Postprint of: Naumenko K., Eremeev V., A non-linear direct peridynamics plate theory, COMPOSITE STRUCTURES, Vol. 279 (2021), 114728,
DOI: 10.1016/j.compstruct.2021.114728

A non-linear direct peridynamics plate theory

Konstantin Naumenko [a,*], Victor A. Eremeyev [b,c,d]

[a] Otto-von-Guericke-University Magdeburg, Institute of Mechanics, PF 4120, Magdeburg, D-39016, Germany
[b] Faculty of Civil and Environmental Engineering Gdansk University of Technology, ul. Gabriela Narutowicza 11/12, Gdansk, 80-233, Poland
[c] DICAAR, University of Cagliari, Via Marengo 2, Cagliari, 09123, Italy
[d] Don State Technical University, Gagarina sq. 1, Rostov on Don, 344000, Russia

ARTICLE INFO

Dedicated to Prof. Dr.-Ing. habil. Dr. h. c. mult. Holm Altenbach on the occasion of his 65th birthday

Keywords:
Thin plates
Peridynamics
Non-local theory
Balance equations
Bond moment density
Bond force density

ABSTRACT

In this paper a direct non-local peridynamics theory for thin plates is developed. Peridynamic points are assumed to behave like rigid bodies with independent translation and finite rotation degrees of freedom. The non-local mechanical interaction between points is characterized by force and moment vectors. The balance equations including the linear momentum, the angular momentum and the energy are presented. Peridynamic deformation states of the plate are introduced including the actual bond vector (relative translation of two points within the bond) and the relative finite rotation tensor (actual relative orientation of two points in the bond). The corresponding power-conjugate bond force and bond moment states are derived. The framework to develop constitutive equations for the plate peridynamic states is addressed. Special cases of the theory including plates with zero drilling moments, membranes, soft interlayers as well as stiff plate layers with small relative rotations are considered.

1. Introduction

Peridynamics is a nonlocal continuum theory in which the material points interact through long-range forces and moments [1,2]. In contrast to the classical continuum mechanics, where contact forces and moments are introduced, the balance equations do not include partial derivatives with respect to spatial coordinates. This makes the peridynamic theory very attractive in modeling highly heterogeneous deformation processes such as fracture including crack initiation, crack growth, crack interactions, crack spiders, etc. Recent review on various applications of peridynamics in solid mechanics is presented in [3].

Thin-walled components are frequently subjected to severe mechanical loading such that the analysis of post-critical states is of practical importance. Among many examples, laminated glass panels and photovoltaic modules should be designed to sustain dynamic impact loadings. These structures are composed from glass skin layers and a core layer from soft polymers [4,5]. The combination of brittle and ductile materials leads to the optimal strength properties of the laminate. The analysis of strength under impact loading would require to consider the complex interaction of different damage mechanisms. Many theories for modeling such laminates are developed within the classical local continuum mechanics [6–8]. Although they are able to reproduce deformation under static and dynamic loadings with high accuracy, damage processes, such as cracks in glass layers are not discussed. In order to model delamination, long-range forces between the interlayer and substrate acting outside the contact area should be taken into account [9,10].

In the classical continuum mechanics two principal approaches to develop a plate theory were established. The first one considers the plate as an approximate model of the three-dimensional theory. The unknown displacements and/or stresses are approximated by continuous and/or piecewise functions with respect to the thickness coordinate. Then the governing equations of the three-dimensional theory are reduced to the two-dimensional plate equations by means of variational methods, asymptotic techniques or through-the thickness integration. The classical example is the Reissner first order shear deformation plate theory derived by means of a mixed variational principle with the assumed stress approximations [11,12]. Further examples include zig-zag through-the thickness approximations of displacements which are used in the theory of laminates [13–15].

Another approach considers a plate as an independent model of continuum mechanics without reductions from the three-dimensional theory. The basic assumption is that the material points of the plate behave like rigid bodies and interact by means of contact forces and moments. Originally introduced by Mindlin in [16] to derive the first order shear deformation theory of plates, this approach found application in many works on non-linear theories of plates and shells [17–20] as well as layer-wise theories of laminates [6,7].

* Corresponding author.
E-mail addresses: konstantin.naumenko@ovgu.de (K. Naumenko), eremeyev.victor@gmail.com (V.A. Eremeyev).

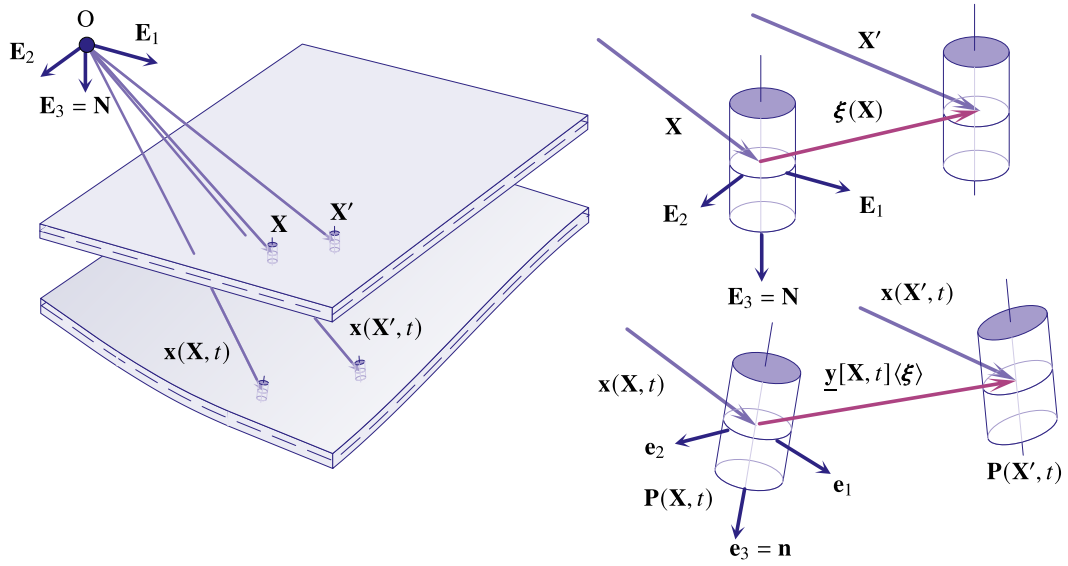

Fig. 1. Reference and actual configurations of a plate layer plate.

Peridynamic plate theories can also be developed by either the reducing three-dimensional equations or by applying the direct approach. The plate model presented in [21] is derived as a two-dimensional approximation of the three-dimensional bond-based theory of peridynamics via an asymptotic analysis. A Mindlin-type peridynamics plate theory with linear translations and rotations is developed in [22,23] applying the variational principle. A two-dimensional linear micropolar peridynamics theory is presented in [24]. In all these developments only linear equations are considered, such that rotations of peridynamic points are assumed small. In view of the application of peridynamics to the analysis of non-local deformation and damage processes in thin plates, a non-linear theory with finite rotations can be required.

In this paper we discuss a framework to the development of the non-linear direct peridynamics plate theory. Towards this goal we address the following problems

- In spirit of the classical direct approaches to the plate theories, peridynamic plate points will be assumed to behave like rigid bodies with translation and finite rotation degrees of freedom. The mechanical interaction between the points will be modeled by non-local forces and moments
- Balance equations of continuum mechanics including linear momentum, angular momentum and energy will be formulated directly for the plate reference surface
- Peridynamics deformation states of the plate will be introduced. These are the actual bond vector and the tensor of relative bond rotation. The corresponding power conjugate bond force and bond moment states will be proposed
- Special cases of the developed theory will be considered. They include plates with zero drilling moments, membranes, soft interlayers as well as stiff layers with linearized relative rotations

