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RESEARCH ARTICLE

2-Outer-Independent Domination in Graphs

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Abstract We initiate the study of 2-outer-independent domination in graphs. A 2-outer-independent dominating set of a graph G is a set D of vertices of G such that every vertex of $V(G)\backslash D$ has at least two neighbors in D, and the set $V(G)\backslash D$ is independent. The 2-outer-independent domination number of a graph G is the minimum cardinality of a 2-outer-independent dominating set of G. We show that if a graph has minimum degree at least two, then its 2-outer-independent domination number equals the vertex cover number. Then we investigate the 2-outer-independent domination in graphs with minimum degree one.

Keywords 2-Outer-independent domination · 2-Domination · Domination

Introduction

Let $G = (V, E)$ be a graph. The number of vertices of G we denote by n and the number of edges we denote by m, thus $|V(G)| = n$ and $|E(G)| = m$. By the *complement* of G, denoted by $\overline{G}$, we mean a graph which has the same vertices as G, and two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a pendant vertex we mean a vertex of degree one, while a support vertex is a vertex adjacent to a pendant vertex. The set of pendant vertices of a graph G we denote by $L(G)$. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two pendant vertices (exactly one pendant vertex, respectively). Let $\delta(G)$ ($\Delta(G)$, respectively) mean the minimum (maximum, respectively) degree among all vertices of G. The *path* (cycle, respectively) on n vertices we denote by P_n (C_n, respectively). A *wheel* W_n, where $n \geq 4$, is a graph with n vertices, formed by connecting a vertex to all vertices of a cycle C_{n-1}. The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph G, denoted by $\mathrm{diam}(G)$, is the maximum eccentricity among all vertices of G. By $K_{p,q}$ we denote a *complete bipartite graph* the partite sets of which have cardinalities p and q. By a *star* we mean the graph $K_{1,m}$ where $m \geq 2$. Let uv be an edge of a graph G. By subdividing the edge uv we mean removing it, and adding a new vertex, say x, along with two new edges ux and xv. By a subdivided *star* we mean a graph obtained from a star by subdividing each one of its edges. Generally, let $K_{t_1,t_2,\ldots,t_k}$ denote the complete multipartite graph with vertex set $S_1 \cup S_2 \cup \ldots \cup S_k$, where $|S_i| = t_i$ for positive integers $i \leq t$. The *corona* of a graph G on n vertices, denoted by $G \circ K_1$, is the graph on $2n$ vertices obtained from G by adding a vertex of degree one adjacent to each vertex of G. We say that a subset of

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$V(G)$ is independent if there is no edge between any two vertices of this set. The independence number of a graph G, denoted by $\alpha(G)$, is the maximum cardinality of an independent subset of the set of vertices of G. A vertex cover of a graph G is a set D of vertices of G such that for every edge uv of G, either $u \in D$ or $v \in D$. The vertex cover number of a graph G, denoted by $\beta(G)$, is the minimum cardinality of a vertex cover of G. It is well-known that $\alpha(G) + \beta(G) = |V(G)|$, for any graph G [1]. The clique number of G, denoted by $\omega(G)$, is the number of vertices of a greatest complete graph which is a subgraph of G. By G^* we denote the graph obtained from G by removing all pendant and isolated vertices.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \backslash D$ has a neighbor in D, while it is a 2-dominating set of G if every vertex of $V(G) \backslash D$ has at least two neighbors in D. The domination (2-domination, respectively) *number* of a graph G, denoted by $\gamma(G)$ ($\gamma_2(G)$, respectively), is the minimum cardinality of a dominating (2-dominating, respectively) set of G. Note that 2-domination is a type of multiple domination in which each vertex, which is not in the dominating set, is dominated at least k times for a fixed positive integer k. Multiple domination was introduced by Fink and Jacobson [2], and further studied for example in [3–9]. For a comprehensive survey of domination in graphs, see [10].

A subset $D \subseteq V(G)$ is a 2-outer-independent dominating set, abbreviated 2OIDS, of G if every vertex of $V(G) \backslash D$ has at least two neighbors in D, and the set $V(G) \backslash D$ is independent. The 2-outer-independent domination number of G, denoted by $\gamma_2^{oi}(G)$, is the minimum cardinality of a 2-outer-independent dominating set of G. A 2-outer-independent dominating set of G of minimum cardinality is called a $\gamma_2^{oi}(G)$-set. The 2-outer-independent domination number of trees was investigated in [11], where it was proved that it is upper bounded by half of the sum of the number of vertices and the number of pendant vertices.

