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WEAK FORMS OF SHADOWING IN TOPOLOGICAL DYNAMICS* † ‡

Danila Cherkashin — Sergey Kryzhevich

Abstract. We consider continuous maps of compact metric spaces. It is proved that every pseudotrajectory with sufficiently small errors contains a subsequence of positive density that is point-wise close to a subsequence of an exact trajectory with same indices. Later, we study homeomorphisms such that any pseudotrajectory can be shadowed by a finite number of exact orbits. In terms of numerical methods this property (we call it multishadowing) implies possibility to calculate minimal points of the dynamical system. We prove that for the non-wandering case multishadowing is equivalent to density of minimal points. Moreover, it is equivalent to existence of a family of ε-networks ($\varepsilon > 0$) whose iterations are also ε-networks. Relations between multishadowing and some ergodic and topological properties of dynamical systems are discussed.

1. Introduction

Shadowing is a very important property of dynamical systems, closely related to problems of structural stability and modelling. For review on general Shadowing Theory we refer to [27,35–37].

Though the most evident application of shadowing is related to numerical methods, first results involving the concept of pseudotrajectories were obtained by Anosov [2], Bowen [11] and Conley [13] as a tool to study qualitative properties of dynamical systems.

In a nutshell, shadowing is existence of an exact trajectory point-wise near a given pseudotrajectory i.e. a trajectory with errors. This property is closely related to structural stability. Indeed, it is well-known that structural stability implies shadowing [44,48]. Such shadowing is Lipschitz [38].

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Sakai [46] demonstrated that the C^1-interior of the set of all diffeomorphisms with shadowing coincides with the set of all structurally stable diffeomorphisms. Osipov, Pilyugin and Tikhomirov [34,38] demonstrated that the so-called Lipschitz periodic shadowing property is equivalent to Ω-stability, see also [36]. Moreover, the corresponding set of dynamical systems coincides with the interior of the set of systems with periodic shadowing property and with the set of systems with orbital limit shadowing property.

Pilyugin and Tikhomirov [42] demonstrated that Lipschitz shadowing is equivalent to structural stability.

Shadowing is not C^1-generic. Bonatti, Diaz and Turcat [10] demonstrated that there is a C^1-open set of diffeomorphisms of the 3-torus where none of diffeomorphisms satisfies shadowing property. Yuan and Yorke [51] proved a similar result for C^r-diffeomorphisms $(r > 1)$.

Surprisingly, shadowing is generic in the C^0-topology of homeomorphisms of a smooth manifold. This was proved by Pliyugin and Plamenevskaya [39]. Similar results were obtained for continuous mappings of manifolds [25,30] and for continuous maps of Cantor set [5].

This fact inspires studying shadowing by means of topological dynamics. This approach gave many important results mostly obtained in last two decades.

Mai and Ye [28] demonstrated that odometers have shadowing. This is the only example of such type infinite minimal systems. Of course, there are many non-minimal infinite systems with shadowing e.g. Bernoulli shift.

On the other hand, Moothathu [31] proved that minimal points are dense for every non-wandering system with shadowing. Moothathu and Oprocha [32] demonstrated that non-wandering systems with shadowing have a dense set of regularly recurrent points.

Dastjerdi and Hosseini [14,15] studied "almost identical" mappings. They proved that if a chain transitive dynamical system has an equicontinuity point then it is a distal, equicontinuous and minimal homeomorphism (see also [18,20]). Thus any transitive system with shadowing is either sensitive or equicontinuous.

Another version of shadowing (the so-called average shadowing) was introduced by Blank [8]. The so-called ergodic shadowing was studied in [15]. Some other kinds of shadowing ($\underline{d}$-shadowing, weak shadowing, etc.) were discussed in [15,46] and [47], see also references therein.

However, the problem of shadowing in non-smooth dynamical systems is very far from being resolved. Theoretical results in this area may be applied for modelling non-smooth dynamics like vibro-impact systems [4,22], systems with dry friction [6,17], biological problems [1] and many other problems [29].

In this paper we demonstrate that for a very general dynamical system, any numerical method, even an inappropriate one, can give some useful information on asymptotical behavior of solutions. First of all, it can be used to find an invariant measure (Theorem 3.1). If we take a random point of a pseudotrajectory, obtained by this "incorrect" numerical method, the probability to find a minimal point in a neighborhood of the selected point (Theorem 3.1, Corollary 4.6) is positive. In some generic assumptions (see Theorem 3.3) it is equal to 1. We show that for any dynamical system and any pseudotrajectory there is a subsequence that can be shadowed by a subsequence of a precise trajectory with same indices. This is the first key result of our paper.

Then it is natural to ask, if any pseudotrajectory can be traced by a finite number of trajectories. This is the so-called multishadowing (Definition 2.20). We demonstrate that this property is C^1-generic. We study a generalisation of equicontinuous systems i.e. systems with almost invariant ε-networks e.g. ones whose iterations are all ε-networks. The second central statement of our research is Theorem 3.3. We prove that for a nonwandering system multishadowing is equivalent to existence of almost invariant ε-networks for any $\varepsilon > 0$. Moreover, both these properties are equivalent to the so-called Bronstein condition [12] i.e. density of minimal points in the set of nonwandering points (Definition 2.9).

Usually, applying numerical methods, one takes initial conditions, applies a number of iterations and claim there is a minimal point in a neighborhood of the last iteration. We demonstrate that this is correct if and only if the considered diffeomorphism satisfies multishadowing property (Theorem 13.1). This is our principal motivation to study this property.

The paper is organised as follows. First of all, we recall the terminology, related to Shadowing Theory and Topological Dynamics (Section 2). In Section 3 we list principal results of the paper. We improve the main result of [26] in Section 4. It is proved that for any continuous mapping of a compact metric space into itself and for any one-sided pseudotrajectory x_k there exists a sequence k_n and a precise trajectory $\{y_k = T^k(y_0)\}$ such that points x_{k_n} and y_{k_n} are uniformly close. The density of $\{k_n\}$ in $\mathbb{N}$ is positive (Theorem 3.1). In Sections 5 and 6 we study nonwandering systems. We prove that multishadowing is equivalent to Bronstein condition. In Section 7 we prove that multishadowing is equivalent to existence of almost invariant ε-networks for all $\varepsilon > 0$. Moreover, for nonwandering homeomorphisms, multishadowing implies existence of an invariant measure, supported on all the phase space (Section 8). In Section 9 we prove that if every chain recurrent point is nonwandering and Bronstein condition holds on the nonwandering set, the considered system satisfies multishadowing property. The converse statement is proved in Section 10. In Section 11 we study networks that are almost invariant almost everywhere with respect to an invariant measure. In Section 12 we demonstrate that multishadowing is C^0 and C^1-generic. In Sections 14 we discuss possible applications of the main results of the paper. Conclusion is given in Section 15.

In this paper, we consider three types of dynamical systems: continuous maps that can be non-invertible, homeomorphisms (both of metric compact spaces) and diffeomorphisms of compact Riemannian manifolds. In order to avoid confusion, we make the following agreement. In Theorem 3.1, all Section 4 and Corollary 14.1 we consider continuous maps. In Theorem 3.3 and all related results – Lemmas 3.5–3.10, Sections 5–11 and 13, Corollaries 14.2 and 14.4 we study homeomorphisms of metric compact sets. In Section 12 and Corollary 14.5 we discuss properties of diffeomorphisms.

2. Definitions

Recall some standard definitions from Topological Dynamics. Consider a compact metric space X endowed with the metric ρ. Let a map $T : X \to X$ be continuous. The pair (X, T) is *dynamical system*.

DEFINITION 2.1. Let $d > 0$. A sequence $\{x_k\}_{k\in\mathbb{N}}$ is *d-pseudotrajectory* if

$$\rho(x_{k+1}, T(x_k)) \leq d$$

for all $k \in \mathbb{N}$.

DEFINITION 2.2. We say that the mapping T satisfies *shadowing property* if for any $\varepsilon > 0$ there is a $d > 0$ such that for any d-pseudotrajectory $\{x_k\}$ there exists an exact trajectory $\{y_k = T^k(y_0), k \in \mathbb{N}\}$ such that $\rho(x_k, y_k) < \varepsilon$ for all $k \in \mathbb{N}$.

Also, we say that shadowing property is satisfied on a subset $Y \subset X$ if it is true for the dynamical system $(Y, T|_Y)$.