2. Preliminaries

Fig. 1 illustrates a plate layer with Cartesian base vectors $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3 = \mathbf{N}$ to specify the position vectors for points in the reference state. Let $\mathbf{X}$ be the position vector of a point in a basis plane, for example midplane, of the plate layer in the reference configuration. In a finite neighborhood consider a point with the vector $\mathbf{X}'$. Following the notation of peridynamics [1,2] a bond vector is specified by $\boldsymbol{\xi} = \mathbf{X}' - \mathbf{X}$. In the actual configuration the corresponding vectors are denoted by $\mathbf{x}$ and $\mathbf{x}'$ respectively. The actual state for the bond $\boldsymbol{\xi}$ will be specified by the state vector $\underline{\mathbf{y}}[\boldsymbol{X}, t]\langle\boldsymbol{\xi}\rangle$. To capture the orientation of peridynamic points in the actual configuration let us introduce the orthonormal vectors $\mathbf{e}_i$ and the rotation tensor $\mathbf{P}(\mathbf{X}, t) = \mathbf{e}_k \otimes \mathbf{E}_k$ such that

$$\mathbf{e}_k = \mathbf{P} \cdot \mathbf{E}_k \tag{1}$$

Here and in the sequel the direct tensor calculus in the sense of Gibbs [25] and Lagally [26] will be applied. A second rank tensor is a finite sum of dyads of vectors $\mathbf{A} = \mathbf{a} \otimes \mathbf{b} + \mathbf{c} \otimes \mathbf{d} + \cdots + \mathbf{e} \otimes \mathbf{f}$. The basic operations for dyads can be summarized as follows

$$(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} = \alpha \mathbf{a}, \quad \alpha = \mathbf{b} \cdot \mathbf{c}, \tag{2}$$

$$\mathbf{c} \cdot (\mathbf{a} \otimes \mathbf{b}) = \beta \mathbf{b}, \quad \beta = \mathbf{c} \cdot \mathbf{a}, \tag{3}$$

$$\mathbf{a} \otimes \mathbf{b} \cdot\cdot\, \mathbf{c} \otimes \mathbf{d} = \alpha\beta, \quad \alpha = \mathbf{b} \cdot \mathbf{c}, \quad \beta = \mathbf{a} \cdot \mathbf{d}, \tag{4}$$

$$(\mathbf{a} \otimes \mathbf{b}) \times \mathbf{c} = \mathbf{a} \otimes \mathbf{d}, \quad \mathbf{d} = \mathbf{a} \times \mathbf{b}, \tag{5}$$

$$\mathbf{c} \times (\mathbf{a} \otimes \mathbf{b}) = \mathbf{e} \otimes \mathbf{b}, \quad \mathbf{e} = \mathbf{c} \times \mathbf{a}, \tag{6}$$

$$\mathrm{tr}\,(\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}, \quad (\mathbf{a} \otimes \mathbf{b})_\times = \mathbf{a} \times \mathbf{b} \tag{7}$$

Hereinafter "$\cdot$", "$\times$", and "$\otimes$" stand for inner, cross and dyadic products, respectively. Operations (2)–(7) are generalized for finite sums of dyads. The direct tensor calculus is widely used in continuum mechanics and rheology, e.g. [27–29].

Any peridynamic point of the plate is considered as a rigid body with both translation and rotation degrees of freedom. The linear velocity vector $\mathbf{v}$ and the spatial angular velocity vector $\boldsymbol{\omega}$ are defined as follows [30,31]

$$\mathbf{v} = \dot{\mathbf{x}}, \quad \dot{\mathbf{P}} = \boldsymbol{\omega} \times \mathbf{P}, \quad -2\boldsymbol{\omega} = (\dot{\mathbf{P}} \cdot \mathbf{P}^{\mathrm{T}})_\times \tag{8}$$

In Eq. (8) the overdot stands for the derivative with respect to time t. The Gibbs cross $(\ldots)_\times$ is computed applying Eq. $(7)_2$. Since the tensor $\mathbf{P}$ is orthogonal, the tensors $\dot{\mathbf{P}} \cdot \mathbf{P}^{\mathrm{T}}$ and $\dot{\mathbf{P}}^{\mathrm{T}} \cdot \mathbf{P}$ are skew-symmetric with the following representations

$$\dot{\mathbf{P}} \cdot \mathbf{P}^{\mathrm{T}} = \boldsymbol{\omega} \times \boldsymbol{I}, \quad \dot{\mathbf{P}}^{\mathrm{T}} \cdot \mathbf{P} = \boldsymbol{\varpi} \times \boldsymbol{I}, \tag{9}$$

where $\boldsymbol{I}$ is the second rank unit tensor. The body angular velocity vector $\boldsymbol{\varpi}$ is defined as follows

$$\boldsymbol{\varpi} = \mathbf{P}^{\mathrm{T}} \cdot \boldsymbol{\omega}, \quad \dot{\mathbf{P}} = \mathbf{P} \times \boldsymbol{\varpi} \tag{10}$$

Let $\varrho_{\mathrm{R}}(\mathbf{X}) \geq 0$ be the surface mass density of the plate in the reference configuration. The corresponding mass density in the actual configuration is specified by $\varrho(\mathbf{X}, t) \geq 0$. The mass of the elementary part of the plate is determined as follows

$$\mathrm{d}m = \varrho_{\mathrm{R}} \mathrm{d}A_{\mathrm{R}} = \varrho \mathrm{d}A, \tag{11}$$

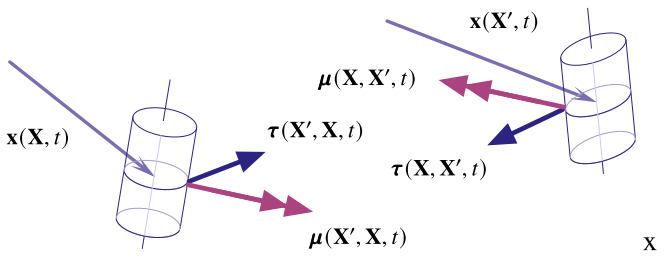

Fig. 2. Force and moment vectors for the points $\mathbf{x}$ and $\mathbf{x}'$.

where $\mathrm{d}A_{\mathrm{R}}$ and $\mathrm{d}A$ are infinitesimal area elements in the reference and actual configurations, respectively. The mass density in peridynamic theory is, in general assumed to be non-local due to non-locality of deformation and damage states. As mentioned in [32] mass distribution could be based on the measure theory. Let us introduce the relative area a as follows

$$a = \frac{\varrho_{\mathrm{R}}}{\varrho} \tag{12}$$

3. Balance equations for linear and angular momentum

The linear momentum $\mathbf{p}$ for a basis surface part $\mathcal{P}$ in the actual configuration is defined as follows

$$\mathbf{p}(\mathcal{P}) = \int_{(\mathcal{P})} \varrho \boldsymbol{\pi} \mathrm{d}A, \quad \mathbf{v} = \dot{\mathbf{x}}, \quad \boldsymbol{\pi} = \mathbf{v} + \boldsymbol{\Theta}_1 \cdot \boldsymbol{\omega} \tag{13}$$

For the angular momentum $\mathbf{L}$ with respect to point O of the reference frame we obtain

$$\mathbf{L}(\mathcal{P}) = \int_{(\mathcal{P})} \varrho(\boldsymbol{\lambda} + \mathbf{x} \times \boldsymbol{\pi}) \mathrm{d}A, \quad \boldsymbol{\lambda} = \boldsymbol{\Theta}_1^{\mathrm{T}} \cdot \mathbf{v} + \boldsymbol{\Theta}_2 \cdot \boldsymbol{\omega}, \tag{14}$$

$\boldsymbol{\Theta}_1$, $\boldsymbol{\Theta}_1^{\mathrm{T}} = -\boldsymbol{\Theta}_1$ and $\boldsymbol{\Theta}_2$, $\boldsymbol{\Theta}_2^{\mathrm{T}} = \boldsymbol{\Theta}_2$ are the tensors of inertia. They characterize the distribution of mass for the points of the plate. Specifying the reference mass density by ρ_{R} the following relations can be obtained [17]

$$\varrho \boldsymbol{\Theta}_1^{\mathrm{T}} = \frac{\theta_1}{a} \mathbf{I} \times \mathbf{n}, \quad \varrho \boldsymbol{\Theta}_2 = \frac{\theta_2}{a} (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}), \quad \mathbf{n} = \mathbf{P} \cdot \mathbf{N}, \tag{15}$$

where

$$\varrho_{\mathrm{R}} = \int_{(h)} \rho_{\mathrm{R}} \mathrm{d}Z, \quad \theta_1 = \int_{(h)} \rho_{\mathrm{R}} Z \mathrm{d}Z, \quad \theta_2 = \int_{(h)} \rho_{\mathrm{R}} Z^2 \mathrm{d}Z,$$

and Z is the coordinate along the thickness direction $\mathbf{N}$ in the reference configuration. The mechanical interactions between points $\mathbf{x}$ and $\mathbf{x}'$ are characterized by forces and moments, Fig. 2. The force and the moment vectors acting on $\mathbf{X}$ from $\mathbf{X}'$ are specified by

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\mathbf{X}', \mathbf{X}, t), \quad \boldsymbol{\mu} = \boldsymbol{\mu}(\mathbf{X}', \mathbf{X}, t)$$

while the corresponding vectors acting on $\mathbf{X}'$ from $\mathbf{X}$ are denoted as follows

$$\boldsymbol{\tau}' = \boldsymbol{\tau}(\mathbf{X}, \mathbf{X}', t), \quad \boldsymbol{\mu}' = \boldsymbol{\mu}(\mathbf{X}, \mathbf{X}', t)$$