In a distributed network, some vertices act as resource centers, or servers, while other vertices are clients. If a set D of servers is a dominating set, then every client in $V(G) \backslash D$ has direct (one hop) access to at least one server. 2-dominating sets represent a higher level of service, since every client has guaranteed access to at least two servers. The outer-independence condition means that the clients are not able to connect with each other directly. This may be useful for example for security, when we allow clients to communicate with each other only through servers.

We initiate the study of 2-outer-independent domination in graphs. We show that if a graph has minimum degree at least two, then its 2-outer-independent domination number equals the vertex cover number. Then we investigate the 2-outer-independent domination in graphs with minimum degree one. We find the 2-outer-independent domination numbers for several classes of graphs. Next we prove some lower and upper bounds on the 2-outer-independent domination number of a graph, and we characterize the extremal graphs. Then we study the influence of removing or adding vertices and edges. We also give Nordhaus–Gaddum type inequalities [12].

Preliminary Results

If G is a disconnected graph with connected components $G_1, G_2, \ldots, G_k$, then we can easily see that $\gamma_2^{oi}(G) = \gamma_2^{oi}(G_1) + \gamma_2^{oi}(G_2) + \ldots + \gamma_2^{oi}(G_k)$.

We have the following inequalities.

Proposition 1 *Let G be a graph. Then:*

(i) $\gamma_2^{oi}(G) \geq \gamma_2(G)$;
(ii) $\gamma_2^{oi}(G) \geq \omega(G) - 1$;
(iii) $\gamma_2^{oi}(G) \geq \beta(G)$.

Proof *(i)* Any 2-outer-independent dominating set of a graph is a 2-dominating set of this graph, and thus $\gamma_2(G) \leq \gamma_2^{oi}(G)$.

(ii) Let D be a $\gamma_2^{oi}(G)$-set, and let A be a maximum clique in G. Since $V(G) \backslash D$ is independent, we have $|(V(G) \backslash D) \cap A| \leq 1$. This implies that $|D| \geq |A| - 1$. We now get $\gamma_2^{oi}(G) = |D| \geq |A| - 1 = \omega(G) - 1$.

(iii) Note that the definition of 2-outer-independent domination implies that every 2OIDS of a graph is a vertex cover of this graph, and thus the result follows. □

Note that the bounds of the above proposition are tight. It is easy to see that for every integer $n \geq 3$ we have $\gamma_2^{oi}(K_n) = \gamma_2(K_n) + n - 3$, for every integer $m \geq 2$ we have $\gamma_2^{oi}(K_{1,m}) = \omega(K_{1,m}) + m - 2$ and $\gamma_2^{oi}(K_{1,m}) = \beta(K_{1,m}) + m - 1$, while $\gamma_2^{oi}(K_3) = 2 = \beta(K_3)$.

We next prove that if a graph has no pendant or isolated vertices, then its 2-outer-independent domination number and vertex cover number are equal.

Theorem 2 *Let G be a graph. If* $\delta(G) \geq 2$, *then* $\gamma_2^{oi}(G) = \beta(G)$.

Proof Let D be a minimum vertex cover of G, and let $x \in V(G) \backslash D$. Clearly, $N_G(x) \subseteq D$. Since $\delta(G) \geq 2$, the vertex x is adjacent to at least two vertices of D. There are no edges between any two vertices of $V(G) \backslash D$, thus the set $V(G) \backslash D$ is independent. This implies that D is a 2OIDS of the graph G. Consequently, $\gamma_2^{oi}(G) \leq \beta(G)$. On the other hand, by Proposition 1 we have $\gamma_2^{oi}(G) \geq \beta(G)$. Thus $\gamma_2^{oi}(G) = \beta(G)$. □

Corollary 3 *Let G be a graph. If* $\gamma_2^{oi}(G) \neq \beta(G)$, *then* $\delta(G) \in \{0, 1\}$.