If $T : X \to X$ is homeomorphism, we may consider "two-sided" pseudotrajectories $\{x_k\}_{k\in\mathbb{Z}}$ and study "two-sided shadowing", defined similarly to Definition 2.2. Abusing notations, we say "pseudotrajectory" and "shadowing" in both cases. If it is necessary we add words "one-sided" or "two-sided" in order to underline which kind of dynamical systems we deal with.

DEFINITION 2.3. A point $x \in X$ is *wandering* if there exists a neighborhood $U \ni x$ such that $T^k(U) \bigcap U = \emptyset$ for all $k \in \mathbb{N}$.

DEFINITION 2.4. *Non-wandering* points form the *non-wandering set* $\Omega(X, T)$. Let NW be the class of non-wandering systems ($X = \Omega(X, T)$).

DEFINITION 2.5. A point $y \in X$ is ω-limit point for $x \in X$ i.e. $y \in \omega(x)$ if there exists a sequence $n_k \to +\infty$ such that $T^{n_k}(x) \to y$ $(k \to \infty)$. Let $\omega(X, T)$ be the closure of all ω-limit points for all points of X.

Recall some classic notations. Define the positive semiorbit of a point x by formula $O^+(x) = \{T^k(x) : k \geq 0\}$. For homeomorphisms, we consider orbits: $O(x) = \{T^k(x) : k \in \mathbb{Z}\}$.

DEFINITION 2.6. The dynamical system (X, T) is called *minimal*, if $\overline{O^+(x)} = X$ for every $x \in X$.
DEFINITION 2.7. A point $y \in X$ is called *minimal* (or almost periodic) for dynamical system (X, T), if the subsystem $(\overline{O^+(y)}, T)$ is minimal. Let $\mathrm{M}(X, T)$ be the set of all minimal points of (X, T).

We mention a classical result [50,Theorem 1.2.7.], that demonstrates existence of minimal ponts for all dynamical systems.
THEOREM 2.8. *Let X be a metric compact set, $T : X \to X$ is a continuous map. Then $\mathrm{M}(X, T) \neq \emptyset$.*

The idea of the proof is quite simple: we consider all nonempty closed subsets of X, ordered by inclusion and apply Zorn's lemma to find a minimal subsystem.
DEFINITION 2.9. If the set of minimal points is dense in X we say that (X, T) satisfies the *Bronstein condition.*

Let us also recall a definition from Combinatorics and Number Theory.
DEFINITION 2.10. A subset $S \subset \mathbb{N}$ is called *syndetic* if there exists $n = n(S) \in \mathbb{N}$ such that for any $m \in \mathbb{N}$ the intersection $S \bigcap [m, m + n]$ is non-empty. We also use notion of n-syndetic set if we need to specify the value n.

We recall a well-known fact from the theory of minimal sets [18,24].
LEMMA 2.11. *Let $T : X \to X$ be a continuous map. System (X, T) is minimal if and only if the set*

$$N(x, U) = \{m \in \mathbb{N} : T^m(x) \in U\} \tag{2.1}$$

is syndetic for every $x \in X$ and nonempty open set U such that $x \in U \subset X$.

Starting from here we assume up to the end of the section that $T : X \to X$ is a homeomorphism.
DEFINITION 2.12. We say that a point $z \in X$ is an α-limit point for a point $x \in X$ if there exists an integer sequence $n_k \to \infty$ such that $T^{-n_k}(x) \to z$ $(k \to \infty)$. Let $\alpha(X, T)$ be the closure of all α-limit points for all points of X.
DEFINITION 2.13. Let T be a homeomorphism. A point $x \in X$ is *recurrent* if $x \in \alpha(x) \bigcap \omega(x)$. Let $\mathrm{R}(X, T)$ be the set of all recurrent points of system (X, T).
REMARK 2.14. Here we use notions from [23]. However, sometimes recurrent points are called *Poisson stable in both directions* and minimal points are called recurrent [33].
DEFINITION 2.15. The *chain recurrent set* $\mathrm{CR}(X, T)$ is the set of points $x \in X$ such that for any $d > 0$ there exists a finite d-pseudotrajectory $x = x_1, x_2, \ldots, x_k = x$, $k > 1$.

We recall a well-known result from Topological Dynamics.
LEMMA 2.16. *Let $T : X \to X$ be a homeomorphism,* $\mathrm{Per}\,(X, T)$ *be the set of all periodic points of T. Then*

1. *sets* $\Omega(X,T)$ *and* $\mathrm{CR}(X,T)$ *are closed;*

2. $\mathrm{Per}(X,T) \subset \mathrm{M}(X,T) \subset \mathrm{R}(X,T) \subset \alpha(X,T) \bigcup \omega(X,T) \subset \Omega(X,T) \subset \mathrm{CR}(X,T)$*;*

3. *[23, Proposition 4.1.18] if* μ *is a Borel probability invariant measure for* (X,T) *then* $\mathrm{supp}\,\mu \subset \overline{\mathrm{R}(X,T)}$.

Here we recall that the support $\mathrm{supp}\,\mu$ of a Borel measure μ is the intersection of all closed subsets $Y \subset X$ such that $\mu(Y) = 1$.

DEFINITION 2.17. A subset $Y \subset X$ is an ε-network in X if for any $x \in X$ there exists a $y \in Y$ such that $\rho(x,y) \leq \varepsilon$.

DEFINITION 2.18. An ε-network Y is *almost invariant* if for every $n \in \mathbb{Z}$ the set $T^n(Y)$ is an ε-network. (Fig. 1).

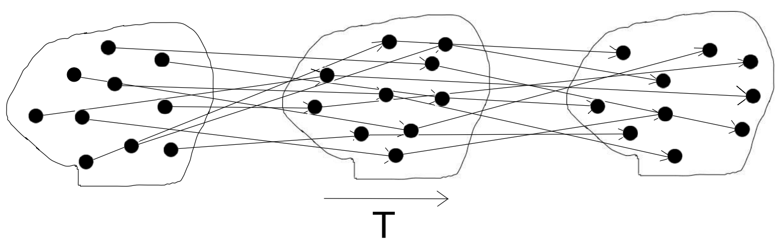

Figure 1. Almost invariant networks.

Denote by Q the class of systems (X,T) (T is homeomorphism) that have finite almost invariant ε-networks for every $\varepsilon > 0$.

LEMMA 2.19. Q $\subset$ NW.

PROOF. If $(X,T) \in$ Q any neighborhood of any point of X contains an ω-limit point, corresponding to a limit point of one of points of an almost invariant network. $\square$

DEFINITION 2.20 (Fig. 2). We say that dynamical system (X,T) satisfies *multishadowing* property if for any $\varepsilon > 0$ there exists a $d = d(\varepsilon) > 0$ as follows: for any d-pseudotrajectory $\{x_k\}$ there exist points $y^1, \ldots, y^N$ ($N = N(\{x_k\}, \varepsilon)$ may depend on $\{x_k\}$ and ε) such that

$$\min_{i=1,\ldots,N} \rho(x_k, T^k(y^i)) < \varepsilon \qquad \text{for all} \quad k \in \mathbb{N}. \tag{2.2}$$

Let W be the class of all systems (X,T) that satisfy the multishadowing property.

The corresponding maximal number of shadowing trajectories $N(\{x_k\}, \varepsilon)$ is called *multishadowing parameter*. Later on, we demonstrate (Corollary 9.3) that for given system (X,T) and $\varepsilon > 0$ the number $N(\{x_k\}, \varepsilon)$ may be selected the same for all $d(\varepsilon)$-pseudotrajectories $\{x_k\}$.

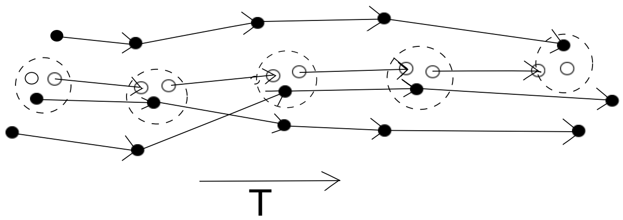

Figure 2. Multishadowing.

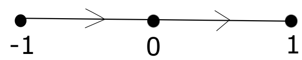

Figure 3. A system with multishadowing and without shadowing.

Of course, shadowing implies multishadowing. The converse statement is not true. For instance, $(X, \mathrm{id}) \in \mathrm{W}$ for any compact metric space X. Another counterexample, one may keep in mind, is a discretisation of the o.d.e. $\dot{x} = x^2 - x^4$, defined on the segment $[-1, 1]$ (Fig. 3). In this case for any $d > 0$ there exists a finite d-pseudotrajectory (d-chain), linking points -1 and 1. On the other hand, exact trajectories that start at $[-1, 0)$ cannot pass through 0. If the space X is not totally disconnected, there exist dynamical systems with no shadowing that belong to the class W (see Lemma 4.3 below).

