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A LOWER BOUND ON THE DOUBLE OUTER-INDEPENDENT DOMINATION NUMBER OF A TREE

Abstract. A vertex of a graph is said to dominate itself and all of its neighbors. A double outer-independent dominating set of a graph G is a set D of vertices of G such that every vertex of G is dominated by at least two vertices of D, and the set $V(G) \setminus D$ is independent. The double outer-independent domination number of a graph G, denoted by $\gamma_d^{oi}(G)$, is the minimum cardinality of a double outer-independent dominating set of G. We prove that for every nontrivial tree T of order n, with l leaves and s support vertices we have $\gamma_d^{oi}(T) \geq (2n+l-s+2)/3$, and we characterize the trees attaining this lower bound. We also give a constructive characterization of trees T such that $\gamma_d^{oi}(T) = (2n+2)/3$.

1. Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a subset of $V(G)$ is independent if there is no edge between every two its vertices.

A vertex of a graph is said to dominate itself and all of its neighbors. A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of G is dominated by at least one vertex of D, while it is a double dominating set of G if every vertex of G is dominated by at least two vertices of D. The domination (double domination, respectively) number of G, denoted by $\gamma(G)$ ($\gamma_d(G)$, respectively), is the minimum cardinality of a dominating (double dominating, respectively) set of G. Note that double domination is a type of k-tuple domination in which each vertex of a graph is dominated at least k times for a fixed positive integer k. The study of k-tuple domination was ini-

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tiated by Harary and Haynes [3]. For a comprehensive survey of domination in graphs, see [4, 5].

A subset $D \subseteq V(G)$ is a double outer-independent dominating set, abbreviated DOIDS, of G if every vertex of G is dominated by at least two vertices of D, and the set $V(G) \setminus D$ is independent. The double outer-independent domination number of a graph G, denoted by $\gamma_d^{oi}(G)$, is the minimum cardinality of a double outer-independent dominating set of G. A double outer-independent dominating set of G of minimum cardinality is called a $\gamma_d^{oi}(G)$-set. Double outer-independent domination in graphs was introduced in [6].

Chellali and Haynes [2] proved the following lower bound on the total domination number of a tree. For every nontrivial tree T of order n with l leaves we have $\gamma_t(T) \geq (n - l + 2)/2$. They also characterized the extremal trees. Blidia, Chellali, and Favaron [1] established the following lower bound on the 2-domination number of a tree. For every nontrivial tree T of order n with l leaves we have $\gamma_2(T) \geq (n + l + 2)/3$. The extremal trees were also characterized.

We prove the following lower bound on the double outer-independent domination number of a tree. For every nontrivial tree T of order n, with l leaves and s support vertices we have $\gamma_d^{oi}(T) \geq (2n + l - s + 2)/3$. We also characterize the trees attaining this lower bound. We also give a constructive characterization of trees T such that $\gamma_d^{oi}(T) = (2n + 2)/3$.

2. Results

Since the one-vertex graph does not have double outer-independent dominating set, in this paper, by a tree we mean only a connected graph with no cycle, and which has at least two vertices.

We begin with the following two straightforward observations.

Observation 1. *Every leaf of a graph G is in every $\gamma_d^{oi}(G)$-set.*

Observation 2. *Every support vertex of a graph G is in every $\gamma_d^{oi}(G)$-set.*

We show that if T is a nontrivial tree of order n, with l leaves and s support vertices, then $\gamma_d^{oi}(T)$ is bounded below by $(2n + l - s + 2)/3$. For the purpose of characterizing the trees attaining this bound we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let T_1 be a path P_2 with vertices labeled x and y, and let $A(T_1) = \{x, y\}$. Let H be a path P_3 with leaves labeled u and z, and the support vertex labeled w. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a vertex v by joining it to any support vertex of T_k. Let $A(T_{k+1}) = A(T_k) \cup \{v\}$.
- Operation $\mathcal{O}_2$: Attach a copy of H by joining u to a vertex of $A(T_k)$ which has degree at least two. Let $A(T_{k+1}) = A(T_k) \cup \{w, z\}$.
- Operation $\mathcal{O}_3$: Attach a copy of H by joining u to a leaf of T_k which is the only one leaf among neighbors of its neighbor. Let $A(T_{k+1}) = A(T_k) \cup \{w, z\}$.

Now we prove that for every tree T of the family $\mathcal{T}$, the set $A(T)$ defined above is a DOIDS of minimum cardinality equal to $(2n + l - s + 2)/3$.