Fig. 3 illustrates a free-body diagram for an infinitesimal part $\mathrm{d}A$ of the plate. The resultant force and moment vectors acting on a finite part $\mathcal{P}$ of the surface from the remaining part $\mathcal{R} = A \backslash \mathcal{P}$ are computed as follows

$$\begin{aligned} \boldsymbol{\mathcal{F}}(\mathcal{R}, \mathcal{P}) &= \int_{(\mathcal{P})} \int_{(\mathcal{R})} \boldsymbol{\tau} \mathrm{d}A' \mathrm{d}A, \\ \boldsymbol{\mathcal{M}}(\mathcal{R}, \mathcal{P}) &= \int_{(\mathcal{P})} \int_{(\mathcal{R})} (\mathbf{x} \times \boldsymbol{\tau} + \boldsymbol{\mu}) \mathrm{d}A' \mathrm{d}A \end{aligned} \tag{16}$$

The body force density $\mathbf{f}$ and the body moment density be $\boldsymbol{\mu}$ are introduced to model the mechanical action of environment on the plate. The integral forms of the balances of momentum and angular momentum for the part $\mathcal{P}$ read

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{(\mathcal{P})} \varrho \boldsymbol{\pi} \mathrm{d}A &= \int_{(\mathcal{P})} \int_{(\mathcal{R})} \boldsymbol{\tau} \mathrm{d}A' \mathrm{d}A + \int_{(\mathcal{P})} \varrho \mathbf{f} \mathrm{d}A, \\ \frac{\mathrm{d}}{\mathrm{d}t} \int_{(\mathcal{P})} \varrho(\boldsymbol{\lambda} + \mathbf{x} \times \boldsymbol{\pi}) \mathrm{d}A &= \int_{(\mathcal{P})} \int_{(\mathcal{R})} (\mathbf{x} \times \boldsymbol{\tau} + \boldsymbol{\mu}) \mathrm{d}A' \mathrm{d}A + \int_{(\mathcal{P})} \varrho(\mathbf{x} \times \mathbf{f} + \mathbf{m}) \mathrm{d}A \end{aligned} \tag{17}$$

Consider two disjoint finite parts $\mathcal{P}_I$ and $\mathcal{P}_{II}$ and the part $\mathcal{P} = \mathcal{P}_I \bigcup \mathcal{P}_{II}$. Subtracting the balance equations (17) for the parts $\mathcal{P}_I$ and $\mathcal{P}_{II}$ from the corresponding equations for the part $\mathcal{P}$ leads to[1]

$$\begin{aligned} \int_{(\mathcal{P}_I)} \int_{(\mathcal{P}_{II})} (\boldsymbol{\tau} + \boldsymbol{\tau}') \mathrm{d}A' \mathrm{d}A &= \mathbf{0}, \\ \int_{(\mathcal{P}_I)} \int_{(\mathcal{P}_{II})} (\mathbf{x} \times \boldsymbol{\tau} + \boldsymbol{\mu} + \mathbf{x}' \times \boldsymbol{\tau}' + \boldsymbol{\mu}') \mathrm{d}A' \mathrm{d}A &= \mathbf{0} \end{aligned} \tag{18}$$

After localization we obtain

$$\boldsymbol{\tau}' = -\boldsymbol{\tau}, \quad \mathbf{x}' \times \boldsymbol{\tau}' + \boldsymbol{\mu}' = -(\mathbf{x} \times \boldsymbol{\tau} + \boldsymbol{\mu}) \tag{19}$$

Furthermore, by setting $\mathcal{P}_I = \mathcal{P}_{II} = \mathcal{P}$ the balances lead to

$$\begin{aligned} \int_{(\mathcal{P})} \int_{(\mathcal{P})} \boldsymbol{\tau} \mathrm{d}A' \mathrm{d}A &= \mathbf{0}, \\ \int_{(\mathcal{P})} \int_{(\mathcal{P})} (\mathbf{x} \times \boldsymbol{\tau} + \boldsymbol{\mu}) \mathrm{d}A' \mathrm{d}A &= \mathbf{0} \end{aligned} \tag{20}$$

With Eqs. (20) the resultant force and resultant moment (16) can be computed as follows

$$\begin{aligned} \boldsymbol{\mathcal{F}}(\mathcal{R}, \mathcal{P}) = \boldsymbol{\mathcal{F}}(A, \mathcal{P}) &= \int_{(\mathcal{P})} \int_{(A)} \boldsymbol{\tau} \mathrm{d}A' \mathrm{d}A, \\ \boldsymbol{\mathcal{M}}(\mathcal{R}, \mathcal{P}) = \boldsymbol{\mathcal{M}}(A, \mathcal{P}) &= \int_{(\mathcal{P})} \int_{(A)} (\mathbf{x} \times \boldsymbol{\tau} + \boldsymbol{\mu}) \mathrm{d}A' \mathrm{d}A \end{aligned} \tag{21}$$

With Eqs. (11) and (21), the balance equations (17) can be put into the following local form

$$\begin{aligned} \varrho \frac{\mathrm{d}\boldsymbol{\pi}}{\mathrm{d}t} &= \int_{(A)} \boldsymbol{\tau} \mathrm{d}A' + \varrho \mathbf{f}, \\ \varrho \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{\lambda} + \mathbf{x} \times \boldsymbol{\pi}) &= \int_{(A)} (\mathbf{x} \times \boldsymbol{\tau} + \boldsymbol{\mu}) \mathrm{d}A' + \varrho(\mathbf{x} \times \mathbf{f} + \mathbf{m}) \end{aligned} \tag{22}$$

Alternatively the following local equations can be derived

$$\begin{aligned} \varrho_{\mathrm{R}} \frac{\mathrm{d}\boldsymbol{\pi}}{\mathrm{d}t} &= \int_{(A_{\mathrm{R}})} \tilde{\mathbf{T}} \mathrm{d}A'_{\mathrm{R}} + \varrho_{\mathrm{R}} \mathbf{f}, \\ \varrho_{\mathrm{R}} \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{\lambda} + \mathbf{x} \times \boldsymbol{\pi}) &= \int_{(A_{\mathrm{R}})} (\mathbf{x} \times \tilde{\mathbf{T}} + \tilde{\mathbf{M}}) \mathrm{d}A'_{\mathrm{R}} + \varrho_{\mathrm{R}}(\mathbf{x} \times \mathbf{f} + \mathbf{m}) \end{aligned} \tag{23}$$

with the force and moment vectors per unit area of the reference configuration

$$\tilde{\mathbf{T}} = a a' \boldsymbol{\tau}, \quad \tilde{\mathbf{M}} = a a' \boldsymbol{\mu}$$

4. Energy balance equations

Let us derive the non-local analogue to the kinetic energy theorem for the plate. To this end the scalar products of linear momentum Eq. $(22)_1$ with $\mathbf{v}$ and of angular momentum Eq. $(22)_2$ with $\boldsymbol{\omega}$ are computed. By adding the results and after the integration over the part of the plate surface we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{K}(\mathcal{P}) = \int_{(\mathcal{P})} \int_{(A)} \ell \mathrm{d}A' \mathrm{d}A + \int_{(\mathcal{P})} \varrho(\mathbf{f} \cdot \mathbf{v} + \mathbf{m} \cdot \boldsymbol{\omega}) \mathrm{d}A, \tag{24}$$

where the kinetic energy is defined as follows

$$\mathcal{K}(\mathcal{P}) = \int_{(\mathcal{P})} \varrho \left(\frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \boldsymbol{\Theta}_1 \cdot \boldsymbol{\omega} + \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{\Theta}_2 \cdot \boldsymbol{\omega} \right) \mathrm{d}A, \tag{25}$$

and the power of force and moment vectors is

$$\ell = \boldsymbol{\tau} \cdot \mathbf{v} + \boldsymbol{\mu} \cdot \boldsymbol{\omega} \tag{26}$$

[1] Details of derivations for three-dimensional solids are presented in [1].