Definition 2.21. We say that system (X, T) is *equicontinuous* if the family of maps $T^k : X \to X$, $k \in \mathbb{Z}$ is equicontinuous. This means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(x, y) < \delta$ implies $\rho(T^n x, T^n y) < \varepsilon$ for all $k \in \mathbb{Z}$.

Remark 2.22. The class Q is a natural generalisation of equicontinuous systems. Evidently, all equicontinuous systems belong to Q. Meanwhile, the introduced class is much reacher, it includes some expansive systems e.g. dynamics on non-wandering sets for Axiom A diffeomorphisms [49].

Let $x \in X$, $\varepsilon > 0$. Define the ε-ball, centered at x by formula $B_\varepsilon(x) = \{y \in X : \rho(x, y) < \varepsilon\}$. For a subset $Y \subset X$ introduce ε-neighborhood of Y:

$$U_\varepsilon(Y) = \{x \in X : \inf_{y \in Y} \rho(x, y) < \varepsilon\}.$$

Definition 2.23. Let T be a homeomorphism of a compact metric space X, μ be a Borel probability invariant measure on X. We say that a finite set A is an ε-network, *almost invariant with respect to* μ if $\mu(U_\varepsilon(T^n(A))) > 1 - \varepsilon$ for any $n \in \mathbb{Z}$.

3. Main results

THEOREM 3.1. *Let $T : X \to X$ be a continuous map of a compact metric space X. For any $\varepsilon > 0$ there exists a $d > 0$ such that for any one-sided d-pseudotrajectory $\{x_k, k \geq 0\}$ there exists a subsequence $K := \{k_n, n \in \mathbb{N}\} \subset \mathbb{N}$ and a point $y \in \mathrm{M}(X, T)$ such that $\rho(x_{k_n}, T^{k_n}(y)) < \varepsilon$. The sequence k_n may be taken so that*

$$a := \limsup_{N \to \infty} \frac{\# K \bigcap [0, N]}{N} > 0. \tag{3.1}$$

If Eq. (3.1) is satisfied, we say that the set K has *positive density* in $\mathbb{Z}^+$.

This result is proved and discussed in Section 4. In fact we do not prove that for a given pseudotrajectory there is a trajectory that traces it. We just prove that both the pseudotrajectory and the "shadowing" trajectory return to a neighborhood of the same point along the same sequence of instants of time.

REMARK 3.2. A result very similar to Theorem 3.1 was proved by one of co-authors in [26]. However, the statement of Theorem 3.1 is stronger. In [26] it was not proved that the sequence $\{k_n\}$ can be chosen so that (3.1) is satisfied. In other words, we prove that the sequence $\{k_n\}$ does not grow too fast, that may be important for applications. In order to obtain inequality (3.1) we have to modify the proof (see Section 4).

Let Br be the class of all systems, corresponding to homeomorphisms of X that satisfy Bronstein condition (see Definition 2.9). Recall that W is the class of dynamical systems with the multishadowing property and Q is the class of systems that have almost invariant ε-networks for all $\varepsilon > 0$.

THEOREM 3.3.

1. $\mathrm{Q} = \mathrm{Br} = \mathrm{W} \bigcap \mathrm{NW}$.
2. *For any homeomorphism from the class* Q *there exists a probability invariant measure, supported on all X.*
3. $(X, T) \in \mathrm{W}$ *if and only if*

$$\mathrm{CR}(X, T) = \overline{\mathrm{M}(X, T)}. \tag{3.2}$$

REMARK 3.4. It is more convenient for us to deal with the following conditions, both equivalent to (3.2):

1. Chain recurrent set coincides with the non-wandering set i.e.

$$\mathrm{CR}(X, T) = \Omega(X, T); \tag{3.3}$$

2. Bronstein condition holds for system $(\Omega(X,T),T)$.

We split the statement of Theorem 3.3 to several lemmas.

LEMMA 3.5. *Systems (X,T) that satisfy Bronstein condition belong to the class* Q.

LEMMA 3.6. *Let K be a compact invariant set for system (X,T). Assume that for any $\varepsilon > 0$ there exists a finite set $A_\varepsilon \subset X$ such that $K \subset U_\varepsilon(T^k(A_\varepsilon))$ for any $k \in \mathbb{Z}$. Then $K \subset \overline{\mathrm{M}(X,T)}$.*

Observe that here we do not assume that $A_\varepsilon \subset K$. Taking $K = X$, we obtain $\mathrm{Q} \subset \mathrm{Br} \bigcap \mathrm{NW}$.

LEMMA 3.7. $\mathrm{Q} \subset \mathrm{W}$*;* $(X,T) \in \mathrm{W}$ *implies* $(\mathrm{CR}(X,T),T) \in \mathrm{Q}$.

Particularly, $\mathrm{W} \bigcap \mathrm{NW} \subset \mathrm{Q}$. So, the first part of Theorem 3.3 follows from Lemmas 3.5–3.7.

LEMMA 3.8. *If $(X,T) \in \mathrm{Q}$, there exists a Borel probability invariant measure, supported on all X.*

By virtue of [23, Theorems 4.1 and 7.1] existence of such an invariant measure implies that $X = \overline{\mathrm{R}(X,T)}$.

LEMMA 3.9. *Let (3.2) take place. Then system (X,T) has multishadowing property.*

LEMMA 3.10. $(X,T) \in \mathrm{W}$ *implies Eq.* (3.3).

Statements of Lemmas 3.9 and 3.10 imply the third item of Theorem 3.3.

Finally, we formulate an ”ergodic” version of Lemmas 3.5 and 3.6.

THEOREM 3.11. *Let T be a homeomorphism of a compact metric space X, μ be a Borel probability invariant measure on X. Then the following statements hold.*

1. *If for any $\delta > 0$ there exists a finite δ-network A_δ, almost invariant with respect to μ, then* $\operatorname{supp}\mu \subset \overline{\mathrm{M}(X,T)}$.

2. *If* $\operatorname{supp}\mu \subset \overline{\mathrm{M}(X,T)}$ *we can take an almost invariant ε-network $A_\varepsilon \subset \operatorname{supp}\mu$ for any $\varepsilon > 0$.*

REMARK 3.12. Density of minimal points for nonwandering systems with shadowing was proved by Moothathu [31,Theorem 1]. Theorem 3.3 demonstrates that so it is for nonwandering systems with multishadowing.

REMARK 3.13. It follows from Lemma 3.7 that ”regular” shadowing (Definition 2.2) implies (3.3).

REMARK 3.14. Third statement of Theorem 3.3 implies that for any $(X,T) \in \mathrm{W}$

$$\mathrm{CR}(X,T) = \mathrm{CR}(\mathrm{CR}(X,T), T|_{\mathrm{CR}(X,T)}).$$

4. Partial shadowing. Proof of Theorem 3.1

First, we prove an auxiliary statement.

LEMMA 4.1. *For any $\varepsilon > 0$, any positive sequence $\delta_m \to 0$ $(m \to \infty)$, and any sequence $\{p_k^m\}$ of δ_m-pseudotrajectories there exists a point $\bar{x} \in \mathrm{M}(X, T)$ such that sets $S_m = \{k : p_k^m \in B_{\varepsilon/2}(\bar{x})\}$ where m is sufficiently big have positive densities in $\mathbb{Z}^+$.*

PROOF OF LEMMA 4.1. We use some ideas of the proof of the Krylov-Bogolyubov Theorem [23, Theorem 4.1.1]. Fix corresponding sequences δ_m and p_k^m. Let $C^0(X \to \mathbb{R})$ be the space of all continuous functions on X with the norm

$$\|\varphi\| = \sup_{x \in X} |\varphi(x)|.$$

Since X is compact, the space $C^0(X \to \mathbb{R})$ is separable [43, Section III.3]. Take $\Phi = \{\varphi_k : k \in \mathbb{N}\}$ be a countable sets of continuous functions on X, dense in $C^0(X \to \mathbb{R})$. Using diagonal sequence method, we obtain an integer sequence $s_j \to \infty$ $(j \to \infty)$ such that for any function $\varphi \in \Phi$ there exists a limit

$$J_m(\varphi) := \lim_{j \to \infty} \frac{1}{s_j} \sum_{i=0}^{s_j - 1} \varphi(p_i^m). \tag{4.1}$$

Moreover, we can take the diagonal sequence so that the set $\{s_j\}$ is the same for all m.

