of the polytope $\hat{\mathcal{F}}$: $\hat{\mathbf{T}}$ vectors realized by periodic tilings create some irregular subset of the lattice and while $N \to \infty$ they densely cover some convex subset of $\hat{\mathcal{F}}$.

Finally, the *rescaled* set of realizable count vectors (*types*) is

$$\hat{\mathcal{P}} \equiv \hat{\mathcal{P}}(\mathcal{A}, \mathcal{S}) = \bigcup_{\mathbf{n}} \mathcal{P}_{\mathbf{n}}(\mathcal{A}, \mathcal{S})/N(\mathbf{n}). \tag{22}$$

Obviously, $\hat{\mathcal{P}} \subset \hat{\mathcal{F}}$.

Observe that $\hat{\mathcal{F}}$ is a compact polyhedron, hence (from basic convex analysis [21]) a polytope, i.e. the convex hull of its *extremal points*. These extremal points are the intersections of the linear subspace $\{\mathbf{T} : \mathbf{CT} = \mathbf{0},\ \sum_{\mathfrak{t}} \mathbf{T}(\mathfrak{t}) = 1\}$ with some μ of D conditions of type $T(\mathfrak{t}) = 0$. The number of the extremal points obtained this way is finite and at most $\binom{D}{\mu}$.

Example 8. *Polytopes in the 2D Case.*
For the L-shape in the 2D case with $m = 2$, we have $\mu = 5$ dimensional polytope in $D = 8$ dimensional space. Among $\binom{8}{5} = 56$ possible ways to choose zero coordinates, it turns out that there are only 7 vertices with all nonnegative coordinates. These seven vertices have the following coordinates:

$$\{(0,0,0,0,0,0,0,1),\ (0,0,0,\frac{1}{3},0,\frac{1}{3},\frac{1}{3},0),\ (0,0,0,\frac{1}{2},\frac{1}{2},0,0,0),\ (0,0,\frac{1}{2},0,0,\frac{1}{2},0,0),$$

$$(0,\frac{1}{2},0,0,0,0,\frac{1}{2},0),\ (0,\frac{1}{3},\frac{1}{3},0,\frac{1}{3},0,0,0),\ (1,0,0,0,0,0,0,0)\}.$$

All of these points correspond to periodic tilings (see cases 1 to 7 in Figure 3). On the other hand, for the 2×2 square shape and $m = 2$, we have 21 vertices of $\mu = 10$ dimensional polytope in $D = 16$ dimensional space as shown in Figure 3. Surprisingly, now some of the vertices do *not* correspond to periodic tilings, so in general not all points in $\hat{\mathcal{F}}$ lead to a realizable tiling and therefore a point in $\hat{\mathcal{P}}$ (see Figure 4). □

Interestingly enough, we can prove that the topological closure of $\hat{\mathcal{P}}$ is still a convex subset of $\hat{\mathcal{F}}$. This is illustrated in Figure 4 and proved below.

Lemma 5. *The closure* $\mathrm{cl}(\hat{\mathcal{P}})$ *of* $\hat{\mathcal{P}}$ *is a convex subset of* $\hat{\mathcal{F}}$.

Proof: To prove convexity of a closed set, it is enough to show that for any two points in this set, the point in the center between them is also in the underlying set. For every point $\hat{\mathbf{T}} \in \mathrm{cl}(\hat{\mathcal{P}})$ one can find a sequence of periodic tilings, whose (rescaled) frequency vectors converge to this points.

Consider two sequences of fields: $\mathbf{x}'_i$ and $\mathbf{x}''_i$ such that their frequency vectors converge to $\hat{\mathbf{T}}'_i \to \hat{\mathbf{T}}'$ and $\hat{\mathbf{T}}''_i \to \hat{\mathbf{T}}''$ as $i \to \infty$. We need to construct a sequence of fields $\mathbf{x}_i$ with frequency vectors $\hat{\mathbf{T}}_i \to \hat{\mathbf{T}} = (\hat{\mathbf{T}}' + \hat{\mathbf{T}}'')/2$. For this purpose, having $\mathbf{x}'_i$ and $\mathbf{x}''_i$ with correspondingly $\hat{\mathbf{T}}'_i$ and

$\hat{\mathbf{T}}_i''$ frequency vectors, we shall construct $\mathbf{x}_i$ field with $\hat{\mathbf{T}}_i$ frequency vector in at most $\epsilon_i > 0$ distance from $(\hat{\mathbf{T}}_i' + \hat{\mathbf{T}}_i'')/2$, where $\epsilon_i \to 0$ is some arbitrary sequence. To accomplish it we cover one half of a large torus with $\mathbf{x}_i'$ tiling and the second with $\mathbf{x}_i''$. If the size of such a torus grows to infinity, the obtained frequency tends to $(\hat{\mathbf{T}}_i' + \hat{\mathbf{T}}_i'')/2$, as desired. ∎

The set $\mathcal{F}_N$ consists of all integer points inside the polytope $N\hat{\mathcal{F}} \cap \mathbb{N}^D$. The volume of $N\hat{\mathcal{F}}$ is proportional to N^μ, and we expect the number of integer points inside also grows asymptotically as N^μ. This is indeed the case by Ehrhart's theorem [8]:

Theorem 6 (Ehrhart, 1967)**.** *If $\hat{\mathcal{F}}$ is a rational polytope (i.e. a polytope with vertices in $\mathbb{Q}^D$), then there exist a period $p \in \mathbb{N}$ and real coefficients $c_{i,j}$ such that $c_{\mu,j} \neq 0$ for some j and*

$$|\mathcal{F}_N| = a_{\mu,j}N^\mu + a_{\mu-1,j}N^{\mu-1} + \dots a_{0,j} \qquad \text{if} \quad N \equiv j \pmod{p}$$

where μ is the dimension of $\hat{\mathcal{F}}$.

Indeed, by the construction the vertices of $\hat{\mathcal{F}}$ are solutions of a system of linear equations with integer coefficients (actually, ± 1), making it a rational polytope. Since $\mathcal{F}_{\mathbf{n}}$ upper bounds $\mathcal{P}_{\mathbf{n}}$ (as a set), the polynomial in N size of $\mathcal{F}_{\mathbf{n}}$ provides an upper bound for the number of types. We need now a matching lower bound. In Section 3 we construct such a bound, leading to the main result of this paper.