Lemma 3. *If $T \in \mathcal{T}$, then the set $A(T)$ defined above is a $\gamma_d^{oi}(T)$-set of size $(2n + l - s + 2)/3$.*

Proof. We use the terminology of the construction of the trees $T = T_k$, the set $A(T)$, and the graph H defined above. To show that $A(T)$ is a $\gamma_d^{oi}(T)$-set of cardinality $(2n + l - s + 2)/3$ we use the induction on the number k of operations performed to construct T. If $T = T_1 = P_2$, then $(2n + l - s + 2)/3 = 2 = \gamma_d^{oi}(T)$. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let n' mean the order of the tree T', l' the number of its leaves, and s' the number of support vertices. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. We have $n = n' + 1$. It is easy to see that $A(T) = A(T') \cup \{v\}$ is DOIDS of the tree T. Of course, $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 1$. If $T' = P_2$, then $l = l'$ and $s = s' - 1$. We get $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 1 = (2n' + l' - s' + 2)/3 + 1 = (2n + l - s + 2)/3$. If $T' \neq P_2$, then $l = l' + 1$ and $s = s'$. Consequently, $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 1 = (2n' + l' - s' + 2)/3 + 1 = (2n + l - s + 2)/3$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. We have $n = n' + 3$, $l = l' + 1$, and $s = s' + 1$. It is easy to see that $A(T) = A(T') \cup \{w, z\}$ is a DOIDS of the tree T. Let us observe that $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 2$. Consequently, $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 2 = (2n' + l' - s' + 2)/3 = (2n + l - s + 2)/3$.

Now assume that T is obtained from T' by operation $\mathcal{O}_3$. We have $n = n' + 3$, $l = l'$, and $s = s'$. Similarly as when considering operation $\mathcal{O}_2$ we conclude that $A(T)$ is a DOIDS of the tree T and $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 2$. Consequently, $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 2 = (2n' + l' - s' + 2)/3 + 2 = (2n + l - s + 2)/3$. ∎

Now we establish the main result, a lower bound on the double outer-independ-ent domination number of a tree together with the characterization of the extremal trees.

Theorem 4. *If T is a tree of order n, with l leaves and s support vertices, then $\gamma_d^{oi}(T) \geq (2n+l-s+2)/3$ with equality if and only if $T \in \mathcal{T}$.*

Proof. If $\mathrm{diam}(T) = 1$, then $T = P_2$. Thus $T \in \mathcal{T}$, and by Lemma 3 we have $\gamma_d^{oi}(T) = (2n+l-s+2)/3$. Now assume that $\mathrm{diam}(T) = 2$. Thus T is a star $K_{1,m}$. If $T = P_3$, then $T \in \mathcal{T}$ as it can be obtained from P_2 by operation $\mathcal{O}_1$. If T is different than P_3, then it is easy to see that T can be obtained from P_3 by a proper number of operations $\mathcal{O}_1$. Thus every star T belongs to the family $\mathcal{T}$, and by Lemma 3 we have $\gamma_d^{oi}(T) = (2n+l-s+2)/3$. Now assume that $\mathrm{diam}(T) = 3$. Thus T is a double star. Observations 1 and 2 imply that every DOIDS of the tree T contains all leaves and all support vertices. Therefore the set $V(T)$ is the only one DOIDS of the tree T. This implies that $\gamma_d^{oi}(T) = n$. We have $l = n-2$ and $s = 2$. Consequently, $(2n+l-s+2)/3 = (2n+n-2-2+2)/3 = (3n-2)/3 = n-2/3 < n = \gamma_d^{oi}(T)$, whence $T \notin \mathcal{T}$.

Now assume that $\mathrm{diam}(T) \geq 4$. Thus the order of the tree T is an integer $n \geq 5$. If $T \in \mathcal{T}$, then by Lemma 3 we have $\gamma_d^{oi}(T) = (2n+l-s+2)/3$. The result we obtain by the induction on the number n. Assume that the theorem is true for every tree T' of order $n' < n$, with l' leaves and s' support vertices.