Let us demonstrate that functionals J_m can be continuously extended to $C^0(X \to \mathbb{R})$. Indeed, let $\psi \in C^0(X \to \mathbb{R})$ and $\varepsilon > 0$. Take a function $\varphi \in \Phi$ so that $\|\psi - \varphi\|_{C^0} \le \varepsilon$. Then, for any $j \in \mathbb{N}$ we have

$$\left| \frac{1}{s_j} \sum_{i=0}^{s_j - 1} (\varphi(p_i^m) - \psi(p_i^m)) \right| \le \varepsilon.$$

This demonstrates that the value $J_m(\psi)$ is correctly defined by the formula, similar to (4.1). Moreover, $|J_m(\psi)| \le \|\psi\|_{C^0}$. So, all functionals $J_m : C^0(X \to \mathbb{R})$ are linear, continuous and positive. By virtue of Riesz Representation Theorem [21], they uniquely define probability measures μ_m on X according to the formula

$$J_m(\varphi) = \int_X \varphi \, d\mu_m \qquad \text{for all} \quad \varphi \in C^0(X \to \mathbb{R}). \tag{4.2}$$

By virtue of Banach-Alaoglu Theorem, the set of all Borel probability measures is compact in the $*$-weak topology. Without loss of generality, we can suppose that the considered sequence $*$-weakly converges to a Borel probability measure μ_*. Let us demonstrate that μ_* is an invariant measure. Fix a $\varphi \in C^0(X \to \mathbb{R})$, then

$$((\varphi \circ T) - \varphi)\, d\mu_* = \lim_{m\to\infty} (J_m(\varphi \circ T) - J_m(\varphi)) =$$

(4.3)
$$\lim_{m\to\infty} \lim_{j\to\infty} \frac{1}{s_j} \left(\sum_{i=1}^{s_j-1} (\varphi(T(p^m_{i-1})) - \varphi(p^m_i)) + \varphi(T(p^m_{s_j-1})) - \varphi(p^m_0) \right) = 0.$$

Indeed, given a function $\varphi \in C^0(X \to \mathbb{R})$ and a value $\sigma > 0$ we may find $m_0 \in \mathbb{N}$ such that $m > m_0$ implies $|\varphi(x) - \varphi(y)| \le \sigma/2$ for all x, y such that $\rho(x, y) \le \delta_m$. Select $j_0 \in \mathbb{N}$ so big that $2\|\varphi\|/s_j < \sigma/2$ for any $j > j_0$. Then the absolute value of the expression in the second line of Eq. (4.3) does not exceed σ. Since σ can be taken arbitrarily small, Eq. (4.3) is satisfied.

Take a point $\bar{x} \in \operatorname{supp} \mu_*$. By definition, $\mu_*(B) \neq 0$, where $B = B_{\varepsilon/2}(\bar{x})$ is an $\varepsilon/2$-ball, centered at $\bar{x}$. The set $\operatorname{supp} \mu_*$ is closed and invariant. By Theorem 2.8, it contains a minimal subset. Hence we may assume that $\bar{x} \in \mathrm{M}(X, T)$.

Since $\mu_*(B) > 0$, there exists an $m_0 > 0$ such that $J_m(\chi_B) = \mu_m(B) > 0$ for all $m > m_0$. Here χ_B is the characteristic function for the set B. By definition of J_m we see that the corresponding set S_m has a positive density in $\mathbb{Z}^+$. Lemma 4.1 is proved. □

Now we suppose that the statement of Theorem 3.1 is wrong. Then there exist a constant $\varepsilon > 0$, a positive sequence $\delta_m \to 0$ $(m \to \infty)$ and a sequence p^m_k of δ_m-pseudotrajectories such that for any $m \in \mathbb{N}$, any point $y \in \mathrm{M}(X, T)$ and any sequence $\{k_n\} \subset \mathbb{N}$ satisfying (3.1) there exists $l \in \mathbb{N}$ such that $\rho(p^m_{k_l}, T^{k_l}(y)) \ge \varepsilon$.

Take the point $\bar{x}$ and the ball $B = B_{\varepsilon/2}(\bar{x})$ that exist for this by Lemma 4.1. By $*$-week converrgence of measures μ_m there exists $m > 0$ such that $\mu_m(B) > 0$. Let the increasing sequence $\mathcal{I}_m = \{i_j\}$ be such that $p^m_{i_j} \in B$ for all $j \in \mathbb{N}$. By definition of μ_m we may select $\bar{x}$ so that

(4.4)
$$N(\bar{x}, B) = \limsup_{n\to\infty} \frac{\#(\mathcal{I}_m \bigcap [0, n])}{n} > 0.$$

The set $\{k : T^k(\bar{x}) \in B\}$ is syndetic (see Lemma 2.11). So, there exists $P > 0$ such that for any $k \in \mathbb{N}$ there exists an $s \in \{0, \dots, P\}$ such that $T^k(y_s) \in B$. Here $y_j = T^j(\bar{x})$, $j = 0, \dots, P$. Let $K_s = \{k^s_n\} \subset \mathcal{I}_m$ be sets such that $T^{k^s_n}(y_s) \in B$, $s = 0, \dots, P$. Evidently,

$$\mathcal{I}_m = \bigcup_{s=0}^{P} K_s$$

and, by virtue of (4.4) at least one of values

$$a_r = \limsup_{n\to\infty} \frac{\#(K_r \bigcap [0, n])}{n}$$

is positive. Then we take $y = y_r$.

To finish the proof, it suffices to observe that $p_k^m, T^k(y) \in B$ implies $\rho(p_k^m, T^k(y)) < \varepsilon$. This gives a contradiction to our assumptions on pseudotrajectories p_k^m. □

REMARK 4.2. For our proof it is crucial that the space X is compact. There is a simple counterexample to the "non-compact" version of the theorem: $X = \mathbb{R}$, $T = \mathrm{id}$, $x_k = dk$, d is a small parameter.

LEMMA 4.3. *Let X be a compact infinite metric space that is not totally disconnected, $T : X \to X$ be an invertible equicontinuous map. Then there exists an $\varepsilon_0 > 0$ such that for any $d > 0$ there exists a double-sided d-pseudotrajectory x_k where none of its double-sided subsequences x_{k_n}, $k_n \to \pm\infty$ as $n \to \pm\infty$ could be ε_0 shadowed by the subsequence y_{n_k} of a trajectory $\{y_k = T^k(y_0) : k \in \mathbb{Z}\}$.*

PROOF. Fix a point $y \in X$ whose connected component Y is not a singleton. Take $z \in Y$, $z \neq y$. Fix $\sigma > 0$ so small that $\rho(y, z) > 2\sigma$.

PROPOSITION 4.4. *For any $\kappa > 0$ there exist $N \in \mathbb{N}$ and a finite sequence $\{x_k\}$, $k = 0, \ldots, N$ such that $x_0 = y$, $x_N = z$ and*

$$\rho(x_{k-1}, x_k) < \kappa \qquad \textit{for all} \quad k = 1, \ldots, N. \tag{4.5}$$

PROOF OF PROPOSITION 4.4. Fix a $\kappa > 0$. Let V_κ be the set of all points of Y that can be linked with y by a finite chain $\{x_k\}$, satisfying (4.5). The set U_κ is open in Y and non-empty since it contains y. On the other hand, the completion $W_\kappa := Y \setminus V_\kappa$ is also open (if $\zeta \in W_\kappa$ then $B_\kappa(\zeta) \subset W_\kappa$). Since Y is connected, $W_k = \emptyset$ and $z \in V_\kappa$. Proposition is proved. □

Take $\varepsilon > 0$ so that $\rho(x, y) < \varepsilon$ implies $\rho(T^n(x), T^n(y)) < \sigma$ for all $n \in \mathbb{Z}$. Fix a $d \in (0, \varepsilon)$. Take $\kappa > 0$ so that $\rho(x, y) < \kappa$ implies $\rho(T^n(x), T^n(y)) < \delta$, $n \in \mathbb{Z}$. Then $\kappa < d < \varepsilon \leq \sigma$. For this κ we take a sequence $\{x_k\}$, $k = 0, \ldots, N$ that exists by Proposition 4.4. Now we define a sequence $\{p_k\}$ by formulae:

$$p_k = \left[\begin{array}{ll} T^k(y) & \text{if} \quad k \leq 0, \\ T^k(x_k) & \text{if} \quad 0 < k < N, \\ T^k(z) & \text{if} \quad k \geq N. \end{array} \right.$$

Observe that $\rho(T(p_k), p_{k+1}) = \rho(T^{k+1}(p_k), T^{k+1}(p_{k+1})) \leq d$ for all $k = 0, \ldots, N-1$. Hence $\{p_k\}$ is a d-pseudotrajectory. If there existed a trajectory $\{q_k = T^k(q_0)\}$ such that $\rho(p_k, q_k) \leq \varepsilon$ for any $k \in \mathbb{Z}$, we would have

$$\rho(y, z) \leq \rho(y, q_0) + \rho(q_0, z) = \rho(p_0, q_0) + \rho(T^{-N}(q_N), T^{-N}(p_N)) \leq \varepsilon + \sigma \leq 2\sigma. \tag{4.6}$$

Here we recall that $y = p_0$, $z = T^{-N}(p_N)$, that $\rho(p_0, q_0) \leq \varepsilon$, and that $\rho(p_N, q_N) < \varepsilon$ implies $\rho(T^{-N}(q_N), T^{-N}(p_N)) < \sigma$. Inequality (4.6) contradicts to (4.4) $\Box$.