Theorem 7. *Consider the torus $\mathcal{O}_{\mathbf{n}}$. There exist constants $0 < c^- \leq c^+$ such that for n_i large enough we have*

$$c^- N^\mu \leq |\mathcal{P}_{\mathbf{n}}(\mathcal{A}, \mathcal{S})| \leq c^+ N^\mu \tag{23}$$

where, we recall, $N = n_1 \cdots n_d$.

We should point out that in [11] for $d = 1$ it was proved that $|\mathcal{F}_n|$ is asymptotically equivalent to $|\mathcal{P}_n|$, that is, $|\mathcal{F}_n| \sim |\mathcal{P}_n|$ as $n \to \infty$: the set of realizable types in dimension 1 is essentially given by the conservation laws. Remarkably, this turns out not to be true in general in higher dimensions. However, in some special cases we can say more about $|\mathcal{F}_N|$ but not necessary about $|\mathcal{P}_N|$. This is discussed next.

2.5 Analytic Approach

In [11] an analytic approach was used to enumerate $\mathcal{F}_N$ (here $N = n = n_1$, the length of the underlying sequence) for $d = 1$. We recall it here and extend it to any dimension and shapes. We should point out, however, that through this approach we will only get better asymptotics for $|\mathcal{F}_N|$ but not for $|\mathcal{P}_N|$. This is actually of interest on its own since it allows us to enumerate precisely the number of nonnegative solutions of a multidimensional system of linear Diophantine equations; not an easy task, as argued in [25].

Let us recall some facts from [11]. We first assume $d = 1$ and enumerate $\mathcal{F}_N$. We accomplish it by finding the following generating function

$$F_m^*(z) = \sum_{N \geq 0} |\mathcal{F}_N(m)| z^N$$

and then taking the coefficient at z^N which is written as $[z^N]F_m^*(z) = |\mathcal{F}_N(m, \mathcal{S})|$.

Let $\mathbf{z} = \{z_{\mathfrak{t}}\}_{\mathfrak{t} \in \mathfrak{T}}$ where in our case $\mathfrak{t} = (\alpha, \beta) \in \mathcal{A}^2$ is a pair of symbols or in other words the shape is $\mathcal{S} = \{0, 1\}$. We also write $\mathbf{z}^{\mathbf{T}} = \prod_{\alpha\beta} z_{\alpha\beta}^{\mathbf{T}(\alpha\beta)}$. We introduce a multidimensional generating function $\sum_{\mathbf{T}} \mathbf{z}^{\mathbf{T}}$ that we estimate in two different ways for $z_{\alpha\beta} = z\frac{y_\alpha}{y_\beta}$ for some $(y_\alpha)_{\alpha\in\mathcal{A}}$ vector:

$$\begin{aligned}\sum_{\mathbf{T}} \mathbf{z}^{\mathbf{T}} &= \prod_{\alpha\beta}\left(\sum_{\mathbf{T}(\alpha\beta)}\left(z\frac{y_\alpha}{y_\beta}\right)^{\mathbf{T}(\alpha\beta)}\right) = \prod_{\alpha\beta}\left(1 - z\frac{y_\alpha}{y_\beta}\right)^{-1},\\ \sum_{\mathbf{T}} \mathbf{z}^{\mathbf{T}} &= \sum_{\mathbf{T}} z^{\sum_{\alpha,\beta}\mathbf{T}(\alpha\beta)} \prod_{\alpha} y_\alpha^{\sum_\beta \mathbf{T}(\alpha\beta) - \sum_\beta \mathbf{T}(\beta\alpha)}.\end{aligned}$$

Now if $\mathbf{T} \in \mathcal{F}_N$, that is,

$$\sum_{\alpha,\beta} \mathbf{T}(\alpha,\beta) = N, \qquad \sum_\beta \mathbf{T}(\alpha\beta) - \sum_\beta \mathbf{T}(\beta\alpha) = 0,$$

then

$$F_m^*(z) = \sum_{N\geq 0} |\mathcal{F}_N(m,\mathcal{S})| z^N = [y_1^0 y_2^0 ... y_m^0] \prod_{\alpha,\beta=1}^{m} \left(1 - z\frac{y_\alpha}{y_\beta}\right)^{-1} \tag{24}$$

where $[\mathbf{y}^0]F(\mathbf{y}) := [y_1^0 y_2^0 ... y_m^0]F(\mathbf{y})$ denotes the zeroes coefficient of $F(\mathbf{y})$.

There is a simple interpretation of formula (24): Its right hand side can be seen as a product of m^2 geometric series, while α, β terms correspond to "$\alpha\beta$" pattern (pair) in our sequence. The auxiliary $\mathbf{y}$ variables are used to restrict $\mathbf{T}$ to those satisfying the conservation laws: each symbol should appear the same number of times on the left and on the right position of $\mathcal{S}$. Thanks to the y_α/y_β term, the power of y_α increases by 1 every time α appears in left position of "$\alpha\beta$", and decreases by 1 when it appears in the right position. However, in addition we have the normalization equation which allows us to eliminate one of the variables, for example by setting $y_m = 1$.

Let us now move to the multidimensional case $d > 1$. Each auxiliary variable corresponds to a single equation of the conservation laws. We can reduce the set of equations by considering only independent variables, as discussed in Theorem 3.