First assume that some support vertex of T, say x, is adjacent to at least two leaves. One of them let us denote by y. Let $T' = T - y$. We have $n' = n-1$, $l' = l-1$, and $s' = s$. Of course, $\gamma_d^{oi}(T') = \gamma_d^{oi}(T) - 1$. Now we get $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 1 \geq (2n'+l'-s'+2)/3 + 1 = (2n-2+l-1-s+2+3)/3 = (2n+l-s+2)/3$. If $\gamma_d^{oi}(T) = (2n+l-s+2)/3$, then obviously $\gamma_d^{oi}(T') = (2n'+l'-s'+2)/3$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Henceforth, we assume that every support vertex of T is adjacent to exactly one leaf.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, u be the parent of v, and w be the parent of u in the rooted tree. By T_x let us denote the subtree induced by a vertex x and its descendants in the rooted tree T. We distinguish between the following two cases: $d_T(u) \geq 3$ and $d_T(u) = 2$.

Case 1. $d_T(u) \geq 3$. First assume that u has a child $b \neq v$ that is a support vertex. Let $T' = T - T_v$. We have $n' = n-2$, $l' = l-1$, and $s' = s-1$. Let D be any $\gamma_d^{oi}(T)$-set. By Observations 1 and 2 we have $t, v, b \in D$. If $u \in D$, then it is easy to observe that $D \setminus \{v, t\}$ is a DOIDS of the tree T'. Now assume that $u \notin D$. We have $d_T(u) \geq 3$, thus $d_{T'}(u) \geq 2$. Since $V(T) \setminus D$ is independent, every neighbor of u belongs to the set D.

Let us observe that $D \setminus \{v, t\}$ is a DOIDS of the tree T' as u has at least two neighbors in $D \setminus \{v, t\}$. Now we conclude that $\gamma_d^{oi}(T') \leq \gamma_d^{oi}(T) - 2$. We get $\gamma_d^{oi}(T) \geq \gamma_d^{oi}(T') + 2 \geq (2n' + l' - s' + 2)/3 + 2 = (2n - 4 + l - 1 - s + 1 + 2 + 6)/3 = (2n + l - s + 4)/3 > (2n + l - s + 2)/3$.

Now assume that v is the only one support vertex among the descendants of u. Thus u is a parent of a leaf, say x. Let $T' = T - T_x$. We have $n' = n - 1$, $l' = l - 1$, and $s' = s - 1$. Let D be a $\gamma_d^{oi}(T)$-set. By Observations 1 and 2 we have $x, u, v \in D$. It is easy to observe that $D \setminus \{x\}$ is a DOIDS of the tree T'. This implies that $\gamma_d^{oi}(T') \leq \gamma_d^{oi}(T) - 1$. Now we get $\gamma_d^{oi}(T) \geq \gamma_d^{oi}(T') + 1 \geq (2n' + l' - s' + 2)/3 + 1 = (2n - 2 + l - 1 - s + 1 + 2 + 3)/3 = (2n + l - s + 3)/3 > (2n + l - s + 2)/3$, whence $T \notin \mathcal{T}$.

Case 2. $d_T(u) = 2$. The parent of w let us denote by d. Let D be any $\gamma_d^{oi}(T)$-set. By Observations 1 and 2 we have $t, v \in D$. If $u \notin D$, then $w \in D$ as $V(T) \setminus D$ is independent. Let $T' = T - T_u$. We have $n' = n - 3$. It is easy to see that $D \setminus \{v, t\}$ is a DOIDS of the tree T'. Now assume that $u \in D$. If $w \in D$, then no neighbor of w besides u belongs to the set D, otherwise $D \setminus \{u\}$ is a DOIDS of the tree T, a contradiction to the minimality of D. It is easy to observe that $D \cup \{d\} \setminus \{u, v, t\}$ is a DOIDS of the tree T'. If $w \notin D$, then it is easy to see that $D \cup \{w\} \setminus \{u, v, t\}$ is a DOIDS of the tree T'. Now we conclude that $\gamma_d^{oi}(T') \leq \gamma_d^{oi}(T) - 2$. We consider the following two possibilities: $d_T(w) = 2$ and $d_T(w) \geq 3$.