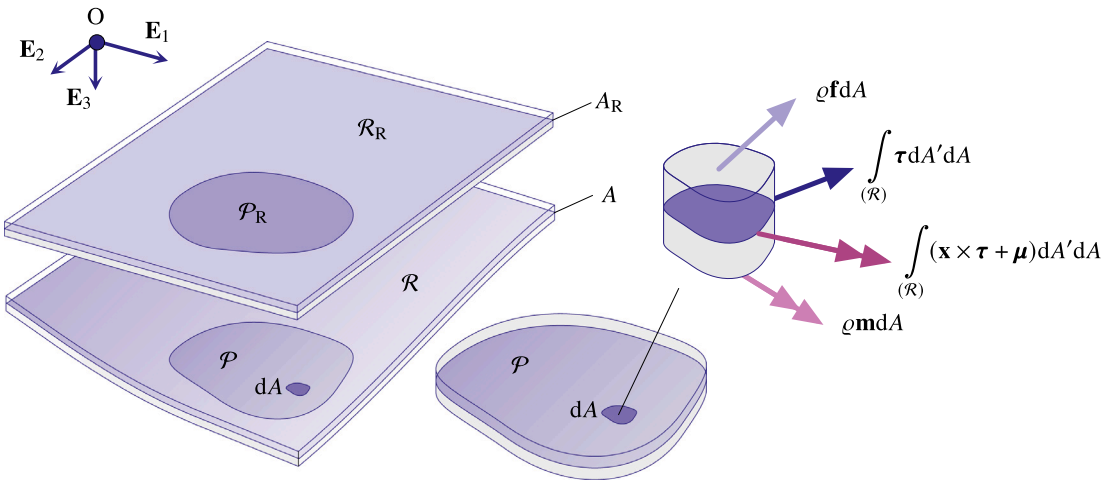

Fig. 3. Forces and moments vectors for an element $\mathrm{d}A$ of the plate.

It can be decomposed into the absorbed (mean) power of the bond ℓ_a and the excessive (supplied from $\mathbf{x}'$ to $\mathbf{x}$) power ℓ_s as follows

$$\begin{aligned}
&\ell(\mathbf{x}',\mathbf{x}) = -\ell_a(\mathbf{x}',\mathbf{x}) + \ell_s(\mathbf{x}',\mathbf{x}),\\
&\ell_a(\mathbf{x}',\mathbf{x}) = \ell_a(\mathbf{x},\mathbf{x}') = -\frac{1}{2}\left(\boldsymbol{\tau}\cdot\mathbf{v} + \boldsymbol{\tau}'\cdot\mathbf{v}' + \boldsymbol{\mu}\cdot\boldsymbol{\omega} + \boldsymbol{\mu}'\cdot\boldsymbol{\omega}'\right),\\
&\ell_s(\mathbf{x}',\mathbf{x}) = -\ell_s(\mathbf{x}',\mathbf{x}) = \frac{1}{2}\left(\boldsymbol{\tau}\cdot\mathbf{v} - \boldsymbol{\tau}'\cdot\mathbf{v}' + \boldsymbol{\mu}\cdot\boldsymbol{\omega} - \boldsymbol{\mu}'\cdot\boldsymbol{\omega}'\right)
\end{aligned} \tag{27}$$

For the mechanical power supplied to the part of the plate surface $\mathcal{P}$ from the remaining part $\mathcal{R} = A\backslash\mathcal{P}$ we obtain

$$\mathcal{L}_s(\mathcal{R},\mathcal{P}) = \int_{(\mathcal{P})}\int_{(\mathcal{R})} \ell_s \mathrm{d}A'\mathrm{d}A \tag{28}$$

From the asymmetry of ℓ_s it follows

$$\mathcal{L}_s(\mathcal{R},\mathcal{P}) = -\mathcal{L}_s(\mathcal{P},\mathcal{R}), \quad \mathcal{L}_s(\mathcal{P},\mathcal{P}) = 0, \quad \mathcal{L}_s(A,\mathcal{P}) = \mathcal{L}_s(\mathcal{R},\mathcal{P}) \tag{29}$$

With Eqs. (27)–(29), Eq. (24) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}(\mathcal{P}) = -\mathcal{L}_a(A,\mathcal{P}) + \mathcal{L}_s(\mathcal{R},\mathcal{P}) + \mathcal{L}_b(\mathcal{P}), \tag{30}$$

where

$$\begin{aligned}
&\mathcal{L}_a(A,\mathcal{P}) = \int_{(\mathcal{P})}\int_{(A)} l_a \mathrm{d}A'\mathrm{d}A, \quad \mathcal{L}_a(A,\mathcal{P}) = \mathcal{L}_a(\mathcal{P},\mathcal{P}) + \mathcal{L}_a(\mathcal{R},\mathcal{P}),\\
&\mathcal{L}_a(\mathcal{R},\mathcal{P}) = \mathcal{L}_a(\mathcal{P},\mathcal{R})
\end{aligned} \tag{31}$$

is the absorbed power and

$$\mathcal{L}_b(\mathcal{P}) = \int_{(\mathcal{P})} \varrho(\mathbf{f}\cdot\mathbf{v} + \mathbf{m}\cdot\boldsymbol{\omega})\mathrm{d}A \tag{32}$$

is the power of body forces and moments. The energy balance equation can be formulated as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\mathcal{P}) = \int_{(\mathcal{P})}\int_{(\mathcal{R})} (\ell_s + q)\mathrm{d}A'\mathrm{d}A + \int_{(\mathcal{P})} \varrho(\mathbf{f}\cdot\mathbf{v} + \mathbf{m}\cdot\boldsymbol{\omega} + r)\mathrm{d}A, \tag{33}$$

where the total energy $\mathcal{E}$ and the internal energy $\mathcal{U}$ are represented as follows

$$\begin{aligned}
&\mathcal{E}(\mathcal{P}) = \mathcal{K}(\mathcal{P}) + \mathcal{U}(\mathcal{P}), \quad \mathcal{U}(\mathcal{P}) = \int_{(\mathcal{P})}\int_{(A)} u_{\mathbf{B}} \mathrm{d}A'\mathrm{d}A,\\
&2u_{\mathbf{B}}(\mathbf{x}',\mathbf{x}) = u(\mathbf{x}',\mathbf{x}) + u(\mathbf{x},\mathbf{x}')
\end{aligned}$$

with $u_{\mathbf{B}}$ being the bond energy density. $q(\mathbf{x}',\mathbf{x})$ is the rate of heat supply from $\mathbf{x}'$ to $\mathbf{x}$ and r is the heat source at $\mathbf{x}$. Subtracting Eq. (25) from (33) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}(\mathcal{P}) = \int_{(\mathcal{P})}\int_{(A)} \ell_a \mathrm{d}A'\mathrm{d}A + \int_{(\mathcal{P})}\int_{(\mathcal{R})} q\mathrm{d}A'\mathrm{d}A + \int_{(\mathcal{P})} \varrho r \mathrm{d}A, \tag{34}$$

Applying Eq. (34) to two disjoint finite parts $\mathcal{P}_I$ and $\mathcal{P}_{II}$ as well as to the part $\mathcal{P} = \mathcal{P}_I \bigcup \mathcal{P}_{II}$ results in

$$\int_{(\mathcal{P}_I)}\int_{(\mathcal{P}_{II})} (q + q')\mathrm{d}A'\mathrm{d}A = 0, \quad q + q' = 0 \tag{35}$$

Specifying the heat flux from $\mathcal{R}$ to $\mathcal{P}$ by $\mathcal{Q}(\mathcal{R},\mathcal{P})$ we obtain

$$\mathcal{Q}(\mathcal{R},\mathcal{P}) = \int_{(\mathcal{P})}\int_{(\mathcal{R})} q\mathrm{d}A'\mathrm{d}A, \quad \mathcal{Q}(\mathcal{P},\mathcal{R}) = -\mathcal{Q}(\mathcal{R},\mathcal{P}), \quad \mathcal{Q}(\mathcal{P},\mathcal{P}) = 0 \tag{36}$$

With Eqs. (35) and (36) the following local form of Eq. (34) can be derived

$$\int_{(A)} \left(\dot{u}_{\mathrm{B}} + u_{\mathrm{B}}\dot{\alpha}\right)\mathrm{d}A' = \int_{(A)} (\ell_a + q)\mathrm{d}A' + \varrho r, \quad \dot{\alpha} = \left(\frac{\dot{a}}{a} + \frac{\dot{a}'}{a'}\right) \tag{37}$$

With respect to the reference configuration the local form of the energy balance is

$$\begin{aligned}
&\int_{(A_{\mathrm{R}})} \dot{U}_{\mathrm{B}}\mathrm{d}A'_{\mathrm{R}} = \int_{(A_{\mathrm{R}})} (L_a + Q)\mathrm{d}A'_{\mathrm{R}} + \varrho_{\mathrm{R}} r, \quad U_{\mathrm{B}} = aa' u_{\mathrm{B}},\\
&L_a = aa'\ell_a, \quad Q = aa' q
\end{aligned} \tag{38}$$