As an example, one can consider identical mapping or a rotation of the circle.

Example 4.5. We give an example of a homeomorphism that does not belong to the class W. Take the unit circle endowed with the angular coordinate φ with the flow defined by ODE $\dot{\varphi} = \sin^2 \varphi$ (Fig. 4). Let T be a discretisation of the considered flow. Map T has exactly two fixed points: the west end of the circle $O_w = \{\varphi = \pi\}$ and the east one $O_e = \{\varphi = 0\}$. Trajectories of T that do not coincide with one of those points, entirely appertain to the "northern" or to the "southern" semicircle. In spite of this, pseudotrajectories can "jump" through fixed points and, consequently, rotate infinitely many times around the circle. This proves that $T \notin$ W. The same example illustrates that lim sup cannot be replaced by lim inf in (3.1). Indeed, for the considered system, pseudotrajectories may stay arbitrarily long in a neighborhood of one fixed point and then leave for another one. So, we can spend 10 steps in a neighborhood of O_w, then (after a fixed number of steps, necessary to proceed from O_w to O_e), we wait 10^{10} steps in O_e, then we go to O_w and spend there $10^{10^{10}}$ steps and so on. In this case, all corresponding lower limits are zero, whatever we select as a shadowing trajectory.

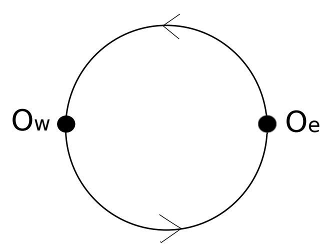

Figure 4. No multishadowing for a diffeomorphism of a circle.

Corollary 4.6 (to Theorem 3.1). *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any δ-pseudotrajectory $\Xi = \{x_k\}$ of the map T the set $\{k \in \mathbb{Z} : x_k \in U_\varepsilon(\overline{\mathrm{M}(X,T)})\}$ is syndetic. Recall that U_ε stands for ε-neighborhood of a set in the topology of X.*

Observe that this statement is very close to one proved by Pilyugin and Sakai [40,41]. The difference is that we take the set $\overline{\mathrm{M}(X,T)}$ instead of $\Omega(X,T)$.

5. Proof of Lemma 3.5

Recall that starting from here we always deal with homeomorphisms of compact metric spaces. Let us prove first of all, that all minimal systems have almost invariant networks.

Lemma 5.1. *Any minimal dynamical system (X, T) belongs to the class Q.*

Proof. Take a point $x \in X$. Due to minimality of (X,T) we have $\overline{O(x)} = X$. Fix $\varepsilon > 0$, cover X by a finite number of $\varepsilon/2$-balls $B_1, \ldots, B_K$ and take $n = \max_j n(N(x, B_j))$, see Eq. (2.1) and Lemma 2.11. Here $n(N(x, B_j))$ is the constant that exists by definition of syndetic sets (Definition 2.10) or, in other words, the maximal possible length of a chain

$$\{T^k(x), T^{k+1}(x), \ldots, T^{m-1}(x), T^m(x)\} \subset X \setminus B_j.$$

Then points x, $T(x)$, …, $T^n(x)$ form an almost invariant ε-network. □

Now we return to the proof of Lemma 3.5. Consider a finite set $\{b_i\}$ that is an $\varepsilon/2$-network in X. By the Bronstein condition, for every i we can find an invariant set A_i such that the system $(A_i, T|_{A_i})$ is minimal and $B_{\varepsilon/2}(b_i) \cap A_i \neq \emptyset$. For every i we select a finite almost invariant $\varepsilon/2$-network $N_i \subset A_i$ and take $N = \cup_i N_i$. Obviously N is an almost invariant finite ε-network in X. □

6. Proof of Lemma 3.6

Take an invariant compact subset $K \subset X$ that satisfies conditions of the lemma. Take a point $x_0 \in K$ and $\varepsilon > 0$. We demonstrate that the set $\overline{B_\varepsilon} = \overline{B_\varepsilon(x_0)}$ contains a minimal point. Let $B_{\varepsilon/2} = B_{\varepsilon/2}(x_0)$.

Lemma 6.1. *There exists a point $\xi \in B_\varepsilon$ such that the set $N(\xi, B_\varepsilon)$ is syndetic.*

Proof. Let $A = \{a_i : i = 1, \ldots, n\}$ be a finite subset of X or "vector". We say that A belongs to the class $\mathcal{H}_\varepsilon$ if for any $k \in \mathbb{Z}$ there exists $j = j(k)$ such that $T^k(a_j) \in \overline{B_{\varepsilon/2}}$.

Observe two evident properties of the class $\mathcal{H}_\varepsilon$.

1. $A \in \mathcal{H}_\varepsilon$ if and only if $T^k(A) \in \mathcal{H}_\varepsilon$ for any $k \in \mathbb{Z}$.

2. Let $A_k = \{a_i^k,\, i = 1, \ldots, n\} \in \mathcal{H}_\varepsilon$, $k \in \mathbb{N}$ be a "vector" of $X^n := X \times \ldots \times X$ (n times) converging to a "vector" A_*. Then $A_* \in \mathcal{H}_\varepsilon$.

We start with a set $A = \{a_1, \ldots, a_n\}$ such that $K \subset \bigcup_{i=1}^n B_{\varepsilon/2}(T^j(a_i))$ for any $j \in \mathbb{Z}$. Evidently, $A \in \mathcal{H}_\varepsilon$. Since $x_0 \in U_{\varepsilon/2}(A)$, we may assume that $a_1 \in B_{\varepsilon/2}$.

Suppose that the set $N(a_1, B_\varepsilon)$ is non-syndetic (otherwise, we set $\xi = a_1$). Then there exists an increasing sequence $q_m \in \mathbb{N}$ such that $T^{q_m+j}(a_1) \notin B_\varepsilon$ for all $j = 1, \ldots, m$. Without loss of generality we may suppose that the sequence $T^{q_m}(A)$ converges to a "vector" $A_* = \{a_j^*, j = 1, \ldots, n\} \in X^n$. Still $A_* \in \mathcal{H}_\varepsilon$. Observe that $T^m(a_1^*) \notin B_\varepsilon$ for any $m \in \mathbb{Z}$. Then the $n-1$ point set $A_1 = \{a_j^*, j = 2, \ldots, n\}$ belongs to the class $\mathcal{H}_\varepsilon$. Similarly, either the set $N(a_2^*, B_\varepsilon)$ is syndetic or there exists an $n-2$ point set $A_2 \in \mathcal{H}_\varepsilon$. Repeating this procedure, we must stop after n steps at most and thus obtain the desired point ξ. □

Fix the obtained point ξ. Let $m \in \mathbb{N}$ be such that the set $N(\xi, \overline{B_\varepsilon}) = \{n_k\}$ is m-syndetic. Let $\widetilde{\omega}_\xi$ be the set of all limit points for the sequence $T^{n_k}(\xi)$, ω_ξ be the ω-limit set for the trajectory $O(\xi)$. Let us prove that

$$\widetilde{\omega}_\xi \subset \overline{B_\varepsilon}, \qquad \omega_\xi = \widetilde{\omega}_\xi \bigcup T(\widetilde{\omega}_\xi) \bigcup \ldots \bigcup T^m(\widetilde{\omega}_\xi). \tag{6.1}$$

Indeed, $\widetilde{\omega}_\xi \subset \overline{B_\varepsilon}$ since $T^{n_k}(\xi) \in \overline{B_\varepsilon}$ that is true by definition of $N(\xi, \overline{B_\varepsilon})$. Now take a point $\chi \in \omega_\xi$. There exists a sequence p_l such that $T^{p_l}(\xi) \to \chi$ $(l \to \infty)$. Since the set $N(\xi, \overline{B}_\varepsilon) = \{n_k\}$ is m-syndetic, for any $l \in \mathbb{N}$ we can represent $p_l = n_{k_l} + r_l$ where $r_l \in \{0, \ldots m\}$ for all l. There is $r \in \{0, \ldots, m-1\}$ such that $r_l = r$ for infinitely many values of l. We can suppose, proceeding to a subsequence, that $r_l = r$ for all l. Then $T^{p_l}(\xi) = T^r(T^{n_{k_l}}(\xi))$ converges to a point of the set $T^r(\widetilde{\omega}_\xi)$. So, $\chi \in T^r(\widetilde{\omega}_\xi)$. The set ω_ξ is closed and invariant. Then, by Theorem 2.8, it contains a minimal point ζ. By (6.1), there is an iteration $T^q(\zeta)$, $q \in \mathbb{Z}$ that is a point of $\overline{U}$. This $T^q(\zeta)$ is the desired point. □

7. Proof of Lemma 3.7

Inclusion Q ⊂ W is obvious: iterations of an almost invariant ε-networks trace any sequence, not only pseudotrajectories.