Henceforth, we study only connected graphs G with $\delta(G) = 1$, that is, connected graphs having at least one pendant vertex. Since a pendant vertex has only one neighbor in the graph, it cannot have two neighbors in the dominating set. Thus we have the following property of pendant vertices.

Observation 4 *Every pendant vertex of a graph G belongs to every $\gamma_2^{oi}(G)$-set.*

Connected Graphs with Minimum Degree One

Throughout this section we consider only connected graphs with minimum degree one. We have the following relation between the 2-outer-independent domination number of a graph and the independence number of the graph obtained from it by removing all pendant vertices.

Lemma 5 *For every graph G with n vertices we have $\gamma_2^{oi}(G) = n - \alpha(G^*)$.*

Proof Let D be any $\gamma_2^{oi}(G)$-set. By Observation 4, all pendant vertices belong to the set D. Therefore $V(G)\backslash D \subseteq V(G^*)$. The set $V(G)\backslash D$ is independent, thus $\alpha(G^*) \geq |V(G)\backslash D| = n - \gamma_2^{oi}(G)$. Now let D^* be any $\alpha(G^*)$-set. Let us observe that in the graph G every vertex of D^* has at least two neighbors in the set $V(G)\backslash D^*$. Thus $V(G)\backslash D^*$ is a 2OIDS of G. We now get $\gamma_2^{oi}(G) \leq |V(G)\backslash D^*| = n - \alpha(G^*)$. This implies that $\gamma_2^{oi}(G) = n - \alpha(G^*)$. □

It is obvious that for every graph G we have $2 \leq \gamma_2^{oi}(G) \leq n$. We now characterize the graphs attaining these bounds.

Proposition 6 *Let G be a graph. We have:*

(i) $\gamma_2^{oi}(G) = 2$ *if and only if* $G \in \{P_2, P_3\}$;
(ii) $\gamma_2^{oi}(G) = n$ *if and only if* $G = P_2$.

Proof Obviously, $\gamma_2^{oi}(P_2) = 2 = n$ and $\gamma_2^{oi}(P_3) = 2$. Assume that for some graph G we have $\gamma_2^{oi}(G) = 2$. Let D be a $\gamma_2^{oi}(G)$-set. If all vertices of G belong to the set D, then the graph G has two vertices. Consequently, $G = P_2$. Now let x be a vertex of $V(G)\backslash D$. The vertex x has to be dominated twice, thus $d_G(x) \geq 2$. Since the set $V(G)\backslash D$ is independent, the vertex x cannot have more than two neighbors in G. This implies that G is a path P_3 as no other vertices can be dominated twice.

Now assume that for some graph G we have $\gamma_2^{oi}(G) = n$. If G has at least three vertices, then it has a vertex, say x, of degree at least two. Let us observe that $D\backslash\{x\}$ is a 2OIDS of the graph G. This implies that $\gamma_2^{oi}(G) \leq n - 1$. Therefore the graph G has exactly two vertices, and consequently, it is a path P_2. □

Corollary 7 *For every graph G with at least three vertices we have $\gamma_2^{oi}(G) \leq n - 1$.*

We now consider graphs G such that $3 \leq \gamma_2^{oi}(G) \leq n - 1$.

Theorem 8 *Let G be a graph of order $n \geq 3$, and let k be an integer such that $3 \leq k \leq n - 1$. We have $\gamma_2^{oi}(G) = k$ if and only if G can be obtained from a connected graph H of order k with $|L(H)| \leq n - k$ and $\alpha(H) = n - k$, by attaching $n - k$ vertices to H in a way such that every pendant vertex of H is a support vertex of G.*

Proof Assume that $\gamma_2^{oi}(G) = k$. Lemma 5 implies that $\alpha(G^*) = n - k$. Clearly, every vertex of $V(G)\backslash V(G^*)$ is a pendant vertex in G. Let us also observe that every pendant vertex of G^* is a support vertex of G. Thus $|L(G^*)| \leq n - |V(G^*)|$.