First assume that $d_T(w) = 2$. We have $l' = l$. If d is adjacent to a leaf in T, then $s' = s - 1$. Consequently, $\gamma_d^{oi}(T) \geq \gamma_d^{oi}(T') + 2 \geq (2n' + l' - s' + 2)/3 + 2 = (2n - 6 + l - s + 1 + 2 + 6)/3 = (2n + l - s + 3)/3 > (2n + l - s + 2)/3$. Now assume that d is not adjacent to any leaf in T. Thus $s' = s$. Now we get $\gamma_d^{oi}(T) \geq \gamma_d^{oi}(T') + 2 \geq (2n' + l' - s' + 2)/3 + 2 = (2n - 6 + l - s + 2 + 6)/3 = (2n + l - s + 2)/3$. If $\gamma_d^{oi}(T) = (2n + l - s + 2)/3$, then obviously $\gamma_d^{oi}(T') = (2n' + l' - s' + 2)/3$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Now assume that $d_T(w) \geq 3$. We have $l' = l - 1$ and $s' = s - 1$. Now we get $\gamma_d^{oi}(T) \geq \gamma_d^{oi}(T') + 2 \geq (2n' + l' - s' + 2)/3 + 2 = (2n - 6 + l - 1 - s + 1 + 2 + 6)/3 = (2n + l - s + 2)/3$. If $\gamma_d^{oi}(T) = (2n + l - s + 2)/3$, then obviously $\gamma_d^{oi}(T') = (2n' + l' - s' + 2)/3$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$. ∎

Since the number of leaves of a tree is greater than or equal to the number of its support vertices, we get the following corollary.

Corollary 5. *For every tree T we have $\gamma_d^{oi}(T) \geq (2n + 2)/3$.*

Now we characterize the trees attaining this bound. For this purpose we introduce a family $\mathcal{F}$ of trees $T = T_k$ that can be obtained as follows. Let T_1 be a path P_2 with vertices labeled x and y, and let $B(T_1) = \{x, y\}$. Let H be a path P_3 with leaves labeled u and z, and the support vertex labeled w. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{X}_1$: Attach a copy of H by joining u to a vertex of $B(T_k)$ which has degree at least two. Let $B(T_{k+1}) = B(T_k) \cup \{w, z\}$.
- Operation $\mathcal{X}_2$: Attach a copy of H by joining u to a leaf of T_k which is the only one leaf among neighbors of its neighbor. Let $B(T_{k+1}) = B(T_k) \cup \{w, z\}$.

Now we prove that for every tree T of the family $\mathcal{F}$, the set $B(T)$ defined above is a DOIDS of minimum cardinality equal to $(2n + 2)/3$.

Lemma 6. *If $T \in \mathcal{F}$, then the set $B(T)$ defined above is a $\gamma_d^{oi}(T)$-set of size $(2n + 2)/3$.*

Proof. The definitions of the families $\mathcal{T}$ and $\mathcal{F}$ imply that $\mathcal{F} \subseteq \mathcal{T}$. Thus $T \in \mathcal{T}$. By Lemma 3, the set $A(T) = B(T)$ is a $\gamma_d^{oi}(T)$-set of size $(2n + l - s + 2)/3$. Obviously, for $T_1 = P_2$ we have $l = s$. Let us observe that performing neither the operation $\mathcal{X}_1$ nor the operation $\mathcal{X}_2$ disturbs the equality $l = s$. Therefore $l = s$, and consequently, $(2n + l - s + 2)/3 = (2n + 2)/3$. ■

Now we prove a lower bound on the double outer-independent domination number of a tree in terms of the number of vertices, together with the characterization of the extremal trees.

Theorem 7. *If T is a tree of order n, then $\gamma_d^{oi}(T) \geq (2n + 2)/3$ with equality if and only if $T \in \mathcal{F}$.*

Proof. The bound is true by Corollary 5. If $T \in \mathcal{F}$, then by Lemma 6 we have $\gamma_d^{oi}(T) = (2n + 2)/3$. Now assume that for a tree T we have $\gamma_d^{oi}(T) = (2n + 2)/3$. The number of leaves of every tree is greater than or equal to the number of its support vertices, thus $l \geq s$. By Theorem 4 we have $\gamma_d^{oi}(T) \geq (2n + l - s + 2)/3$. This implies that $l = s$. We have $\gamma_d^{oi}(T) = (2n + 2)/3 = (2n + l - s + 2)/3$. By Theorem 4 we have $T \in \mathcal{T}$. Suppose that T is obtained from $T_1 = P_2$ in a way such that the operation $\mathcal{O}_1$ is used at least once. Let us observe that $l > s$ as $l(P_2) = s(P_2)$, the operation $\mathcal{O}_1$ increases l not changing s, and the operations $\mathcal{O}_2$ and $\mathcal{O}_3$ do not disturb the equality $l = s$. This is a contradiction to that $l = s$. Therefore the operation $\mathcal{O}_1$ was not used to obtain the tree T. Since the operations $\mathcal{O}_2$ and $\mathcal{O}_3$ are identical to operations $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively, we conclude that $T \in \mathcal{F}$. ■

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