5. Plate peridynamics states

Following the peridynamics theory [1] the bond vector $\boldsymbol{\xi} = \mathbf{X}' - \mathbf{X}$ connected to the point $\mathbf{X}$ is introduced, Fig. 1. For the considered plate theory the bond vectors belong to the circular neighborhoods $\mathcal{H}_{\mathrm{R}}$ within the basis plane such that $\boldsymbol{\xi}\cdot\mathbf{N} = 0$. By analogy to the force and moment tensors in the classical plate theory and following the notation of peridynamics [1,33] let us introduce the force $\underline{\tilde{\mathbf{T}}}[\mathbf{X},t]$ and moment $\underline{\tilde{\mathbf{M}}}[\mathbf{X},t]$ vector state fields such that

$$\begin{aligned}
&\tilde{\mathbf{T}}(\mathbf{X}',\mathbf{X},t) = \underline{\tilde{\mathbf{T}}}[\mathbf{X},t]\langle\boldsymbol{\xi}\rangle - \underline{\tilde{\mathbf{T}}}[\mathbf{X}',t]\langle-\boldsymbol{\xi}\rangle, \quad \boldsymbol{\xi} = \mathbf{X}' - \mathbf{X}\\
&\tilde{\mathbf{M}}(\mathbf{X}',\mathbf{X},t) = \underline{\tilde{\mathbf{M}}}[\mathbf{X},t]\langle\boldsymbol{\xi}\rangle - \underline{\tilde{\mathbf{M}}}[\mathbf{X}',t]\langle-\boldsymbol{\xi}\rangle + \underline{\mathbf{y}}[\mathbf{X},t]\langle\boldsymbol{\xi}\rangle \times \underline{\tilde{\mathbf{T}}}[\mathbf{X},t]\langle\boldsymbol{\xi}\rangle,\\
&\underline{\mathbf{y}}[\mathbf{X},t]\langle\boldsymbol{\xi}\rangle = \boldsymbol{x}[\mathbf{X}+\boldsymbol{\xi},t] - \boldsymbol{x}[\mathbf{X},t]
\end{aligned} \tag{39}$$

With Eqs. (39) the absorbed power of forces and moments in (21) is computed as follows

$$\begin{aligned}
2L_a &= -(\tilde{\mathbf{T}}\cdot\mathbf{v} + \tilde{\mathbf{T}}'\cdot\mathbf{v}' + \tilde{\mathbf{M}}\cdot\boldsymbol{\omega} + \tilde{\mathbf{M}}'\cdot\boldsymbol{\omega}')\\
&= \underline{\tilde{\mathbf{T}}}\cdot(\underline{\dot{\mathbf{y}}} - \boldsymbol{\omega}\times\underline{\mathbf{y}}) + \underline{\tilde{\mathbf{T}}}'\cdot(\underline{\dot{\mathbf{y}}}' - \boldsymbol{\omega}'\times\underline{\mathbf{y}}') + \underline{\tilde{\mathbf{M}}}\cdot(\boldsymbol{\omega}' - \boldsymbol{\omega}) + \underline{\tilde{\mathbf{M}}}'\cdot(\boldsymbol{\omega} - \boldsymbol{\omega}')
\end{aligned} \tag{40}$$

Let us define the following tensor state of relative rotations

$$\underline{\mathbf{R}}[\mathbf{X},t]\langle\boldsymbol{\xi}\rangle = \mathbf{P}^{\mathrm{T}}[\mathbf{X},t]\cdot\mathbf{P}[\mathbf{X}+\boldsymbol{\xi},t], \quad \underline{\dot{\mathbf{R}}} = \underline{\boldsymbol{\Omega}}\times\underline{\mathbf{R}} \tag{41}$$

With the following relative angular velocity vector state we obtain

$$\underline{\boldsymbol{\Omega}}[\mathbf{X},t]\langle\boldsymbol{\xi}\rangle = \mathbf{P}^{\mathrm{T}}[\mathbf{X},t]\cdot(\boldsymbol{\omega}' - \boldsymbol{\omega}) \tag{42}$$

With Eqs. (41) and (42) the absorbed power (40) can be computed as follows

$$2L_a = \Lambda + \Lambda', \quad \Lambda = \underline{\mathbf{T}}\cdot\underline{\dot{\mathbf{Y}}} + \underline{\mathbf{M}}\cdot\underline{\boldsymbol{\Omega}}, \quad \Lambda' = \underline{\mathbf{T}}'\cdot\underline{\dot{\mathbf{Y}}}' + \underline{\mathbf{M}}'\cdot\underline{\boldsymbol{\Omega}}', \tag{43}$$

where

$$\underline{\mathbf{Y}} = \mathbf{P}^{\mathrm{T}} \cdot \underline{\mathbf{y}} \tag{44}$$

is the deformation state rotated to the reference configuration and

$$\underline{\mathbf{T}} = \mathbf{P}^{\mathrm{T}} \cdot \underline{\tilde{\mathbf{T}}}, \quad \underline{\mathbf{M}} = \mathbf{P}^{\mathrm{T}} \cdot \underline{\tilde{\mathbf{M}}} \tag{45}$$

are rotated force and moment vector states. In the classical local theory of shells and plates the corresponding states are defined by Piola–Kirchhoff type force and moment tensors [17,34].

Let us decompose deformation state into the plane part $\underline{\mathbf{Y}}_{\mathrm{p}}$ and the transverse part $\underline{Y}_{\mathrm{q}}$ as follows

$$\underline{\mathbf{Y}} = \underline{\mathbf{Y}}_{\mathrm{p}} + \underline{Y}_{\mathrm{q}}\mathbf{N}, \quad \underline{\mathbf{Y}}_{\mathrm{p}} \cdot \mathbf{N} = 0 \tag{46}$$

Fig. 4 illustrates the deformation states $\underline{\mathbf{y}}$ and $\underline{\mathbf{Y}}$ as well as the corresponding plane and transverse components.

The force state can be decomposed into the in-plane (membrane) $\underline{\mathbf{T}}_{\mathrm{p}}$ and the transverse (shear) $\underline{T}_{\mathrm{q}}$ states as follows

$$\underline{\mathbf{T}} = \underline{\mathbf{T}}_{\mathrm{p}} + \underline{T}_{\mathrm{q}}\mathbf{N}, \quad \underline{\mathbf{T}}_{\mathrm{p}} \cdot \mathbf{N} = 0 \tag{47}$$

For the angular velocity and moment states the decompositions read

$$\underline{\boldsymbol{\Omega}} = \underline{\boldsymbol{\Omega}}_{\mathrm{p}} + \underline{\Omega}_{\mathrm{N}}\mathbf{N}, \quad \underline{\mathbf{M}} = \underline{\mathbf{M}}_{\mathrm{p}} + \underline{M}_{\mathrm{N}}\mathbf{N} \tag{48}$$

With the introduced plate states the power of internal forces Λ reads

$$\Lambda = \underline{\mathbf{T}}_{\mathrm{p}} \cdot \dot{\underline{\mathbf{Y}}}_{\mathrm{p}} + \underline{T}_{\mathrm{q}}\dot{\underline{Y}}_{\mathrm{q}} + \underline{\mathbf{M}}_{\mathrm{p}} \cdot \underline{\boldsymbol{\Omega}}_{\mathrm{p}} \tag{49}$$

6. Constitutive equations for elastic materials

Constitutive equations of the plate theory should relate the in-plane force state $\underline{\mathbf{T}}_{\mathrm{p}}$, the out-of-plane force state $\underline{T}_{\mathrm{q}}$ and the moment state $\underline{\mathbf{M}}$ with corresponding deformation states, $\underline{\mathbf{Y}}_{\mathrm{p}}$, $\underline{Y}_{\mathrm{q}}$, $\underline{\mathbf{R}}$, their rates and a set of state variables, such as hardening or damage to consider the loading history. In this paper we consider a class of constitutive equations for elastic plates. To this end we assume that the introduced forces and moment states depend on the current deformation states only. Furthermore we assume that the force states have a potential. As usual in the continuum mechanics we introduce the stored bond deformation energy density W_{B}, as a part of the internal energy U_{B} such that

$$\int_{(A_{\mathrm{R}})} \dot{W}_{\mathrm{B}} \mathrm{d}A'_{\mathrm{R}} = \int_{(A_{\mathrm{R}})} L_{\mathrm{a}} \mathrm{d}A'_{\mathrm{R}}, \qquad 2W_{\mathrm{B}} = W + W' \tag{50}$$

The strain energy density for a point of the plate is defined by

$$\mathcal{W} = \int_{(A_{\mathrm{R}})} W_{\mathrm{B}} \mathrm{d}A'_{\mathrm{R}} \tag{51}$$

The strain energy density of the plate is computed as follows

$$\mathcal{W}(\mathcal{A}_{\mathrm{R}}) = \int_{(A_{\mathrm{R}})}\int_{(A_{\mathrm{R}})} W_{\mathrm{B}} \mathrm{d}A'_{\mathrm{R}} \mathrm{d}A_{\mathrm{R}} = \int_{(A_{\mathrm{R}})}\int_{(A_{\mathrm{R}})} W \mathrm{d}A'_{\mathrm{R}} \mathrm{d}A_{\mathrm{R}} \tag{52}$$