Now assume that G is a graph obtained from a connected graph H of order k with $|L(H)| \leq n - k$ and $\alpha(H) = n - k$, by attaching $n - k$ vertices to H in a way such that every pendant vertex of H is a support vertex of G. Let us observe that $G^* = H$. Let D be a maximum independent set of H. Clearly, $V(G)\backslash D$ is a 2OIDS of G, and therefore $\gamma_2^{oi}(G) \leq n - \alpha(H) = k$. Suppose that $\gamma_2^{oi}(G) < k$. Using Lemma 5 we obtain $\alpha(H) > n - k$, a contradiction. Thus $\gamma_2^{oi}(G) = k$. □

Bounds

We have the following upper bound on the 2-outer-independent domination number of a graph in terms of its vertex cover number and the number of pendant vertices.

Proposition 9 *If G is a graph with l pendant vertices, then $\gamma_2^{oi}(G) \leq \beta(G) + l$.*

Proof Let us observe that vertices of any minimum vertex cover of G together with all pendant vertices of G form a 2OIDS of the graph G. □

Let us observe that the bound from the previous proposition is tight. Let l be a positive integer, and let $H = C_6$. Let x be a vertex of H, and let G be a graph obtained from H by attaching l new vertices and joining them to the vertex x. It is straightforward to see that $\beta(G) = 3$, while $\gamma_2^{oi}(G) = 3 + l$.

We have the following upper bound on the 2-outer-independent domination number of a graph in terms of its vertex cover number and maximum degree.

Proposition 10 *For every graph G we have $\gamma_2^{oi}(G) \leq \beta(G)\Delta(G)$.*

Proof Let S be a minimum vertex cover of G. The vertices of S together with all pendant vertices of G form a 2OIDS of the graph G. Every vertex of S is adjacent to at most $\Delta(G)$ pendant vertices. Thus $\gamma_2^{oi}(G) \leq \beta(G)\Delta(G)$. □

Let us observe that the bound from the previous proposition is tight. For stars $K_{1,m}$ we have $\gamma_2^{oi}(K_{1,m}) = m = 1 \cdot m = \beta(K_{1,m})\Delta(K_{1,m})$.

We have the following upper bound on the 2-outer-independent domination number of a graph.

Proposition 11 *For every graph G with l pendant vertices we have*

$$\gamma_2^{oi}(G) \le \frac{n\Delta(G) + l}{\Delta(G) + 1}.$$

Proof By Lemma 5 we have $\gamma_2^{oi}(G) = n - \alpha(G^*)$. Since every maximal independent set of a graph is a dominating set of this graph, we have $\gamma(G^*) \le \alpha(G^*)$. We now get

$$\alpha(G^*) \ge \gamma(G^*) \ge \frac{|V(G^*)|}{\Delta(G^*) + 1} \ge \frac{n - l}{\Delta(G) + 1}.$$

□

We have the following upper bound on the 2-outer-independent domination number of a graph in terms of its diameter.

Proposition 12 *If G is a graph of diameter d, then* $\gamma_2^{oi}(G) \le n - \lfloor d/2 \rfloor$.

Proof Let $v_0, v_1, \ldots, v_d$ be a diametrical path in G. If d is even, then let $D = \{v_{2i-1} : 1 \le i \le d/2\}$, while if d is odd, then let $D = \{v_{2i-1} : 1 \le i \le (d-1)/2\}$. Let us observe that $V(G) \backslash D$ is a 2OIDS of the graph G. □

Let us observe that the bound from the previous proposition is tight. We have $\gamma_2^{oi}(P_n) = \lfloor n/2 \rfloor + 1 = n - \lfloor (n-1)/2 \rfloor - 1 + 1 = n - \lfloor (n-1)/2 \rfloor = n - \lfloor d/2 \rfloor$.

We have the following upper bound on the 2-outer-independent domination number of a tree in terms of its independence number and the number of support vertices.

Theorem 13 *For every tree T of order at least three with s support vertices we have* $\gamma_2^{oi}(T) \le \alpha(T) + s - 1$.

Proof Let n mean the number of vertices of the tree T. We proceed by induction on this number. If $\text{diam}(T) = 1$, then $T = P_2$. We have $\gamma_2^{oi}(P_2) = 2 = 1 + 2 - 1 = \alpha(P_2) + s - 1$. Now assume that $\text{diam}(T) = 2$. Thus T is a star $K_{1,m}$. We have $\gamma_2^{oi}(K_{1,m}) = m < m + 1 = m + 2 - 1 \le 2m - 1 = m + m - 1 = \alpha(K_{1,m}) + s(K_{1,m}) - 1$. Now let us assume that $\text{diam}(T) = 3$. Thus T is a double star. We have $\gamma_2^{oi}(T) = n - 1 = n - 2 + 2 - 1 = \alpha(T) + s(T) - 1$.