Let us start with some examples. For the `L` shape as in Example 2 we have $\mathcal{S} = \{(0,0),(0,1),(1,0)\}$, and

$$F_m^*(z) = [x_1^0 x_2^0 ... x_m^0 y_1^0 y_2^0 ... y_m^0] \prod_{\alpha,\beta,\gamma} \left(1 - z\frac{x_\alpha}{x_\beta}\frac{y_\alpha}{y_\gamma}\right)^{-1},$$

where the auxiliary variables $\mathbf{x}$ now guard the conservation law in one direction, $\mathbf{y}$ in the other direction. In other words, the `L` shape tile `t` is marked as follows

$$\begin{pmatrix} \gamma & \\ \alpha\beta \end{pmatrix}$$

and then the conservation laws are

$$\sum_{\beta\gamma} \mathbf{T}\begin{pmatrix} \gamma \\ \alpha\beta \end{pmatrix} = \sum_{\beta\gamma} \mathbf{T}\begin{pmatrix} \gamma \\ \beta\alpha \end{pmatrix}, \qquad \sum_{\beta\gamma} \mathbf{T}\begin{pmatrix} \gamma \\ \alpha\beta \end{pmatrix} = \sum_{\beta\gamma} \mathbf{T}\begin{pmatrix} \alpha \\ \gamma\beta \end{pmatrix}.$$

We can choose $x_m = y_m = 1$. Interestingly enough, in this case $\mathrm{cl}(\hat{\mathcal{P}}) = \hat{\mathcal{F}}$ since both are spanned on 7 vertices in $D = 8$ dimensional space, as shown in Figure 2. This, however, does not imply that $|\mathcal{F}_N| \sim |\mathcal{P}_N|$.

For the analogous 3D `L` shape $\mathcal{S} = \{(0,0,0),(0,0,1),(0,1,0),(1,0,0)\}$, we find

$$F_m^* = [x_1^0 x_2^0 ... x_m^0 y_1^0 y_2^0 ... y_m^0 u_1^0 u_2^0 ... u_m^0] \prod_{\alpha,\beta,\gamma,\delta} \left(1 - z\frac{x_\alpha}{x_\beta}\frac{y_\alpha}{y_\gamma}\frac{u_\alpha}{u_\delta}\right)^{-1}$$

and we can set $x_m = y_m = u_m = 1$. Using partial fraction expansions, as in [11], we can obtain asymptotic expressions for $|\mathcal{F}_N|$, as illustrated below.

Example 8. For $m = 2$ in the 2D case and the `L` shape, we have

$$F_2^*(z) = \frac{1 - z + z^2}{(z-1)^6(z+1)^2(z^2+z+1)}.$$

Using the partial fraction expression we find after some algebra

$$|\mathcal{F}_N(2, L)| = \frac{N^5}{12 \cdot 5!} + O(N^4).$$

For the analogous 3D `L` shape when $m = 2$ we arrive at

$$F_2^*(z) = \frac{Q(z)}{(z-1)^{13}(z+1)^5(z^2+1)(z^2+z+1)^3(z^4+z^3+z^2+z+1)}$$

with $Q(z) = 1 + 2z + 22z^2 + 50z^3 + 94z^4 + 138z^5 + 175z^6 + 184z^7 + 163z^8 + 120z^9 + 76z^{10} + 38z^{11} + 16z^{12} + 2z^{13} + z^{14}$. Using the partial fraction decomposition and Cauchy's integral we find

$$|\mathcal{F}_N(2, L)| = \frac{541}{4320} \frac{N^{12}}{12!} + O(N^{11})$$

for large N. □

Let us now look at a situation with a more subtle dependence between the conservation laws, for example for the 2×2 square shape in 2D discussed in Example 3. The first approach could be:

$$F_m^*(z) = [x_{11}^0 x_{12}^0 \ldots x_{mm}^0 y_{11}^0 y_{12}^0 \ldots y_{mm}^0] \prod_{\alpha,\beta,\gamma,\delta} \left(1 - z \frac{x_{\alpha\beta}}{x_{\gamma\delta}} \frac{y_{\alpha\gamma}}{y_{\beta\delta}}\right)^{-1}$$

where we can initially set $x_{mm} = y_{mm} = 1$. This formula corresponds to $\mathcal{S}'$ selected as $\mathcal{S}' = \{(0,0),(1,0)\}$, $\mathbf{s} = (0,1)$ and $\mathcal{S}' = \{(0,0),(0,1)\}, \mathbf{s} = (1,0)$ or the following marking of the square tile

$$\begin{pmatrix} \delta & \gamma \\ \alpha & \beta \end{pmatrix}.$$

It leads to a complete set of conservation laws, but with some linear dependencies, as the conservation law for $\mathcal{S}' = \{(0,0)\}$, $\mathbf{s} = (1,1)$ can be induced in two ways. To ensure using only independent variables/conservation laws, we use Theorem 3 to deduce the set of independent conservation laws. This leads to

$$F_m^* = [u_1^0 .. u_m^0 v_1^0 .. v_m^0 w_1^0 .. w_m^0 x_{11}^0 .. x_{mm}^0 y_{11}^0 .. y_{mm}^0] \prod_{\alpha,\beta,\gamma,\delta} \left(1 - z \frac{u_\alpha}{u_\beta} \frac{v_\alpha}{v_\gamma} \frac{w_\alpha}{w_\delta} \frac{x_{\alpha\beta}}{x_{\gamma\delta}} \frac{y_{\alpha\gamma}}{y_{\beta\delta}}\right)^{-1}$$

where $u_m = v_m = w_m = 1$ and $x_{im} = x_{mi} = y_{im} = y_{mi} = 1$ for any $i \in \mathcal{A}$.

Example 9. Consider $m = 2$. Then both approaches lead to

$$F_2^*(z) = \frac{1 + 2z^3 + 5z^4 + 2z^6 + 6z^7 + 8z^8 + 6z^9 + 2z^{10} + 5z^{12} + 2z^{13} + z^{16}}{(z-1)^{11}(z+1)^7(z^2+1)^3(z^2-z+1)(z^2+z+1)^3}$$

from which we find

$$|\mathcal{F}_{Size}(2, \square)| = \frac{5}{3456} \frac{N^{10}}{10!} + O(N^9)$$

for large N. $\square$

For a general shape we consider the conservation laws (14), attach a variable ($\mathbf{y}$) to each of them, and choose a fraction of some of these variables in the product of $m^{|\mathcal{S}|}$ geometric series to enforce the conservation laws by zeroing the power of these variables. This allows us to find a general expression for the underlying generating function, that is,

$$F_m^*(z) = [\mathbf{y}^0] \prod_{\mathfrak{t}:\mathcal{S}\to\mathcal{A}} \left(1 - z \prod_{\mathcal{S}' \text{ embeddable in } \mathcal{S},\ \boldsymbol{\varepsilon}\neq\boldsymbol{\varepsilon}_{\mathcal{S}'}} \frac{y_{\boldsymbol{\varepsilon}^*(\mathfrak{t})}}{y_{\boldsymbol{\varepsilon}^*_{\mathcal{S}'}(\mathfrak{t})}}\right)^{-1} \tag{25}$$

where $\boldsymbol{\varepsilon}_{\mathcal{S}'}$ is the canonical embedding, $[\mathbf{y}^0]$ denotes taking zeroth power of all used y_i.