For elastic materials the stored energy density W is a function of the deformation states, i.e.

$$W = \hat{W}(\underline{\mathbf{Y}}_{\mathrm{p}}, \underline{Y}_{\mathrm{q}}, \underline{\mathbf{R}}), \quad W' = \hat{W}'(\underline{\mathbf{Y}}'_{\mathrm{p}}, \underline{Y}'_{\mathrm{q}}, \underline{\mathbf{R}}') \tag{53}$$

The time derivative is computed as follows

$$\dot{W} = W_{\underline{\mathbf{Y}}_{\mathrm{p}}} \cdot \dot{\underline{\mathbf{Y}}}_{\mathrm{p}} + W_{\underline{Y}_{\mathrm{q}}}\dot{\underline{Y}}_{\mathrm{q}} + (\underline{\mathbf{R}} \cdot W_{\underline{\mathbf{R}}})_{\times} \cdot \underline{\boldsymbol{\Omega}}, \tag{54}$$

where $W_{\underline{\mathbf{Y}}_{\mathrm{p}}}$, $W_{\underline{Y}_{\mathrm{q}}}$ and $W_{\underline{\mathbf{R}}}$ are the Fréchet derivatives of W with respect to the deformation states, as defined in [1]. With Eqs. (54), (50), (43) and (49) we obtain

$$\begin{aligned}
&\int_{(A_{\mathrm{R}})} \left\{ \left(\underline{\mathbf{T}}_{\mathrm{p}} - W_{\underline{\mathbf{Y}}_{\mathrm{p}}}\right) \cdot \dot{\underline{\mathbf{Y}}}_{\mathrm{p}} + \left(\underline{T}_{\mathrm{q}} - W_{\underline{Y}_{\mathrm{q}}}\right)\dot{\underline{Y}}_{\mathrm{q}} + \left(\underline{\mathbf{M}} - \left(\underline{\mathbf{R}} \cdot W_{\underline{\mathbf{R}}}\right)\right)_{\times} \cdot \underline{\boldsymbol{\Omega}} \right\} \mathrm{d}A'_{\mathrm{R}} \\
&+ \int_{(A_{\mathrm{R}})} \left\{ \left(\underline{\mathbf{T}}'_{\mathrm{p}} - W'_{\underline{\mathbf{Y}}'_{\mathrm{p}}}\right) \cdot \dot{\underline{\mathbf{Y}}}'_{\mathrm{p}} + \left(\underline{T}'_{\mathrm{q}} - W'_{\underline{Y}'_{\mathrm{q}}}\right)\dot{\underline{Y}}'_{\mathrm{q}} + \left(\underline{\mathbf{M}}' - \left(\underline{\mathbf{R}}' \cdot W'_{\underline{\mathbf{R}}}\right)\right)_{\times} \cdot \underline{\boldsymbol{\Omega}}' \right\} \mathrm{d}A'_{\mathrm{R}} = 0
\end{aligned} \tag{55}$$

Eq. (55) is a linear form with respect to velocities. Taking into account that the velocities can be arbitrary, the following constitutive equations follow from Eq. (55)

$$\underline{\mathbf{T}}_{\mathrm{p}} = W_{\underline{\mathbf{Y}}_{\mathrm{p}}}, \quad \underline{T}_{\mathrm{q}} = W_{\underline{Y}_{\mathrm{q}}}, \quad \underline{\mathbf{M}} = \left(\underline{\mathbf{R}} \cdot W_{\underline{\mathbf{R}}}\right)_{\times} \tag{56}$$

By specifying the strain energy density, various cases of constitutive equations can be derived.

7. Final system of equations and special cases

The balance equations for linear momentum $(23)_1$ and angular momentum $(23)_2$ can be rewritten as follows

$$\varrho_{\mathrm{R}}\ddot{\boldsymbol{\pi}} = \int_{(A_{\mathrm{R}})} \tilde{\mathbf{T}} \mathrm{d}A'_{\mathrm{R}} + \varrho_{\mathrm{R}}\mathbf{f}, \quad \varrho_{\mathrm{R}}(\dot{\boldsymbol{\lambda}} + \mathbf{v} \times \boldsymbol{\pi}) = \int_{(A_{\mathrm{R}})} \tilde{\mathbf{M}} \mathrm{d}A'_{\mathrm{R}} + \varrho_{\mathrm{R}}\mathbf{m} \tag{57}$$

Applying Eqs. (13) and (14) as well as the tensors of inertia (15), Eqs. (57) take the following form

$$\begin{aligned}
\varrho_{\mathrm{R}}\mathbf{P}^{\mathrm{T}} \cdot \ddot{\boldsymbol{\pi}} &= \int_{(\mathcal{H}_{\mathrm{R}})} \left(\underline{\mathbf{T}}[\mathbf{X}, t]\langle \mathbf{X} - \mathbf{X}' \rangle - \underline{\mathbf{T}}[\mathbf{X}', t]\langle \mathbf{X}' - \mathbf{X} \rangle \right) \mathrm{d}A'_{\mathrm{R}} + \varrho_{\mathrm{R}}\mathbf{P}^{\mathrm{T}} \cdot \mathbf{f}, \\
\varrho_{\mathrm{R}}\mathbf{P}^{\mathrm{T}} \cdot (\dot{\boldsymbol{\lambda}} + \mathbf{v} \times \boldsymbol{\pi}) &= \int_{(\mathcal{H}_{\mathrm{R}})} \left(\underline{\mathbf{M}}[\mathbf{X}, t]\langle \mathbf{X} - \mathbf{X}' \rangle - \underline{\mathbf{M}}[\mathbf{X}', t]\langle \mathbf{X}' - \mathbf{X} \rangle \right) \mathrm{d}A'_{\mathrm{R}} \\
&\quad + \int_{(\mathcal{H}_{\mathrm{R}})} \left(\underline{\mathbf{Y}}[\mathbf{X}, t]\langle \mathbf{X} - \mathbf{X}' \rangle \times \underline{\mathbf{T}}[\mathbf{X}, t]\langle \mathbf{X} - \mathbf{X}' \rangle \right) \mathrm{d}A'_{\mathrm{R}} + \varrho_{\mathrm{R}}\mathbf{P}^{\mathrm{T}} \cdot \mathbf{m},
\end{aligned} \tag{58}$$

The rates of change of linear and angular momentum vectors in (58) can be computed as follows

$$\begin{aligned}
&\mathbf{P}^{\mathrm{T}} \cdot \ddot{\boldsymbol{\pi}} = \dot{\boldsymbol{\pi}}_{\mathrm{R}} + \boldsymbol{\varpi} \times \boldsymbol{\pi}_{\mathrm{R}}, \\
&\mathbf{P}^{\mathrm{T}} \cdot (\dot{\boldsymbol{\lambda}} + \mathbf{v} \times \boldsymbol{\pi}) = \dot{\boldsymbol{\lambda}}_{\mathrm{R}} + \boldsymbol{\varpi} \times \boldsymbol{\lambda}_{\mathrm{R}} + \boldsymbol{\upsilon} \times \boldsymbol{\pi}_{\mathrm{R}}
\end{aligned} \tag{59}$$

with

$$\begin{aligned}
&\boldsymbol{\pi}_{\mathrm{R}} = \boldsymbol{\upsilon} + \boldsymbol{\Theta}_{1\mathrm{R}} \cdot \boldsymbol{\varpi}, \quad \boldsymbol{\lambda}_{\mathrm{R}} = \boldsymbol{\Theta}_{1\mathrm{R}}^{T} \cdot \boldsymbol{\upsilon} + \boldsymbol{\Theta}_{2\mathrm{R}} \cdot \boldsymbol{\varpi}, \\
&\boldsymbol{\upsilon} = \mathbf{P}^{\mathrm{T}} \cdot \boldsymbol{v}, \quad \varrho_{\mathrm{R}}\boldsymbol{\Theta}_{1\mathrm{R}}^{\mathrm{T}} = \theta_1 \mathbf{I} \times \mathbf{N}, \quad \varrho_{\mathrm{R}}\boldsymbol{\Theta}_{2\mathrm{R}} = \theta_2(\mathbf{I} - \mathbf{N} \otimes \mathbf{N})
\end{aligned} \tag{60}$$

7.1. Plates with negligible drilling moments

In the classical local theories of plates and shells the drilling moments, i.e. the moments about the normals to the basis plane $\mathbf{N}$ are neglected. This assumption is valid for plates made from non-polar materials [19]. Analysis of shells with drilling moments is presented in [35].