Now assume that $\text{diam}(T) \ge 4$. Thus the order n of the tree T is at least five. We obtain the result by the induction on the number n. Assume that the theorem is true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say n, is strong. Let y be a pendant vertex adjacent to x. Let $T' = T - y$. We have $s' = s$. Let D' be any $\gamma_2^{oi}(T')$-set. Obviously, $D' \cup \{y\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 1$. Let us observe that there exists a maximum independent set of T' that contains the vertex x. Let A' be such a set. It is easy to see that $D' \cup \{y\}$ is an independent set of the tree T. Thus $\alpha(T) \ge \alpha(T') + 1$. We now get $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 1 \le \alpha(T') + s' = \alpha(T') + s \le \alpha(T) + s - 1$. Henceforth, we can assume that all support vertices of T are weak.

We now root T at a vertex r of maximum eccentricity $\text{diam}(T)$. Let t be a pendant vertex at maximum distance from r, v be the parent of t, u be the parent of v, and w be the parent of u in the rooted tree. By T_x let us denote the subtree induced by a vertex x and its descendants in the rooted tree T.

Assume that among the children of u there is a support vertex, say x, different from v. Let $T' = T - T_v$. We have $s' = s - 1$. Let us observe that there exists a $\gamma_2^{oi}(T')$-set that contains the vertex u. Let D' be such a set. It is easy to observe that $D' \cup \{t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 1$. Now let A' be a maximum independent set of T'. It is easy to observe that $D' \cup \{t\}$ is an independent set of T. Thus $\alpha(T) \ge \alpha(T') + 1$. We now get $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 1 \le \alpha(T') + s' = \alpha(T') + s \le \alpha(T) + s - 1$.

Now assume that u is adjacent to a pendant vertex, say x. It suffices to consider only the possibility when $d_T(u) = 3$. Let $T' = T - x$. We have $s' = s - 1$. Obviously, $\alpha(T) \ge \alpha(T')$. Let D' be any $\gamma_2^{oi}(T')$-set. Obviously, $D' \cup \{x\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 1$. We now get $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 1 \le \alpha(T') + s' = \alpha(T') + s - 1 \le \alpha(T) + s - 1$.

Now assume that $d_T(u) = 2$. Let $T' = T - T_v$. We have $s' \le s$. Let D' be any $\gamma_2^{oi}(T')$-set. By Observation 4 we have $u \in D'$. It is easy to observe that $D' \cup \{t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 1$. Now let A' be a maximum independent set of T'. It is easy to see that $D' \cup \{t\}$ is an independent set of the tree T. Thus $\alpha(T) \ge \alpha(T') + 1$. We now get $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 1 \le \alpha(T') + s' \le \alpha(T') + s \le \alpha(T) + s - 1$. □

We have the following bounds on the 2-outer-independent domination number of a graph in terms of its order and size.

Proposition 14 *For every graph G we have*

$$\frac{2n + 1 - \sqrt{(2n-1)^2 - 8(m-1)}}{2} \le \gamma_2^{oi}(G) \le \frac{2n + 1 + \sqrt{(2n-1)^2 - 8(m-1)}}{2}.$$

Proof Let D be a $\gamma_2^{oi}(G)$-set. Let t denote the number of edges between the vertices of D and the vertices of

$V(G)\backslash D$. Obviously, $m \le t + |E(G[D])|$. Since G has at least one pendant vertex, we have $t \le (|D|-1)\cdot |V(G)\backslash D| + 1$. Notice that $|E(G[D])| \le (|D|-1)(|D| - 2)/2$. Now simple calculations imply the result. ☐

We also have the following lower bound on the 2-outer-independent domination number of a graph in terms of its order and size.

Proposition 15 *For every graph G we have $\gamma_2^{oi}(G) \ge n - m/2$.*

Proof Let D be a $\gamma_2^{oi}(G)$-set. Since every vertex of $V(G)\backslash D$ has at least two neighbors in D, have $m \ge 2|V(G)\backslash D|$. ☐

Let us observe that the bound from the previous proposition is tight. For positive integers n we have $\gamma_2^{oi}(P_n) = \lfloor n/2 \rfloor + 1 = (n+1)/2 = n - (n-1)/2 = n - m/2$.