2.6 Number of Types in a Box – An Upper Bound

Finally, we comment on the number of types $\tilde{\mathcal{P}}_{\mathbf{n}}(m,S)$ in the box $\mathcal{I}_{\mathbf{n}} = I_{n_1} \times I_{n_2} \times \ldots \times I_{n_d} \subset \mathbb{Z}^d$. We only discuss an upper bound, leaving establishing the proper growth to a forthcoming paper. Let $\mathbf{x} = \mathbf{x}^{\mathbf{n}}$ be a configuration in the box $\mathcal{I}_{\mathbf{n}}$. Its type in the box is now defined by shifts (embeddings) that fit into the box, that is,

$$\tilde{\mathbf{T}}(\mathfrak{t}) = |\{\mathbf{s} \in \tilde{\mathcal{I}}_{\mathbf{n}} : \mathbf{x}|_{\mathcal{S}+\mathbf{s}} = \mathfrak{t}\}| \qquad \text{where} \qquad \tilde{\mathcal{I}}_{\mathbf{n}} = \{\mathbf{s} : \mathcal{S} + \mathbf{s} \subset \mathcal{I}_{\mathbf{n}}\}.$$

We assume that $\mathbf{0} \in \mathcal{S}$ and $\tilde{\mathcal{I}}_{\mathbf{n}} \subset \mathcal{I}_{\mathbf{n}}$. We know that $\mathbf{T}$ in the torus satisfies the conservation laws $\mathbf{C} \cdot \mathbf{T} = \mathbf{0}$. For the box, however, we must re-define the type $\tilde{\mathbf{T}}$ by taking into account the boundary effect on $\mathbf{T}$, that is,

$$\tilde{\mathbf{T}}(\mathfrak{t}) = \mathbf{T}(\mathfrak{t}) - |\{\mathbf{s} \in \mathcal{I}'_{\mathbf{n}} : \mathbf{x}|_{\mathcal{S}+\mathbf{s}} = \mathfrak{t}\}| \qquad \text{where} \qquad \mathcal{I}'_{\mathbf{n}} = \mathcal{I}_{\mathbf{n}} \setminus \tilde{\mathcal{I}}_{\mathbf{n}}, \tag{26}$$

that is, we need to eliminate that shifts that drive $\mathcal{S}$ outside the box. Multiplying (26) by $\mathbf{C}$ and using $\mathbf{C} \cdot \mathbf{T} = \mathbf{0}$ we find the following conservation laws for the box:

$$\mathbf{C}\tilde{\mathbf{T}} = \mathbf{b} \qquad \text{for} \qquad \mathbf{b} = \mathbf{C} \cdot \mathbf{b}', \quad \mathbf{b}' = \left(-|\{\mathbf{s} \in \mathcal{I}'_{\mathbf{n}} : \mathbf{x}|_{\mathcal{S}+\mathbf{s}} = \mathfrak{t}\}|\right)_{\mathfrak{t}\in\mathfrak{T}}.$$

Notice that the norm of $\mathbf{b}'$ vector is bounded by the size of the boundary, that is, $\sum_{\mathfrak{t}} |\mathbf{b}'(\mathfrak{t})| \leq |\mathcal{I}'_{\mathbf{n}}|$, which is of order $\Theta(N/\min_i n_i)$. Furthermore, matrix $\mathbf{C}$ does not depend on the size (only on $\mathcal{S}$ and m), therefore the number of $\mathbf{b}$ is bounded by $\Theta(N/\min_i n_i)$. Finally, by the normalization

$$\sum_{\mathfrak{t}} \tilde{\mathbf{T}}(\mathfrak{t}) = |\tilde{\mathcal{I}}_{\mathbf{n}}| =: \tilde{N},$$

types $\tilde{\mathbf{T}}$ in the box are in a $D-1$ dimensional linear subspace. For every $\mathbf{b}$, the conservation laws $\mathbf{C}\tilde{\mathbf{T}} = \mathbf{b}$ have $O(N^\mu)$ nonnegative solutions inside a polygon. The freedom of choosing $\mathbf{b} \in \mathbb{N}^{\mathrm{rk}(\mathbf{C})}$ allows us to shift this polygon in the remaining $\mathrm{rk}(\mathbf{C}) = D - 1 - \mu$ dimensions by at most $\Theta(N/\min_i n_i)$, so that $\mathbf{b}$ is inside the ball of $O((N/\min_i n_i)^{D-1-\mu})$ integer points. This leads to the upper bound $O(N^{D-1}/(\min_i \ n_i)^{\mathrm{rk}(\mathbf{C})})$ on the number of types. However, finding the right order of growth in this case remains an open question.

3 Analysis

In this section we provide the proofs of Theorems 3 and 7.

	$\mathcal{S}'$	s																
1		↑		1		1	-1		-1			1		1	-1		-1	
2		↗			1	1	-1	-1					1	1	-1	-1		
3		→					-1	-1	-1	-1	1	1	1	1				
4		↑		1	1	1	-1				-1				-1			
5		→		-1	1		-1	-1			1		1					

Figure 5: The nonzero coordinates for all 5 conservation laws discussed in Theorem 3 for a 2×2 box shape and $m = 2$: the upper row shows all $D = 16$ squares corresponding to all tiles $\mathfrak{t}$. $\mathcal{S}'$ denotes the canonical embedding and $\mathbf{s}$ denotes shift for the second embedding in $(\hat{\boldsymbol{\varepsilon}}_{\mathcal{S}'}\mathbf{T})(\mathfrak{t}') - (\hat{\boldsymbol{\varepsilon}}\mathbf{T})(\mathfrak{t}') = 0$ conservation law. Reduced alphabet is $\mathcal{A}' = \{1\}$ here, so we need to consider only constant $\mathfrak{t}' = 1$.