The corresponding peridynamics theory can be formulated under following assumptions

$$\int_{(\mathcal{H}_{\mathrm{R}})} (\underline{M}_{\mathrm{N}} - \underline{M}'_{\mathrm{N}}) \mathrm{d}A'_{\mathrm{R}} = 0, \quad m_{\mathrm{N}} = \mathbf{m} \cdot \mathbf{n} = 0 \tag{61}$$

The first equation in (61) sets the resultant drilling interaction moment zero, while the second equation in (61) is the restriction for the external drilling moment. Instead of Eq. (61) the following more strong condition for the bond moment density vector can be applied

$$\underline{M}_{\mathrm{N}} = 0 \tag{62}$$

By computing the scalar product of Eq. $(58)_2$ with $\mathbf{N}$ and taking into account Eqs. (61) we obtain

$$\int_{(\mathcal{H}_{\mathrm{R}})} (\underline{\mathbf{Y}} \times \underline{\mathbf{T}}) \cdot \mathbf{N} \mathrm{d}A'_{\mathrm{R}} = 0$$

Applying the decompositions (46) and (47) results in

$$\int_{(\mathcal{H}_{\mathrm{R}})} (\underline{\mathbf{Y}}_{\mathrm{p}} \times \underline{\mathbf{T}}_{\mathrm{p}}) \cdot \mathbf{N} \mathrm{d}A'_{\mathrm{R}} = 0 \tag{63}$$

Eq. (62) can be satisfied by setting

$$\underline{\mathbf{Y}}_{\mathrm{p}} \times \underline{\mathbf{T}}_{\mathrm{p}} = \mathbf{0} \tag{64}$$

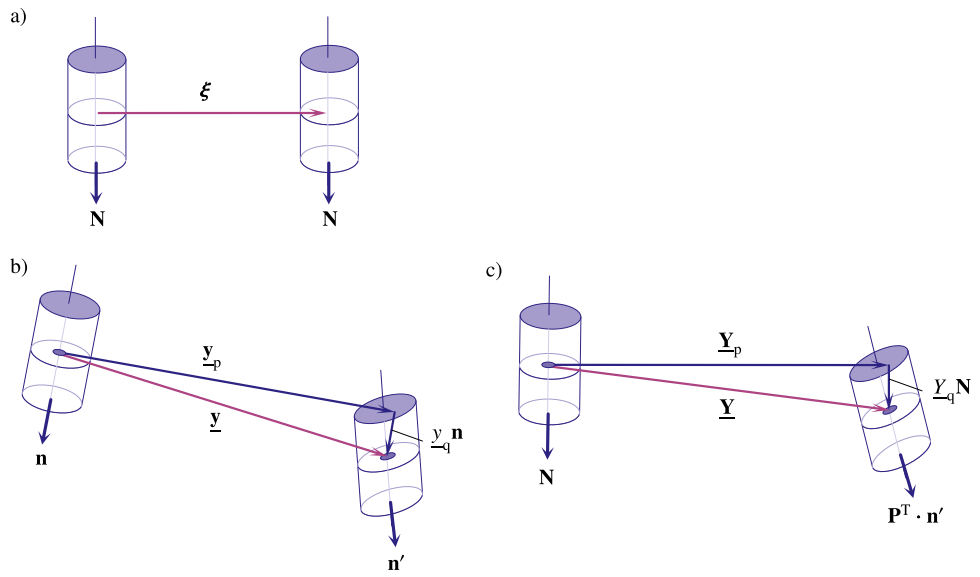

Fig. 4. Deformation states of a bond. (a) Reference configuration, (b) actual configuration, (c) actual configuration rotated by $\mathbf{P}^{\mathrm{T}}$.

From Eq. (64) it follows that the membrane part of the force state vector is collinear to the in-plane deformation state, i.e.

$$\underline{\mathbf{T}}_{\mathrm{p}} = \lambda \underline{\mathbf{Y}}_{\mathrm{p}}, \tag{65}$$

where λ is a scalar function to be defined by the constitutive law. The strain energy density is then a function of the length of the vector $\underline{\mathbf{Y}}_{\mathrm{p}}$. Following the classification introduced in [33] this case of the peridynamic plate theory may be termed semi-ordinary. Indeed the total force state is not collinear with the total deformation state while for the in-plane parts the restriction (64) is valid. The balance of angular momentum $(58)_2$ takes the form

$$\varrho_{\mathrm{R}}\mathbf{P}^{\mathrm{T}} \cdot (\dot{\boldsymbol{\lambda}} + \mathbf{v} \times \boldsymbol{\pi}) = \int_{(\mathcal{H}_{\mathrm{R}})} \left(\underline{\mathbf{M}}_{\mathrm{p}} - \underline{\mathbf{M}}'_{\mathrm{p}} \right) \mathrm{d}A'_{\mathrm{R}}$$
$$+ \int_{(\mathcal{H}_{\mathrm{R}})} \left(\underline{Y}_{\mathrm{q}} \mathbf{N} \times \underline{\mathbf{T}}_{\mathrm{p}} + \underline{\mathbf{Y}}_{\mathrm{p}} \times \mathbf{N} \underline{T}_{\mathrm{q}} \right) \mathrm{d}A'_{\mathrm{R}} + \varrho_{\mathrm{R}}\mathbf{P}^{\mathrm{T}} \cdot \mathbf{m}_{\mathrm{p}}, \tag{66}$$

where the external moment vector is the plane vector, i.e. $\mathbf{m}_{\mathrm{p}} \cdot \mathbf{n} = 0$

7.2. Membranes

Membranes do not support the moments. In this case both the resultant moments of interaction and external moments can be set to zero

$$\int_{(\mathcal{H}_{\mathrm{R}})} (\underline{\mathbf{M}} - \underline{\mathbf{M}}') \mathrm{d}A'_{\mathrm{R}} = \mathbf{0}, \quad \mathbf{m} = \mathbf{0} \tag{67}$$

Furthermore, for thin membranes the rotary inertia is neglected by setting

$$\theta_1 = \theta_2 = 0 \tag{68}$$

Eq. (67) can be replaced by the following more strong condition for the bond moment density vector

$$\underline{\mathbf{M}} = \mathbf{0} \tag{69}$$

With the assumptions (67) and (68), the balance Eq. (66) of angular momentum reduces to the following peristatic equation.

$$\int_{(\mathcal{H}_{\mathrm{R}})} \left(\underline{\mathbf{Y}} \times \underline{\mathbf{T}} \right) \mathrm{d}A'_{\mathrm{R}} = \mathbf{0} \tag{70}$$

Eq. (70) can be satisfied by setting

$$\underline{\mathbf{Y}} \times \underline{\mathbf{T}} = \mathbf{0} \tag{71}$$

The restriction (71) implies that the bond force density state vector is collinear with the deformation state vector. This is classified to be the ordinary peridynamic theory of membranes. The rotation tensor can be set to the reference value, for example $\mathbf{P} = \mathbf{P}_0 = \mathbf{I}$.

For membranes subjected to in-plane loads only, the transverse force and transverse deformation states can be set to zero. Peridynamic theory of membranes is discussed in detail in [36].

7.3. Soft interlayers

Soft interlayers in laminated glass panels and photovoltaic modules are subjected to transverse shear deformation, while the bending/twisting moments are usually neglected [4,37,38]. Layer-wise plate and shell theories were developed to analyze deformation and stress states of glass laminates [6,7]. In view of the application of peridynamics to soft interlayers let us consider this special case. The resultant moment state can be neglected

$$\int_{(\mathcal{H}_{\mathrm{R}})} (\underline{\mathbf{M}} - \underline{\mathbf{M}}') \mathrm{d}A'_{\mathrm{R}} = \mathbf{0} \tag{72}$$

The more strong condition is to neglect the bond moment state

$$\underline{\mathbf{M}} = \mathbf{0} \tag{73}$$

For soft interlayers the rotary inertia can be neglected by applying Eqs. (68). Then instead of Eq. $(58)_2$ the following peristatic equation can be applied

$$\int_{(\mathcal{H}_{\mathrm{R}})} \underline{\mathbf{Y}} \times \underline{\mathbf{T}} \mathrm{d}A'_{\mathrm{R}} + \varrho_{\mathrm{R}}\mathbf{P}^{\mathrm{T}} \cdot \mathbf{m} = \mathbf{0} \tag{74}$$

Equations for the interlayer, for example in a glass laminate are solved together with equations for skin glass layers. Then the external moment vector $\mathbf{m}$ and the rotation tensor $\mathbf{P}$ can be determined.