We have the following necessary condition for that a graph attains the bound from the previous proposition.

Proposition 16 *If for a graph G we have $\gamma_2^{oi}(G) = n - m/2$, then the graph G is bipartite and it has at least $m/2$ vertices of degree two.*

Proof Let D be a $\gamma_2^{oi}(G)$-set. Let t denote the number of edges between the vertices of D and the vertices of $V(G)\backslash D$. If some vertex of $V(G)\backslash D$ has degree at least three, then we get $m \ge t \ge 3 + 2(|V(G)\backslash D| - 1) = 2|V(G)\backslash D| + 1 = 2(n - \gamma_2^{oi}(G)) + 1 = m + 1 > m$, a contradiction. Thus every vertex of $V(G)\backslash D$ has degree two. We have $|V(G)\backslash D| = n - \gamma_2^{oi}(G) = m/2$. Thus there are at least $m/2$ vertices of degree two. If the set D is not independent, then we get $m > t = 2|V(G)\backslash D| = 2(n - \gamma_2^{oi}(G)) = m$, a contradiction. Therefore D is an independent set. Since the set $V(G)\backslash D$ is also independent, the graph G is bipartite. ☐

It is an open problem to characterize the graphs attaining the bound from Proposition 16.

Problem 17 *Characterize graphs G such that $\gamma_2^{oi}(G) = n - m/2$.*

We now study the influence of the removal of a vertex of a graph on its 2-outer-independent domination number.

Proposition 18 *Let G be a graph. For every vertex v of G we have $\gamma_2^{oi}(G) - 1 \le \gamma_2^{oi}(G - v) \le \gamma_2^{oi}(G) + d_G(v) - 1$.*

Proof Let D be a $\gamma_2^{oi}(G)$-set. If $v \notin D$, then observe that D is a 2OIDS of the graph $G - v$. Now assume that $v \in D$. Let us observe that $D \cup N_G(v)\backslash\{v\}$ is a 2OIDS of the graph $G - v$. Therefore $\gamma_2^{oi}(G - v) \le |D \cup N_G(v)\backslash \{v\}| \le |D\backslash\{v\}| + |N_G(v)| = \gamma_2^{oi}(G) + d_G(v) - 1$.

Now let D' be any $\gamma_2^{oi}(G - v)$-set. It is easy to see that $D' \cup \{v\}$ is a 2OIDS of the graph G. Thus $\gamma_2^{oi}(G) \le \gamma_2^{oi}(G - v) + 1$. ☐

Let us observe that the bounds from the previous proposition are tight. For the lower bound, let $G = K_n$, where $n \ge 4$. We have $\gamma_2^{oi}(G) = \gamma_2^{oi}(K_n) = n - 1 = n - 2 + 1 = \gamma_2^{oi}(K_{n-1}) + 1$. For the upper bound, let G be subdivided star. The vertex of minimum eccentricity we denote by v. Let m denote its degree. We have $G - v = mK_2$. Consequently, $\gamma_2^{oi}(G - v) = \gamma_2^{oi}(mK_2) = m\gamma_2^{oi}(K_2) = 2m = m + 1 + m - 1 = \gamma_2^{oi}(G) + d_G(v) - 1$.

We now study the influence of the removal of an edge of a graph on its 2-outer-independent domination number.

Proposition 19 *Let G be a graph. For every edge e of G we have*

$$\gamma_2^{oi}(G - e) \in \{\gamma_2^{oi}(G) - 1, \gamma_2^{oi}(G), \gamma_2^{oi}(G) + 1\}.$$

Proof Let D be a $\gamma_2^{oi}(G)$-set, and let $e = xy$ be an edge of G. Since the set $V(G)\backslash D$ is independent, some of the vertices x and y belongs to the set D. Without loss of generality we may assume that $x \in D$. If $y \in D$, then it is easy to see that D is a 2OIDS of the graph $G - e$. If $y \notin D$, then $D \cup \{y\}$ is a 2OIDS of $G - e$. Thus $\gamma_2^{oi}(G - e) \le \gamma_2^{oi}(G) + 1$. Now let D' be a $\gamma_2^{oi}(G - e)$-set. If some of the vertices x and y belongs to the set D', then D' is a 2OIDS of the graph G. If none of the vertices x and y belongs to the set D', then it is easy to observe that $D' \cup \{x\}$ is a 2OIDS of the graph G. Therefore $\gamma_2^{oi}(G) \le \gamma_2^{oi}(G - e) + 1$. ☐