3.1 Proof of Theorem 3

In Theorem 3 we present a complete set of independent conservation laws. Specifically, we take every subshape $\mathcal{S}'$ embeddable is $\mathcal{S}$ and select one of it as the canonical one. Then we consider all $\mathfrak{t}' : \mathcal{S}' \to \mathcal{A}'$ where $\mathcal{A}' = \mathcal{A} \setminus \{m\}$ and the corresponding conservation laws $(\hat{\boldsymbol{\varepsilon}}_{\mathcal{S}'}\mathbf{T})(\mathfrak{t}') - (\hat{\boldsymbol{\varepsilon}}\mathbf{T})(\mathfrak{t}') = 0$ for all other $\hat{\boldsymbol{\varepsilon}}$ embeddings of this subshape. Observe that the dropped symbol $m \in \mathcal{A}$ is automatically included since the following holds:

$$\mathbf{T}''(\mathfrak{t}' \cup o_m) = \mathbf{T}'(\mathfrak{t}') - \sum_{i \in \mathcal{A}'} \mathbf{T}''(\mathfrak{t}' \cup o_i)$$

where o_i is a single point/position outside $\mathfrak{t}'$ that takes value $i \in \mathcal{A}$ there. Clearly, $\mathbf{T}' = \hat{\boldsymbol{\varepsilon}}'(\mathbf{T})$ and $\mathbf{T}'' = \hat{\boldsymbol{\varepsilon}}''(\mathbf{T})$, where $\boldsymbol{\varepsilon}'$, $\boldsymbol{\varepsilon}''$ are embeddings corresponding to $\mathfrak{t}'$ and $\mathfrak{t}' \cup o_i$.

To prove independence of the conservation laws, we have to define some order among them and show that $\mathbf{C}$ becomes a triangular. We order the conservation laws by the size of $\mathcal{S}'$ (referred as height). We illustrate it in Figure 5 where the ordering of the columns is shown in gray leading to a triangular form of $\mathbf{C}$.

To make this more formal, let us introduce a certain basis in the space $\mathbb{R}^D$ of functions on the configurations on $\mathcal{S}$. Fix the standard basis $\{\mathbf{e}(\mathfrak{t}), \mathfrak{t} \in \mathfrak{T}(\mathcal{A}, \mathcal{S})\}$. For each tiling $\mathfrak{t}$, we can split out the *inessential part*, the cells $\boxed{b} \in \mathcal{S}$ where $\mathfrak{t}(b) = m$, and the *support* of $\mathfrak{t}$, i.e. the collection of boxes where $\mathfrak{t}(b) \neq m$. Alternatively we can enumerate the tilings $\mathfrak{t}$ of $\mathcal{S}$ by the shape $\mathcal{S}'$ off their support, by the embedding $\boldsymbol{\varepsilon}$ of this support into $\mathcal{S}$, and by the tiling of $\mathcal{S}'$ over the reduced alphabet $\mathcal{A} \setminus \{m\} =: \mathcal{A}'$. Such a basis vector we will denote as

$$\mathbf{e}(\mathcal{S}', \boldsymbol{\varepsilon}, \mathfrak{t}).$$

We will call the *height* of a basis vector $\mathbf{e}(\mathcal{S}', \boldsymbol{\varepsilon}, \mathfrak{t})$ the size of its support, that is

$$\mathtt{H}\left(\mathbf{e}(\mathcal{S}', \boldsymbol{\varepsilon}, \mathfrak{t})\right) = |\mathcal{S}'|.$$

Further, we assign to each basis vector its *weight*, defined as follows. We number all the cells of $\mathcal{S}$, and, denoting the number of the cell $\boxed{b}$ as $\#(b)$, we assign the weight of b to be $\epsilon^{\#(b)}$ for some small $\epsilon > 0$. The weight of a basis vector $\mathbf{e}(\mathcal{S}', \boldsymbol{\varepsilon}, \mathfrak{t})$ is the sum

$$\sum_{b \in \boldsymbol{\varepsilon}(\mathcal{S}')} \epsilon^{\#(b)}.$$

There is nothing very specific about this choice of the weights; the only property we will use is that for small enough ϵ, the weights corresponding to different embeddings of the same subshape $\mathcal{S}'$ are all different (which is easy to verify). In particular, for any embeddable $\mathcal{S}'$, there exists a unique embedding $\boldsymbol{\varepsilon}_{\mathcal{S}'}$ of $\mathcal{S}'$ having *maximal weight* among all such embeddings $\boldsymbol{\varepsilon} : \mathcal{S}' \to \mathcal{S}$. We will be calling this basis vector the *anchor* of the embeddable shape $\mathcal{S}'$ and its embedding as *canonical*.

We group the basis vectors $\mathbf{e}(\mathcal{S}', \boldsymbol{\varepsilon}, \mathfrak{t})$ according to their height (increasing left to right), and within each height by the support shape $\mathcal{S}'$ and within each group corresponding to a support shape $\mathcal{S}'$ by the tiling $\mathfrak{t}$ of $\mathcal{S}'$ over the reduced alphabet $\mathcal{A}'$. Finally, within each such group (corresponding to a given subshape $\mathcal{S}'$ and its tiling $\mathfrak{t}$), we order the basis vectors $\mathbf{e}(\mathcal{S}', \boldsymbol{\varepsilon}, \mathfrak{t})$ by the *weight of the embeddings* $\boldsymbol{\varepsilon}$. In particular, the anchor within each group is the rightmost element. This defines a complete ordering on the basis vectors.

Now we are ready to prove Theorem 3. We will be using the basis consisting of the standard vectors $\mathbf{e}(\mathcal{S}', \boldsymbol{\varepsilon}, \mathfrak{t})$ ordered as described above, left to right. The rows of the (sub)matrix $\mathbf{C}_m$ (defined in Theorem 3) correspond to the covectors

$$\boldsymbol{v}_{(\mathcal{S}', \mathfrak{t}, \boldsymbol{\varepsilon}_{\mathcal{S}'}, \boldsymbol{\varepsilon})}.$$

Each such covector has exactly two components,

$$\mathbf{e}(\mathcal{S}', \boldsymbol{\varepsilon}_{\mathcal{S}'}, \mathfrak{t}) - \mathbf{e}(\mathcal{S}', \boldsymbol{\varepsilon}, \mathfrak{t})$$

in the group of height $\mathtt{H}\,(\mathbf{e}(\mathcal{S}', \boldsymbol{\varepsilon}_{\mathcal{S}'}, \mathfrak{t})) = \mathtt{H}\,(\mathbf{e}(\mathcal{S}', \boldsymbol{\varepsilon}, \mathfrak{t}))$; all other components have *higher height*.