Now we fix an $\varepsilon > 0$ and assume that for some $\delta > 0$ any δ-pseudotrajectory of T is ε-multishadowed by a finite set of trajectories. Let us prove existence of an almost invariant 2ε-network in $\mathrm{CR}(X,T)$. Consider a point $x \in \mathrm{CR}(X,T)$. Let $\{y_i := y_{i \bmod k} | i \in \mathbb{Z}\}$ be a periodic δ- pseudotrajectory with $y_0 = y_k = x$. Here $k = k(x)$. Due to multishadowing there exists $A(x) := \{a_1, \ldots, a_r\}$ such that $x = y_{km} \in B_\varepsilon(T^{km}(A(x)))$ for all $m \in \mathbb{Z}$. Select $\{x_1, \ldots, x_N\}$ – a finite ε-network for $\mathrm{CR}(X,T)$. Then

$$A = \bigcup_{j=1}^{N} \bigcup_{i=0}^{k(x_j)-1} T^i(A(x_j))$$

is such that $\mathrm{CR}(X,T) \subset U_\varepsilon(T^m(A))$ for any $m \in \mathbb{Z}$. Demonstrate that we can select $A \subset \mathrm{CR}(X,T)$. Take an increasing sequence $\{k_l \in \mathbb{N}\}$ so that iterations $T^{k_l}(A)$ of the set A converge point-wise to a set A_*. Then sets $T^m(A_*) \subset \omega(X,T) \subset \mathrm{CR}(X,T)$, $m \in \mathbb{Z}$ form 2ε-networks there, so it suffices to replace A with A_* and ε with 2ε. □

8. Proof of Lemma 3.8

Fix a sequence $\varepsilon_m \to 0$ $(m \to \infty)$. For every m, we consider an almost invariant ε_m-network

$$A_m = \{p_{m,j} : j = 1, \ldots, N_m\}.$$

Let μ_m be the probability atomic measure such that $\mu_m(\{p_{m,j}\}) = 1/N_m$ for all $j = 1, \dots, N_m$. Let $T_\#$ be the pushforward operator on Borel probability measures induced by T:

$$(T_\#\mu)(A) = \mu(T^{-1}(A))$$

for any measurable set A. Consider the sequence

$$\mu_{m,n} = \frac{1}{n}\sum_{i=0}^{n-1} T_\#^i \mu_m.$$

There exists an increasing subsequence n_l such that μ_{m,n_l} converges in the $*$-weak topology. The limit (call it μ_m^*) is a Borel invariant measure. Moreover, for any $x \in X$ we have $\mu_m^*(B_{\varepsilon_m}(x)) \geq 1/N_m$. To construct the desired measure μ^*, we can set

$$\mu^* = \sum_{m=1}^{\infty} \frac{1}{2^m}\mu_m^*. \tag{8.1}$$

Observe that by (8.1) $U_{\varepsilon_m}(\operatorname{supp}\mu_m^*) = X$ and

$$\operatorname{supp}\mu^* \supset \bigcup_{m=1}^{\infty} \operatorname{supp}\mu_m^*.$$

So, $\operatorname{supp}\mu^* = X$. This finishes the proof. □

9. Proof of Lemma 3.9

We start with a statement that is a corollary of the definition of chain recurrent sets.

LEMMA 9.1. *For any $\sigma > 0$ there exists a $\delta > 0$ such that for any δ-pseudotrajectory $\Xi = \{x_k\}$ the set $P(X,T,\Xi,\sigma) = \{k \in \mathbb{Z} : x_k \notin U_\sigma(\mathrm{CR}(X,T))\}$ is finite.*

PROOF. Assume that there exists a sequence $\delta_n \to 0$ ($n \to \infty$) and a sequence $P_n = P(X,T,\Xi_n,\sigma)$ of infinite sets that correspond to δ_n-pseudotrajectories Ξ_n. Each of pseudotrajectories Ξ_n has an ω-limit point $p_n \notin \overline{U_\sigma(\mathrm{CR}(X,T))}$. Without loss of generality, we assume that $p_n \to p_*$ ($n \to \infty$). Then $p_* \in \mathrm{CR}(X,T)$ that contradicts to our assumptions. □

Now we start the proof of Lemma 3.9. By (3.2) we have $\mathrm{CR}(X,T) = \overline{\mathrm{M}(X,T)}$. Bronstein condition implies multishadowing on $\overline{\mathrm{M}(X,T)}$ (Lemmas 3.5 and 3.7). Given an $\varepsilon > 0$ we consider $\delta_0 > 0$ so that any δ_0-pseudotrajectory in $\overline{\mathrm{M}(X,T)}$ is $\varepsilon/2$-multishadowed. Take a $\sigma \in (0, \min(\varepsilon/2, \delta_0))$ so that any point-wise σ-perturbation of a $\delta_0/2$-pseudotrajectory is a δ_0-pseudotrajectory. Take $\delta < \delta_0/2$ so that this δ corresponds to σ in the sense of

Lemma 9.1. By this lemma any δ-pseudotrajectory p_k cannot have infinitely many points out of σ-neighborhood of the set $\overline{\mathrm{M}(X,T)} = \mathrm{CR}(X,T)$. Fix a δ-pseudotrajectory $\{p_k\}$ and consider the sequence p'_k defined as follows. We set $p'_k = p_k$ if $p_k \notin U_\sigma((X,T))$. Otherwise, we take a point $p'_k \in \mathrm{M}(X,T)$ such that $\rho(p_k, p'_k) < \sigma$. The sequence $\{p'_k\}$ is a δ_0-pseudotrajectory that consists of two infinite parts inside $\mathrm{M}(X,T)$ and a finite number of points. Such pseudotrajectory can be $\varepsilon/2$-traced by a finite number of exact trajectories. Since $\sigma < \varepsilon/2$, the pseudotrajectory $\{p_k\}$ is ε-traced by same trajectories. □

Though we have already proved Lemma 3.9, notice some important corollaries of Lemma 9.1. Let $\mathcal{P}$ be the set of all δ-pseudotrajectories of T (X, T and σ are fixed).

LEMMA 9.2. *In conditions of Lemma 9.1*

$$L_\sigma := \sup_{\Xi \in \mathcal{P}} P(X,T,\Xi,\sigma) < +\infty.$$

PROOF. Fix a $\sigma > 0$. Let $\varepsilon = \sigma/2$, take $d > 0$ such that Eq. (2.2) is satisfied for any d-pseudotrajectory $\{x_k\}$. Let M_d be the maximal number of points $q_i \in X$ ($i = 1, \dots, M$) such that $\rho(q_i, q_j) \geq d$ for all $i \neq j$. The value M_d is finite, otherwise there is a sequence in X without any converging subsequence. If there exist $M_d + 1$ points of a d-pseudotrajectory $\{x_k\}$ out of $U_\sigma(\mathrm{CR}(X,T))$, at least two of these points, x_i and x_j are such that $\rho(x_i, x_j) < d$. Then there exists a periodic d-pseudotrajectory with a point out of $U_\sigma(\mathrm{CR}(X,T))$. Then, ε-shadowing this pseudotrajectory by a finite number of exact trajectories and proceeding to limit in one of these trajectories, we find an ω limit point out of $U_\varepsilon(\mathrm{CR}(X,T))$ (recall that $\sigma = 2\varepsilon$). So, $L_\sigma \leq M_d$. □

Next statement demonstrates that for any $\varepsilon > 0$ the number N of tracing trajectories $\{T^k(y_l)\}$ in Definition 2.20 can be taken the same for all d-pseudotrajectories $\{x_k\}$ where $d = d(\varepsilon)$.