Let us observe that the bounds from the previous proposition are tight. For the lower bound, let xy be an edge of the complete graph K_4. Let G be a graph obtained from K_4 by adding two vertices x_1, y_1, and joining x to x_1, and y to y_1. Then $\gamma_2^{oi}(G - xy) = \gamma_2^{oi}(G) - 1$. For the upper bound, consider a path P_4, and the central edge of it.

Similarly, we have the following result, which immediately follows from Proposition 19, concerning the influence of adding an edge on the 2-outer-independent domination number of a graph.

Proposition 20 *Let G be a graph. If $e \notin E(G)$, then*

$$\gamma_2^{oi}(G + e) \in \{\gamma_2^{oi}(G) - 1, \gamma_2^{oi}(G), \gamma_2^{oi}(G) + 1\}.$$

Let us observe that the bounds from the previous proposition are tight.

Nordhaus–Gaddum Type Inequalities

A Nordhaus–Gaddum type result is a lower or upper bound on the sum or product of a parameter of a graph and its complement. In 1956 Nordhaus and Gaddum [12] proved the following inequalities for the chromatic number of a graph G and its complement: $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n+1$ and $n \leq \chi(G)\chi(\overline{G}) \leq (n+1)^2/4$.

We now give Nordhaus–Gaddum type inequalities for the sum of the 2-outer-independent domination number of a graph and its complement.

Theorem 21 *For every graph* G *we have* $n-1 \leq \gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) \leq 2n$.

Proof Let D be a $\gamma_2^{oi}(G)$-set. Since $V(G)\setminus D$ is an independent set, the vertices of $V(G)\setminus D$ form a clique in $\overline{G}$. Let $\overline{D}$ be any $\gamma_2^{oi}(\overline{G})$-set. Let us observe that at most one vertex of $V(G)\setminus D$ does not belong to $\overline{D}$. Therefore $|\overline{D}| \geq |V(G)\setminus D| - 1$. We now get $\gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) = |D| + |\overline{D}| \geq |D| + |V(G)\setminus D| - 1 = n-1$.

Obviously, $\gamma_2^{oi}(G) \leq n$ and $\gamma_2^{oi}(\overline{G}) \leq n$. Thus $\gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) \leq 2n$. □

We now prove that the complete graphs of order at most two, and their complements are the only graphs which attain the upper bound from Theorem 21.

Theorem 22 *Let* G *be a graph. We have* $\gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) = 2n$ *if and only if* $G = K_1$ *or* $G = K_2$ *or* $G = K_1 \cup K_1$.

Proof First, it is straightforward to see that $\gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) = 2n$ if $G = K_1$ or $G = K_2$ or $G = K_1 \cup K_1$. Now assume that for some graph G we have $\gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) = 2n$. This implies that $\gamma_2^{oi}(G) = n$ and $\gamma_2^{oi}(\overline{G}) = n$. By Corollary 7, $n \leq 2$. Consequently, $G = K_1$ or $G = K_2$ or $G = K_1 \cup K_1$. □

Corollary 23 *If* G *and* $\overline{G}$ *are different from* K_1 *and* K_2, *then* $\gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) \leq 2n-1$.

We now prove that the path P_3 and its complement are the only graphs which attain the bound from the previous corollary.

Theorem 24 *Let* G *be a graph. We have* $\gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) = 2n-1$ *if and only if* G *or* $\overline{G}$ *is a path* P_3.