It follows that, if one augments $\mathbf{C}_m$ by the rows with basis vectors $\mathbf{e}(\mathcal{S}', \boldsymbol{\varepsilon}_{\mathcal{S}'}, \mathfrak{t})$, running over all embeddable subshapes $\mathcal{S}'$, and their tilings $\mathfrak{t}$, then the leftmost vectors in rows will be all different.Finally, sorting the rows according to these leftmost elements will result in the upper-triangular matrix.

This, in turn, implies that the basis elements

$$\mathbf{e}(\mathcal{S}', \boldsymbol{\varepsilon}_{\mathcal{S}'}, \mathfrak{t})$$

span a complement to the kernel of $\mathbf{C}_m$, and therefore the kernel of $\mathbf{C}$ has dimension at most the number of tilings by symbols of $\mathcal{A}'$ of embeddable shapes $\mathcal{S}'$.

Denote the subspace of $\mathscr{V} \cong \mathbb{R}^D$ spanned by the basis vectors as

$$L_{\mathcal{S}} := \mathbf{e}(\mathcal{S}', \boldsymbol{\varepsilon}_{\mathcal{S}'}, \mathfrak{t}), \mathfrak{t} \in (\mathcal{A}')^{\mathcal{S}'}, \mathcal{S}' \text{embeddable into } \mathcal{S}.$$

To prove that $\operatorname{Ker}\mathbf{C} = \operatorname{Ker}\mathbf{C}_m$, we will produce for any torus $\mathcal{O}_{\mathbf{n}}$ of sufficiently large $\mathbf{n}$, a collection of tilings, of size $\dim(L_{\mathcal{S}})$, such that their frequency vectors, paired with the basis vectors spanning $L_{\mathcal{S}}$, result in a triangular matrix with $\pm$ on the diagonal. This would imply that $\operatorname{Ker}\mathbf{C} = \operatorname{Ker}\mathbf{C}_m$.

Let $\mathbf{n}(\mathcal{S})$ be the smallest vector $\mathbf{n}$ such that the box $\mathcal{I}_{\mathbf{n}}$ contains $\mathcal{S} + \mathcal{S} = \{a + b : a, b \in \mathcal{S}\}$ (understood as the Minkowski sum). We will be always assuming that $\mathcal{S}$ is embedded into this interval (denoted as $\mathcal{I}_{\mathcal{S}}$) in a fixed way.

Let $\mathfrak{t} \in \mathfrak{T}(\mathcal{A}, \mathcal{S})$ be a tiling of the shape $\mathcal{S}$, and $\mathcal{S}'$ its support (i.e. the set of cells $\boxed{b}$ where $\mathfrak{t}(b) \neq m$), and $\mathfrak{t}'$ the corresponding tiling of $\mathcal{S}'$ by symbols of the reduced alphabet $\mathcal{A}'$. For any large enough torus $\mathcal{O}_{\mathbf{n}}$, place a single copy of $\mathfrak{t}$, in an arbitrary way in the torus, extending it to the rest of the torus by the symbol m. Denote the corresponding frequency vector $\mathbf{T}_{\mathcal{S}', \mathfrak{t}'}$.

Lemma 8. *Consider the matrix of scalar products*

$$\left(\mathbf{T}_{\mathcal{S}', \mathfrak{t}'}, \mathbf{e}(\mathcal{S}'', \boldsymbol{\varepsilon}_{\mathcal{S}''}, \mathfrak{t}'')\right)$$

where both $\mathcal{S}', \mathfrak{t}'$ and $\mathcal{S}'', \mathfrak{t}''$ run over all embeddable subshapes and their tilings by the reduced alphabet. Then, if the subshapes are ordered by their heights, the matrix is upper triangular, with $1/N$ on the diagonal.

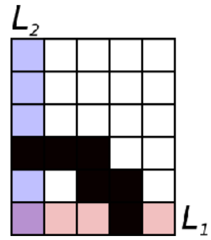


Figure 6: Example of a a 6-cell shape $\mathcal{S}'$ in its lowest position.

Proof. The proof is straightforward: the type vector $\mathbf{T}_{\mathcal{S}',\mathfrak{t}'}$ is produced by scanning through the torus by the shifts of $\mathcal{S}$. There is unique position where the support lands on the anchor of $\mathcal{S}'$, and all other positions are either not anchored (thus yielding zero products with the basis vectors $\mathbf{e}(\mathcal{S}'', \varepsilon_{\mathcal{S}''}, \mathfrak{t}''))$, or have lower height. ■

We remark that one can modify the tiling of the torus: in lieu of a single copy of the interval containing a copy of $\mathcal{S}$, one could tile $\mathcal{O}_{\mathbf{n}}$ with $\Theta(N)$ copies of the shape $\mathcal{S}'$, supporting $\mathfrak{t}'$. In this case, the rescaled frequency vector $\mathbf{T}/N$ would converge, as $\mathbf{n}$ increases, to some vector $\mathbf{T}_{\mathcal{S}',\mathfrak{t}'} \in \hat{\mathcal{P}}$.

3.2 Proof of Corollary 4

By Theorem 3, we need to sum $(m-1)^{|\mathcal{S}'|}$ over all shapes embeddable into $\mathcal{S}$ as in (16). Among all possible embeddings, there is a unique one that is (lexicographically) minimal. One can think about a gravity force pointing along the vector $(-1, \ldots, -1)$ and forcing $\mathcal{S}'$ to slide inside the box $\mathcal{S}$ to its lowest position. Clearly, this lowest position is characterized by the condition that $\mathcal{S}'$ has non-empty intersection with the lowest k-th coordinate layer

$$\mathsf{L}_k = I_{l_1} \times \ldots \times \{1\} \times \ldots \times I_{l_d}$$

(here $\{1\}$ stands in k-th place in the product), for each $k \in \{1, \ldots, d\}$; see Figure 6.