7.4. Small relative rotations

For stiff plate layers the relative rotations within a small horizon can be assumed small. In this case we can set

$$\underline{\mathbf{R}}[\mathbf{X}, t]\langle \boldsymbol{\xi} \rangle = \mathbf{I} + \underline{\boldsymbol{\psi}}[\mathbf{X}, t]\langle \boldsymbol{\xi} \rangle \times \mathbf{I}, \tag{75}$$

where the state variable $\underline{\boldsymbol{\psi}}[\mathbf{X}, t]\langle \boldsymbol{\xi} \rangle$ stands for the vector of small relative bond rotations. The angular velocity state is determined as the time derivative of the rotation vector

$$\underline{\boldsymbol{\Omega}} = \dot{\underline{\boldsymbol{\psi}}} \tag{76}$$

The rotations of peridynamic points can be still finite. For the rotation tensor we obtain

$$\mathbf{P}[\mathbf{X}+\boldsymbol{\xi},t]=\mathbf{P}[\mathbf{X},t]+\underline{\boldsymbol{\psi}}[\mathbf{X},t]\langle\boldsymbol{\xi}\rangle\times\mathbf{P}[\mathbf{X},t], \tag{77}$$

Finally, if the rotations are small than after linearizing Eq. (77) we obtain

$$\mathbf{P}[\mathbf{X},t]=\mathbf{I}+\boldsymbol{\varphi}[\mathbf{X},t]\times\mathbf{I},\quad \underline{\boldsymbol{\psi}}[\mathbf{X},t]\langle\boldsymbol{\xi}\rangle=\boldsymbol{\varphi}'-\boldsymbol{\varphi} \tag{78}$$

where $\boldsymbol{\varphi}$ is the vector of small rotations.

8. Conclusions and outlook

The aim of this paper was to derive a non-linear peridynamics plate theory. To this end a direct approach is applied assuming the peridynamic points to behave like rigid bodies. Balance equations are formulated for the plate basis plane in the reference configuration. The power conjugated bond force (moment) states and bond deformation states are introduced. The general form of elasticity constitutive equations is introduced for the in-plane bond density force vector, out-of-plane (transverse) force vector and the moment vector. By taking into account the symmetry group of the plate material the strain energy density as a function of invariants of the deformation states and the constitutive laws (bond or state based) can be specified. The developed equations can be used inside a layer-wise plate theory for modeling individual layers and non-local interactions between the layers. One feature of the developed theory is the rotation tensor state introduced to capture the relative orientation of peridynamic points inside the bond. In developing the numerical procedure to implement the theory the known parametrization of rotation tensors, e.g. finite rotation vector, quaternion, Tait–Bryan angles [39] can be applied. For plates with zero drilling moment the rotation tensor can be represented by the composition of two rotations about two fixed axes. Examples illustrating the applicability of the developed model will be introduced in a forthcoming paper.

CRediT authorship contribution statement

Konstantin Naumenko: Conceptualization, Investigation, Methodology, Writing – original draft. **Victor A. Eremeyev:** Conceptualization, Investigation, Methodology, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Silling SA, Lehoucq RB. Peridynamic theory of solid mechanics. Adv Appl Mech 2010;44:73–168.
[2] Silling SA. Introduction to peridynamics. In: Handbook of peridynamic modeling, Chapman and Hall/CRC; 2016, p. 63–98.
[3] Javili A, Morasata R, Oterkus E, Oterkus S. Peridynamics review. Math Mech Solids 2019;24:3714–39.
[4] Schulze S, Pander M, Naumenko K, Altenbach H. Analysis of laminated glass beams for photovoltaic applications. Int J Solids Struct 2012;49:2027–36.
[5] Paggi M, Kajari-Schröder S, Eitner U. Thermomechanical deformations in photovoltaic laminates. J Strain Anal Eng Des 2011;46:772–82.
[6] Naumenko K, Eremeyev VA. A layer-wise theory for laminated glass and photovoltaic panels. Compos Struct 2014;112:283–91.
[7] Naumenko K, Eremeyev VA. A layer-wise theory of shallow shells with thin soft core for laminated glass and photovoltaic applications. Compos Struct 2017;178:434–46.
[8] Foraboschi P. Analytical model for laminated-glass plate. Composites B 2012;43:2094–106.
[9] Nase M, Rennert M, Naumenko K, Eremeyev VA. Identifying traction–separation behavior of self-adhesive polymeric films from in situ digital images under t-peeling. J Mech Phys Solids 2016;91:40–55.
[10] Naumenko K, Bagheri B. A direct approach to evaluate interaction forces between self-adhesive polymeric films subjected to t-peeling. Arch Appl Mech 2021;91:629–41.
[11] E. Reissner. The effect of transverse shear deformation on the bending of elastic plates. J Appl Mech 1945;12:A69–77.
[12] E. Reissner. A variational theorem in elasticity. J Math Phys 1950;29:90–5.
[13] Carrera E. Historical review of Zig-Zag theories for multilayered plates and shells. Appl Mech Rev 2003;56:287–308.
[14] Carrera E, Brischetto S, Nali P. Plates and shells for smart structures: classical and advanced theories for modelling and analysis. Chichester: Wiley; 2011.
[15] Carrera E, Ciuffreda A. A unified formulation to assess theories of multilayered plates for various bending problems. Compos Struct 2005;69:271–93.
[16] Mindlin RD. Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates. Trans ASME J Appl Mech 1951;18:31–8.
[17] Altenbach H, Zhilin PA. The theory of simple elastic shells. Berlin, Heidelberg: Springer Berlin Heidelberg; 2004, p. 1–12.
[18] Altenbach H, Naumenko K, Zhilin PA. A direct approach to the formulation of constitutive equations for rods and shells. In: Pietraszkiewicz W, Szymczak C, editors. Shell structures: theory and applications. Leiden: Taylor & Francis; 2005, p. 87–90.
[19] Altenbach J, Altenbach H, Eremeyev VA. On generalized Cosserat-type theories of plates and shells: a short review and bibliography. Arch Appl Mech 2010;80:73–92.
[20] Green A, Naghdi P. On the derivation of shell theories by direct approach. J Appl Mech 1974;41:173.
[21] Taylor M, Steigmann DJ. A two-dimensional peridynamic model for thin plates. Math Mech Solids 2015;20:998–1010.
[22] Vazic B, Oterkus E, Oterkus S. Peridynamic model for a Mindlin plate resting on a Winkler elastic foundation. J Peridyn Nonlocal Model 2020;1–10.
[23] Yang Z, Oterkus E, Oterkus S. A novel peridynamic Mindlin plate formulation without limitation on material constants. J Peridyn Nonlocal Model 2021;1–20.
[24] Diana V, Casolo S. A full orthotropic micropolar peridynamic formulation for linearly elastic solids. Int J Mech Sci 2019;160:140–55.
[25] Wilson EB. Vector analysis. In: Founded upon the lectures of G.W. Gibbs, New Haven: Yale University Press; 1901.
[26] Lagally M. Vorlesungen über Vektorrechnung. Leipzig: Geest & Portig; 1962.
[27] Lebedev LP, Cloud MJ, Eremeyev VA. Tensor analysis with applications in mechanics. New Jersey: World Scientific; 2010.
[28] Naumenko K, Altenbach H. Modeling high temperature materials behavior for structural analysis. part I: continuum mechanics foundations and constitutive models. Switzerland: Springer International Publishing; 2016.
[29] Belytschko T, Liu WK, Moran B, Elkhodary K. Nonlinear finite elements for continua and structures. Wiley; 2014.
[30] Altenbach H, Naumenko K, Pylypenko S, Renner B. Influence of rotary inertia on the fiber dynamics in homogeneous creeping flows. ZAMM-J Appl Math Mech Z Angew Math Mech: Appl Math Mech 2007;87:81–93.
[31] Zhilin P. A new approach to the analysis of free rotations of rigid bodies. ZAMM-J Appl Math Mech Z Angew Math Mech: Appl Math Mech 1996;76:187–204.
[32] Truesdell C. A first course in rational continuum mechanics. New York: Academic Press; 1977.
[33] Silling SA, Epton M, Weckner O, Xu J, Askari E. Peridynamic states and constitutive modeling. J Elasticity 2007;88:151–84.
[34] Libai A, Simmonds JG. The nonlinear theory of elastic shells. Cambridge University Press; 2005.
[35] Pietraszkiewicz W. The resultant linear six-field theory of elastic shells: What it brings to the classical linear shell models? ZAMM-J Appl Math Mech Z Angew Math Mech: Appl Math Mech 2016;96:899–915.
[36] Silling SA, Bobaru F. Peridynamic modeling of membranes and fibers. Int J Non-Linear Mech 2005;40:395–409.
[37] Zemanová A, Zeman J, Šjnoha M. Comparison of viscoelastic finite element models for laminated glass beams. Int J Mech Sci 2017;131:380–95.
[38] Weps M, Naumenko K, Altenbach H. Unsymmetric three-layer laminate with soft core for photovoltaic modules. Compos Struct 2013;105:332–9.
[39] Pietraszkiewicz W, Eremeyev V. On vectorially parameterized natural strain measures of the non-linear Cosserat continuum. Int J Solids Struct 2009;46:2477–80.