Proof We have $\gamma_2^{oi}(P_3) + \gamma_2^{oi}(\overline{P_3}) = 5 = 2n-1$. Now assume that for some graph G we have $\gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) = 2n-1$. This implies that $\gamma_2^{oi}(G) = n-1$ or $\gamma_2^{oi}(\overline{G}) = n-1$. Without loss of generality we assume that $\gamma_2^{oi}(G) = n-1$. By Theorem 8, the graph G is obtained from a complete graph K_r, for some $r \geq 1$, by attaching at least one pendant vertex. We show that $n = 3$. Suppose that $n \geq 4$. Since $\delta(G) = 1$, we may assume that x is a pendant vertex of G. Thus x has at least two neighbors in the graph $\overline{G}$. Therefore $V(G)\setminus\{x\}$ is a 2OIDS of $\overline{G}$, and consequently, $\gamma_2^{oi}(\overline{G}) \leq n-1$. We now get $\gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) \leq 2n-2$, a contradiction. We deduce that $n = 3$. Consequently, $G = P_3$. □

We next improve the lower bound from Theorem 21.

Theorem 25 *For every graph* G *with* l *pendant vertices we have* $\gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) \geq n + l - 2$.

Proof By Theorem 8, the graph G is obtained from a connected graph H with $\alpha(H) = n - \gamma_2^{oi}(G)$, by attaching $n - |V(H)|$ pendant vertices to H such that any pendant vertex of H is a support vertex of G. Let $X = V(G)\setminus V(H)$. By Lemma 5 we have $\gamma_2^{oi}(G) = n - \alpha(H)$. Let S be a maximum independent set in H. Then clearly $V(G)\setminus S$ is a $\gamma_2^{oi}(G)$-set. Let D be a $\gamma_2^{oi}(\overline{G})$-set. Clearly, $\overline{G}[X]$ and $\overline{G}[S]$ are complete graphs. Thus $|D \cap S| \geq |S| - 1$, and $|D \cap X| \geq |X| - 1$. We now get

$$\begin{aligned}\gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) &\geq |V(G)| - |S| + |S| - 1 + |X| - 1\\ &= n + |X| - 2 = n + l - 2.\end{aligned}$$

□

We now characterize graphs attaining the lower bound from Theorem 21, that is, graphs G for which $\gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) = n-1$. Since $\gamma_2^{oi}(G) \geq 2$, we may assume that $\gamma_2^{oi}(G) < n-2$.

Theorem 26 *Let* G *be a graph such that* $\gamma_2^{oi}(G) < n-2$. *Then* $\gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) = n-1$ *if and only if* G *is obtained from a connected graph* H *such that* $\alpha(H) = n - \gamma_2^{oi}(G)$ *and* $|L(H)| \leq 1$, *by attaching one pendant vertex to* H *such that if* H *has a pendant vertex* x, *then* x *is a support vertex in* G.

Proof Assume that for some graph G we have $\gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) = n-1$. By Theorem 8, the graph G is obtained from a connected graph H with $\alpha(H) = n - \gamma_2^{oi}(G)$, by attaching $n - |V(H)|$ pendant vertices to H such that any pendant vertex of H is a support vertex of G. Let $|V(G)\setminus V(H)| = l$. By Theorem 25 we have $n - 1 = \gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) \geq n + l - 2$. This implies that $l \leq 1$, and so $l = 1$. Now the result follows.

Conversely, let G be obtained from a connected graph H with $\alpha(H) = n - \gamma_2^{oi}(G)$ and $|L(H)| \leq 1$, by attaching one pendant vertex (say u) to H such that if H has a pendant vertex x, then x is a support vertex in G. By Theorem 8 we have $\gamma_2^{oi}(G) = n - \alpha(H)$. Let S be a maximum independent set in H. Since $\gamma_2^{oi}(G) < n-2$, we find that $|S| \geq 3$. Let $x, y \in S$. Then $(S - \{x,y\}) \cup \{u\}$ is a 2OIDS for $\overline{G}$, and thus $\gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) \leq n - |S| + |S| - 2 + 1 = n-1$. By Theorem 25, $\gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) \geq n + l - 2 = n-1$, and thus the result follows. □

Similarly we obtain the following result.

Theorem 27 *Let $k \leq n - 1$ be a non-negative integer. If G is a graph of order n, then $\gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) = n + k$ if and only if G is obtained from a connected graph H such that $\alpha(H) = n - \gamma_2^{oi}(G)$ and $|L(H)| \leq t$, by attaching t pendant vertices to H, where $t \leq k + 2$, in a way such that if H has a pendant vertex x, then x is a support vertex in G.*

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