COROLLARY 9.3. *Let $(X,T) \in W$. For any $\varepsilon > 0$ there exists a number $N = N(\varepsilon) \in \mathbb{N}$ such that for any $d(\varepsilon)$-pseudotrajectory $\{x_k\}$ there exist N points $y_1, \dots, y_N$ such that (2.2) is satisfied.*

PROOF. Given $\varepsilon > 0$, we take $d = d(\varepsilon/2)$ and fix the value L_d that exists by Lemma 9.2. Let K be the cardinality of an almost invariant $\varepsilon/2$-network in $\mathrm{CR}(X,T)$. So, we can take $N(\varepsilon) = L_d + K$. of points x_k that are out of $U_{\varepsilon/2}$ are ε-shadowed by themselves, others are ε-shadowed by points of the almost invariant network. □

10. Proof of Lemma 3.10

Let $x \in \mathrm{CR}(X,T)$. Then for any $\delta > 0$ there is a periodic δ-pseudotrajectory

$$\dots, x = x_0, x_1, x_2, \dots x_n = x, x_{n+1} = x_1, \dots$$

where n depends on δ. This pseudotrajectory is ε-shadowed by a finite number of trajectories $\{T^k(y_m)\}$, $m \in \{1,\ldots,r\}$; $\delta = d(\varepsilon)$. There exists a $l \in \{1,\ldots,r\}$ such that $\rho(T^{kn}(y_l), x) \le \varepsilon$ for infinitely many k. Then there exists a point $q \in \omega(y_l)$ such that $\rho(q,x) \le \varepsilon$. Since ε is arbitrary, we have proved that $x \in \Omega(X,T)$. □

11. Proof of Theorem 3.11

1. By definition, $\operatorname{supp}\mu$ is a closed invariant subset of X. Fix an $\varepsilon > 0$ and consider $\delta \in (0,\varepsilon)$ such that $\mu(B_\varepsilon(x)) > \delta$ for all $x \in \operatorname{supp}\mu$. Such δ exists since the set $\operatorname{supp}\mu$ is compact. Let A_δ be a finite δ-network, almost invariant with respect to μ.

Let us prove that

$$\operatorname{supp}\mu \subset U_{2\varepsilon}(T^n(A_\delta)) \tag{11.1}$$

for all $n \in \mathbb{Z}$. If (11.1) is not satisfied there exists an $n \in \mathbb{Z}$ and an ε-ball $B_\varepsilon(x_0)$, $x_0 \in \operatorname{supp}\mu$ such that $U_\varepsilon(T^n(A))$ for all $n \in \mathbb{Z}$. Then by definition of almost invariant networks, $\mu(B_\varepsilon(x)) \le 1 - (1-\delta) = \delta$. This contradicts to the choice of δ.

By Lemma 3.6, any neighborhood of any point of $\operatorname{supp}\mu$ contains a minimal point. So, $\operatorname{supp}\mu \subset \overline{\mathrm{M}(X,T)}$.

2. If minimal points are dense in $\operatorname{supp}\mu$, almost invariant ε-networks exist by Theorem 3.3. Of course, they all are also almost invariant with respect to μ. □

12. Multishadowing is C^1-generic

Certainly, multishadowing is C^0-generic in the space of homeomorphisms of a compact manifold X cause the "regular" shadowing is [39]. However, the "regular" shadowing is not C^1-generic [10].

Here we formulate an important statement that demonstrates a principle difference between multishadowing and classical shadowing.

Theorem 12.1. *Let X be a C^1-smooth compact manifold,* $\mathrm{Diff}^1(X)$ *be the space of C^1-diffeomorphisms of X. Then the set $W \bigcap \mathrm{Diff}^1(X)$ contains a residual subset in* $\mathrm{Diff}^1(X)$.

Proof. Given a diffeomorphism T, let $\mathrm{Per}(X,T)$ be the set of all periodic points. Bonatti and Crovisier [9] demonstrated that for a C^1-generic diffeomorphism T periodic points are dense in the set of chain recurrent ones

$$\overline{\mathrm{Per}(X,T)} = \mathrm{CR}(X,T). \tag{12.1}$$

By Theorem 3.3, Eq. (12.1) implies that $(X,T) \in \mathrm{W}$. □

13. Multishadowing and numerical methods

The main result of this section explains our motivation for studying multishadowing property. Roughly speaking, we demonstrate that minimal points could be found as limit points for iterations of a numerical method if and only if the multishadowing property is satisfied. First of all, observe that classical definitions 2.5 and 2.12 for α and ω-limit points may be spread to pseudotrajectories.

THEOREM 13.1. *Let T be a homeomorphism of a compact metric space X. Then two following statements are equivalent.*

1. $(X, T) \in W$.

2. *For any $\varepsilon > 0$ there exists $d > 0$ such that for any ω-limit point x_* of any d-pseudotrajectory x_k of system (X, T) the ball $\overline{B_\varepsilon(x_*)}$ contains a minimal point.*

Of course, a similar statement is true for α-limit points.

PROOF. **1** $\Rightarrow$ **2**. Given $\varepsilon > 0$, select $d > 0$ from Definition 2.20. Let $\{x_k\}$ be a d-pseudotrajectory, $y_1, \ldots, y_N$ be points of X such that (2.2) is satisfied. Since N is finite, we can proceed to limit in (2.2) along any subsequence $k_j \to \infty$ such that both x_{k_j} and $T^k(y^i_{k_j})$ converge (proceeding to a subsequence we may assume that the number i that provides the minimum is the same for all j). Thus we obtain that if ξ is an ω-limit point for $\{x_k\}$ then

$$\xi \in \overline{\bigcup_{i=1}^{N} U_\varepsilon(\omega_{y_i})} \subset \overline{U_\varepsilon(\mathrm{M}(X, T))}$$

(Theorem 3.3) which finishes the first part of the proof.

2 $\Rightarrow$ **1**. Take a point $x \in \mathrm{CR}(X, T)$. For any $d > 0$ there exists a periodic d-pseudotrajectory that contains x. Of course, x is an ω-limit point for all these trajectories. Thus, any neighborhood of x contains a minimal point that is $x \in \overline{M(X.T)}$. To finish the proof, it suffices to apply Theorem 3.3. □

14. Discussion

Let us discuss possible theoretical applications of obtained results: Theorems 3.1 and 3.3. In this section we give some more or less simple corollaries of these statements in order to demonstrate possible ways of application of obtained results to various domains of Dynamical Systems Theory.

We start with Theorem 3.1. Its main idea is quite simple: even an incorrectly applied numerical method can give a correct information about the dynamical system.

Fix a homeomorphism T of a compact metric space X. First of all, recall Corollary 4.6. It claims that any pseudotrajectory has a syndetic set of numbers that correspond to points of the pseudotrajectory in a neighborhood of the set of minimal point.

Basing on technique of Theorem 3.1 we prove that for a sufficiently precise pseudotrajectory almost all points are near the set of recurrent points.

COROLLARY 14.1. *For any* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that for any* δ*-pseudotrajectory* $p = \{p_k\}$ *of the map* T

$$\liminf_{N\to\infty} \frac{\# K_\varepsilon \bigcap [0,N]}{N} > 1 - \varepsilon. \tag{14.1}$$

Here $K_\varepsilon = \{k \geq 0 : p_k \in U_\varepsilon(\mathrm{R}(X,T))\}$ *where* $U_\varepsilon(\mathrm{R}(X,T))$ *is the* ε*- neighborhood of all recurrent points in* X.

PROOF. Take a sequence $\delta_m \to 0$ $(m \to \infty)$ and a sequence $\{p_k^m\}$ of δ_m pseudotrajectories. We demonstrate that every $\varepsilon > 0$

$$\lim_{m\to\infty} \limsup_{N\to\infty} \frac{\# L_{m,\varepsilon} \bigcap [0,N]}{N} = 0. \tag{14.2}$$

Here $L_{m,\varepsilon}$ is the completion of the corresponding set K_ε i.e.

$$L_{m,\varepsilon} = \{k \geq 0 : p_k^m \notin U_\varepsilon(\mathrm{R}(X,T))\}.$$

Evidently, (14.2) implies (14.1).