Alternatively, the sum we need to evaluate is the total number of all tilings of the box shape $\mathcal{S}$ with each of the layers $\mathsf{L}_k, k = 1, \ldots, d$ containing at least one cell marked with a symbol of the reduced alphabet $\mathcal{A}'$.

The set of tilings with at least one cell in L_k marked by an element of $\mathcal{A}'$ is, clearly, the set of all tilings with the tilings having *all cells in* L_k *marked with* m. Denote the set of such tilings by M_k. The size of the set of tilings we are interested in is therefore,

$$|\mathfrak{T}(\mathcal{S})| - \cup_k |M_k|.$$

By inclusion-exclusion, this is equal to

$$\sum_{J \subset \{1,\ldots,d\}} (-1)^{|J|} | \cap_{j \in J} M_j |,$$

where the summation is over all subsets of $\{1, \ldots, d\}$ (for the empty subset, we take the summand to be $|\mathfrak{T}(\mathcal{S})|$).

The cardinality of the set $\cap_{j \in J} M_j$ is, clearly, the number of all tilings which have cells in $\cup_{j \in J} \mathsf{L}_j$ equal to m, which is, obviously, the number of all tilings of $\mathcal{S} - \cup_{j \in J} \mathsf{L}_j$. Put together, these formulae imply the corollary. ■

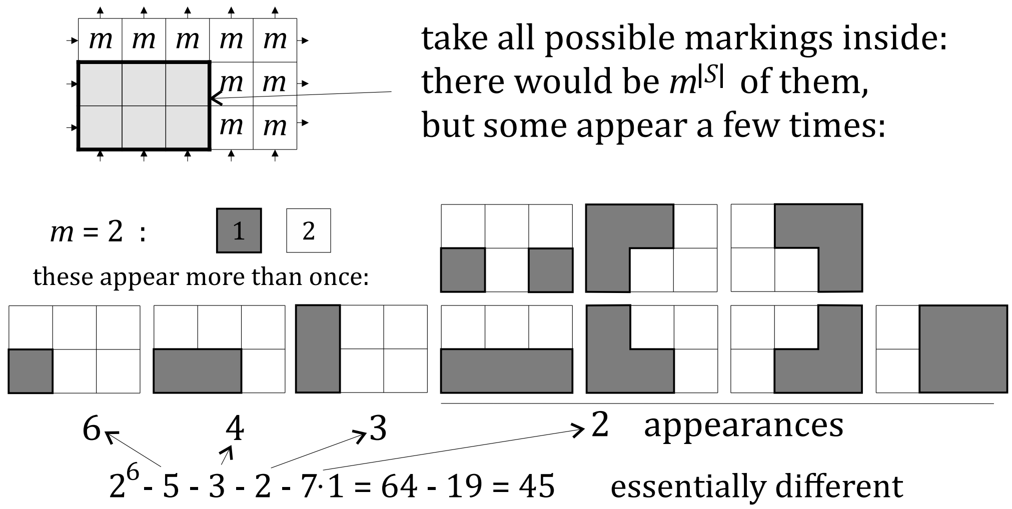


Figure 7: An illustration to the construction in Lemma 9: We place a tile with $m^{|\mathcal{S}|}$ possible patterns in the torus with all remaining positions filled with symbol m. The number of such fields is equal to $D - \mathrm{rk}(\mathbf{C}) = 2^{16} - 19 = 45$, as desired.

3.3 Proof of Theorem 7

Since $\mathcal{P}_\mathbf{n} \subset \mathcal{F}_\mathbf{n}$ and $\hat{\mathcal{F}}$ is a convex polytope, we conclude the upper on $|\mathcal{P}_\mathbf{n}|$ from the Ehrhart's Theorem 6 applied to $\mathcal{F}_\mathbf{n}$. Therefore, we can now focus on establishing a lower bound. We will accomplish it by constructing a family of tilings with a set of frequency vectors growing as N^μ. Specifically, we will first construct building blocks: $\mu + 1$ small linearly independent tilings. Then we will construct large tilings by concatenating these small ones, obtaining a regular lattice of frequency vectors in the μ dimensional simplex on these $\mu + 1$ vertices.

Let us first observe that if the torus is not large enough, there are some additional constraints due to the cyclical boundary condition. For example, in 1D case for $\mathcal{S} = \{0, 1, 2\}$ and torus/cycle $\mathcal{O} = \{0, 1, 2, 3\}$, the tile "111" automatically enforces the tile having "1*1", where "*" is any letter on the remaining position. These additional constraints can reduce the dimension of realizable frequency vectors. For example for 3x2 rectangular shape and $m = 2$, there are only 21 linearly independent possible tilings of a 3x3 torus. For 4x3 torus this number grows to 42, and finally saturates at required $\mu + 1 = 45$ for a 5x3 torus.

In the next lemma we construct $\mu + 1$ linearly independent frequency vectors. To formulate it, we need to define *width* of the shape $\mathcal{S}$ as the smallest $(w_1, .., w_d) \in \mathbb{N}^d$ such that for some shift $S \subset I_{w_1} \times .. \times I_{w_d}$.

Lemma 9. *If $n_i \geq 2w_i - 1$ for all $i = 1, .., d$, then there exist $\mu + 1$ tilings of $\mathcal{O}_\mathbf{n}$ with linearly independent frequency vectors.*

Proof: We will construct these tilings as

$$\mathtt{Conf}^0(\mathcal{D}) := \{\mathbf{x} : \mathcal{O}_\mathbf{n} \to \mathcal{A} \ : \ \mathbf{x}(\mathbf{a}) = m \quad \text{for all} \quad \mathbf{a} \in \mathcal{O}_\mathbf{n} \setminus \mathcal{S}\}.$$

that is, the torus is filled with $m \in \mathcal{A}$ outside the $\mathcal{S}$ shape. The remaining $|\mathcal{S}|$ values $\mathbf{x}|_\mathcal{S}$ can be selected in $m^{|\mathcal{S}|} = D$ ways, which is more than $\mu + 1 = D - \mathrm{rk}(\mathbf{C})$. However, this set contains tilings differing only by a shift and hence having identical frequency vectors. For a given field $\mathbf{x} \in \mathtt{Conf}^0(\mathcal{D})$, let $\mathcal{S}' \subset \mathcal{S}$ be a subset on which the corresponding tiling $\mathfrak{t}'$ has values different than m, that is,

$$\mathcal{S}' = \{\mathbf{a} : \mathbf{x}(\mathbf{a}) \in \{1, .., m-1\}\} \subset \mathcal{S}. \tag{27}$$