Suppose that (14.2) is wrong. Then, without loss of generality, we may select the sequence $\{p_k^m\}$ so that there is $\alpha > 0$ and increasing integer subsequences $\{N_k^m : k \in \mathbb{N}\}$ such that

$$\lim_{m\to\infty} \frac{\# L_{m,\varepsilon} \bigcap [0,N_k^m]}{N_k^m} \geq \alpha. \tag{14.3}$$

For any $m \in \mathbb{N}$ we take a sequence $n_k^m \to \infty$ as $k \to \infty$ $(\{n_k^m : k \in \mathbb{N}\} \subset \{N_k^m : k \in \mathbb{N}\})$ such that the limit

$$J_m(\varphi) := \frac{1}{n_k^m} \sum_{k=0}^{n_k^m - 1} \varphi(p_k^m)$$

is well-defined for any $\varphi \in C^0(X \to \mathbb{R})$ (see Eq. (4.1)).

By Riesz Representation theorem, every functional J_m corresponds to a probability measure μ_m (see Eq. (4.2)). We may assume that the sequence μ_m $*$-weakly converges to a measure μ_* that is invariant (see proof of Theorem 3.1, Section 4). Then $\operatorname{supp} \mu_* \subset \overline{\mathrm{R}(X,T)}$. On the other hand, (14.3) implies that $\mu_m(X \setminus U_\varepsilon(\mathrm{R}(X,T))) \geq \alpha$ for all $m \in \mathbb{N}$. Taking a test function φ such that $\varphi(x) = 0$ for all $x \in \mathrm{R}(X,T)$ and $\varphi(x) = 1$ if $x \notin U_\varepsilon(\mathrm{R}(X,T))$, we obtain

$$\alpha \leq \int_X \varphi \, d\,\mu_m \to \int_X \varphi \, d\mu_* = 0, \qquad m \to \infty.$$

This contradiction finishes the proof. □

The result of Theorem 3.3 provides a link between Shadowing Theory, Topological Dynamics and Ergodic Theory. In order to illustrate this we provide two corollaries of Theorem 3.3 and Lemma 3.8.

COROLLARY 14.2. *For any homeomorphism T of a compact topological space X, such that $(T, X) \in W$ there exists an invariant set Ξ, dense in $\Omega(X, T)$ such that for any $\varphi \in C^0(X \to \mathbb{R})$ and any $x \in \Xi$ there exists a limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(T^k(x)). \tag{14.4}$$

PROOF. Without loss of generality, proceeding to the dynamics on the nonwandering set, we may assume that $\Omega(X, T) = X$. Then, by Lemma 3.8, there exists an invariant probability measure μ such that $\operatorname{supp} \mu = X$. Take a set $\Phi = \{\varphi_i : i \in \mathbb{N}\}$ dense in $C^0(X \to \mathbb{R})$. By Birkhoff's Ergodic Theorem [23, Theorem 4.1.2], for any $i \in \mathbb{N}$ there exists a set Ξ_i such that the limit (14.4) exists for $\varphi = \phi_i$ and for any $x \in \Xi_i$. Let $\Xi = \bigcap_{i\in\mathbb{N}} \Xi_i$. Observe that $\mu(\Xi) = 1$ and, since $\operatorname{supp} \mu = X$, the set Ξ is dense in X. We demonstrated that for any $x \in \Xi$ the limit (14.4) exists for any $\varphi \in \Phi$. So, similarly to the proof of Lemma 4.1, we may prove that the limit exists for all $\varphi \in C^0(X \to \mathbb{R})$, $x \in \Xi$. □

COROLLARY 14.3. *For any C^1-diffeomorphism T of a compact smooth manifold X, $(X, T) \in W$ there exists a set Ψ, dense in $\Omega(X, T)$ and such that for any $x \in \Psi$ there exists limit*

$$\lim_{n\to\infty} \frac{1}{n} \log \|DT^n(x)\|.$$

Of course, this is the greatest Lyapunov exponent of the trajectory of x.

PROOF. We construct the measure μ, the same as in the previous proof. Then the desired statement follows from Kingman's Subadditive Ergodic Theorem [24]. □

Observe that we may select $\Xi = \Psi$ where sets Ξ and Ψ are defined by Corollaries 14.2 and 14.3 respectively. Indeed, $\mu(\Xi \bigcap \Psi) = 1$.

Now we look for possible applications of obtained results in Structural Stability Theory, mostly, for the so-called Ω-stability.

COROLLARY 14.4. *Let $(X, T) \in$ W. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any homeomorphism $S : X \to X$*

$$\max_{x\in X} \rho(S(x), T(x)) < \delta \tag{14.5}$$

implies

$$\mathrm{CR}(X,S) \subset U_\varepsilon(\mathrm{M}(X,T)). \tag{14.6}$$

PROOF. Fix $\varepsilon > 0$ and take $\delta > 0$ so that any 2δ-pseudotrajectory of T can be $\varepsilon/2$ traced by a finite number of trajectories of x. Let $x \in \mathrm{CR}(X,S)$ where the homeomorphism S satisfies (14.5). Then there is a periodic 2δ pseudotrajectory of the map T that contains point x. Since this pseudotrajectory can be $\varepsilon/2$ traced by a finite number of trajectories of T, the closed $\varepsilon/2$-ball, centered at x, contains an ω-limit point of T. Thus, by Theorem 3.3, we obtain (14.6). □

COROLLARY 14.5. *Let X be a C^1-smooth compact manifold. There exists a residual subset $\mathrm{Z} \subset \mathrm{Diff}^1(X)$ such that for any $T \in \mathrm{Z}$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that (14.5)* implies

$$\mathrm{CR}(X,S) \subset U_\varepsilon(\mathrm{Per}(X,T)).$$

PROOF. We can take $\mathrm{Z} = \mathrm{W} \bigcap \mathrm{Diff}^1(X)$ and apply Theorem 3.3 and Theorem 12.1. □

15. Conclusion

First of all, we list principal results of our paper.

We have established a result that is a weaker version of shadowing (Theorem 3.1): any one-sided pseudotrajectory can be shadowed by an exact trajectory along an increasing sequence of time instants. We may assume that points of this trajectory are minimal.

Certainly, Theorem 3.3 is one of central results of our paper. It gives necessary and sufficient condition of multishadowing and, respectively, new necessary conditions to classical shadowing. It was proved by Aoki and Hirade [3, Theorem 3.1.2] that shadowing property on the chain recurrent set $\mathrm{CR}(X,T)$ implies (3.3). Our Theorem 3.3 improves the mentioned result. First, even the multishadowing property on $\mathrm{CR}(X,T)$ implies (3.3) and, moreover, the Bronstein condition. Particularly, for systems of the class W, we have $\Omega(\Omega(X,T),T) = \Omega(X,T)$. Also, there must be a probability invariant measure supported on all $\Omega(X,T)$.

Equalities (3.2) and (3.3) are well-known in Dynamics, particularly in Shadowing Theory and Ω-Stability Theory. In [25], the authors showed that the following are equivalent:

(a) T belongs to the set of diffeomorphisms having the periodic shadowing property,

(b) T belongs to the set of diffeomorphisms having the Lipschitz periodic shadowing property, and

(c) T satisfies both Axiom A and the no-cycle condition.

For Axiom A diffeomorphisms multishadowing is equivalent to (3.3). This follows from Theorem 3.3.

The result of Theorem 13.1 claims that any ω-limit point of any pseudotrajectory of a homeomorphism with multishadowing is close to a minimal point of the modelled system.

Finally, we list some open problems, that are interesting for us in the framework of our research and may be considered as farther development of our results.

1. Generally speaking, the density a in Eq. (3.1) depends on the parameter ε and may tend to zero as ε tends to zero. For which systems (X, T) we can take a greater than a fixed positive constant for all ε and all pseudotrajectories?

2. What does periodic multishadowing property imply?

3. Is there any "two-sided" version of Theorem 3.1?

4. What can we say about topological entropy for diffeomorphisms with multishadowing?

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Danila Cherkashin
Moscow Institute of Physics and Technology,
Laboratory of Advanced Combinatorics and Network Applications,
9 Institutskiy per., Dolgoprudny, Moscow Region, 141701, RUSSIA
St. Petersburg Department of
V.A. Steklov Institute of Mathematics of
the Russian Academy of Sciences;
27, Fontanka, Saint-Petersburg, 191023, RUSSIA;

Sergey Kryzhevich
Faculty of Mathematics and Mechanics,
Saint-Petersburg State University,
28, Universitetskiy pr., Peterhof,
Saint-Petersburg, 198503, RUSSIA;
School of Natural Sciences and Mathematics,
The University of Texas at Dallas,
800 W. Campbell Road, Richardson, Texas, 75080-3021, USA.
School of Science, University of Nova Gorica,
Vipavska 13, SI-5000 Nova Gorica, SLOVENIA

kryzhevicz@gmail.com