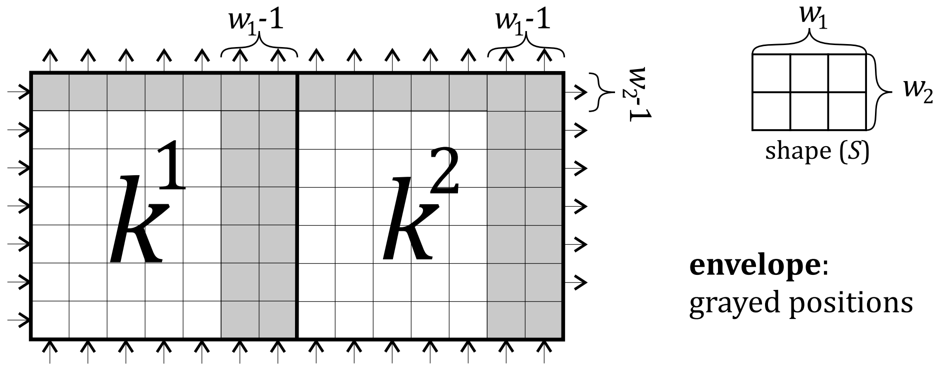


Figure 8: Illustration to Lemma 10.

Observe that if $\mathcal{S}' + \mathbf{s} \subset \mathcal{S}$ for some $\mathbf{s} \in \mathbb{Z}^d$, then there exists an element of $\texttt{Conf}^0(\mathcal{D})$ differing from $\mathbf{x}$ only by shift $\mathbf{s}$. There are $|\{\mathbf{s} : \mathcal{S}' + \mathbf{s} \subset \mathcal{S}\}| - 1$ such elements of $\texttt{Conf}^0(\mathcal{D})$ having identical frequency vector. For given $\mathcal{S}'$ there are $(m-1)^{|\mathcal{S}'|}$ such situations ($\mathbf{x}|_{\mathcal{S}'}$), thus from the initial $|\texttt{Conf}^0(\mathcal{D})| = m^{|\mathcal{S}|}$ tilings we need to subtract

$$\sum_{\mathcal{S}' \subset \mathcal{S}} (m-1)^{|\mathcal{S}'|} \left(|\{\mathbf{s} : \mathcal{S}' + \mathbf{s} \subset \mathcal{S}\}| - 1 \right)$$

redundant shifted copies. This gives exactly $\mathrm{rk}(\mathbf{C})$ as in (16). Finally, if we count only once all elements of $\texttt{Conf}^0(\mathcal{D})$ differing by a shift, hence their number is exactly $m^{|\mathcal{S}|} - \mathrm{rk}(\mathbf{C}) = \mu + 1$ as desired. This is illustrated in Figure 7 for $m = 2$.

It remains to show that this set of $\mu + 1$ frequency vectors is linearly independent. This is in essence already shown in the proof of the Theorem 3: we exhibited a collection of $\mu + 1$ covectors such that the pairing matrix of these covectors with the type vectors we constructed is upper triangular, in the ordering constructed there. The result follows immediately. ■

To continue our construction, we now concatenate just constructed tilings of size $2\mathbf{w} - 1$ designing a set of tilings growing as desired N^μ. Generally, such concatenation can lead to some new tiles near the boundary, but this problem disappears if concatenated tilings are identical on the *envelope* defined as

$$E = I_{n_1} \times .. \times I_{n_d} \setminus I_{n_1 - w_1 + 1} \times .. \times I_{n_d - w_d + 1}$$

which we assume to hold. To complete the proof, we need a small lemma.

Lemma 10. *If $\mathbf{x}^1$, $\mathbf{x}^2$ tilings of size $\mathbf{n}$ and frequency vectors $\hat{\mathbf{T}}^1$, $\hat{\mathbf{T}}^2$ are identical on the envelope E, that is, $\mathbf{x}^1|_E = \mathbf{x}^2|_E$, then the frequency vector of a tiling constructed by concatenating them is $\hat{\mathbf{T}}^{12} = (\hat{\mathbf{T}}^1 + \hat{\mathbf{T}}^2)/2$.*

Proof: Observe that the resulting tile appearing in all positions from both original tilings. But the size of the torus is twice as big leading to the average frequency vector $\hat{\mathbf{T}}^{12} = (\hat{\mathbf{T}}^1 + \hat{\mathbf{T}}^2)/2$. This is illustrated in Figure 8. ■

We are now in the position to complete the **proof of Theorem 7**. For $n_i \geq 2w_i - 1$ size torus, construct a family of periodic tilings for which the set of frequency vectors grows like N^μ. Let us now take $\mu + 1$ such tilings with linearly independent size $2\mathbf{w} - 1$ frequency vectors $\hat{\mathbf{T}}^1, .., \hat{\mathbf{T}}^{\mu+1}$ discussed in Lemma 9. We can concatenate them into larger tori and then the resulting frequency vector corresponds to a convex combination. Thus the resulting frequency vectors are

$$\left\{ \frac{a_1 \hat{\mathbf{T}}^1 + .. + a_{\mu+1} \hat{\mathbf{T}}^{\mu+1}}{a_1 + \cdots + a_{\mu+1}} : \forall_i \ a_i \in \mathbb{N}, \ \sum_i a_i = \frac{N}{N'} \right\}$$

where $N' = \prod_i (2w_i - 1)$ and a_i is the number of tiles with the frequency vector $\hat{\mathbf{T}}_i$. Observe now that the size of this discrete simplex is determined by the number of integer solutions of

$$\sum_i a_i = \frac{N}{N'}$$

which is

$$\binom{N/N' + \mu - 1}{\mu} = O(N^\mu).$$

This implies the existence of a lower bound when n_i are integer multiplicities of $2w_i - 1$. In the general case we can increase the size of single layers of such concatenated tori (partially filled with m) as in the construction of Lemma 9. This completes the construction of a lower bound, and the proof of Theorem 7.

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