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A NOTE ON A WIENER-WINTNER THEOREM FOR MEAN ERGODIC MARKOV AMENABLE SEMIGROUPS

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Abstract. We prove a Wiener-Wintner ergodic type theorem for a Markov representation $\mathcal{S} = \{S_g : g \in G\}$ of a right amenable semitopological semigroup G. We assume that $\mathcal{S}$ is mean ergodic as a semigroup of linear Markov operators acting on $(C(K), \|\cdot\|_{\sup})$, where K is a fixed Hausdorff, compact space. The main result of the paper is necessary and sufficient conditions for mean ergodicity of a distorted semigroup $\{\chi(g)S_g : g \in G\}$, where χ is a semigroup character. Such conditions were obtained before under the additional assumption that $\mathcal{S}$ is uniquely ergodic.

1. Introduction

The paper contributes towards a recently published paper [10] due to M. Schreiber. To avoid redundancy and keep the format of this note appropriately compact we generally follow definitions and notation from [10]. However, for the convenience of the reader we give a brief summary of the topic we deal with. Given a compact Hausdorff space K and the complex Banach lattice $C(K)$ of all continuous complex valued functions on K, a linear contraction operator $S : C(K) \to C(K)$ is called (strongly) mean ergodic if the sequence of its Cesàro means $\frac{1}{n}\sum_{j=1}^{n} S^j f$ converges uniformly on K (i.e. in the sup norm $\|\cdot\|$) to Qf. It is well known that the limit operator Q is a linear projection on the manifold $\mathrm{Fix}(S) = \{f \in C(K) : Sf = f\}$ of S-invariant functions. The characterization of mean ergodicity are today a classical part of operator ergodic theory and can be found in most monographs (cf. [4], pp. 138-139, [5], [8], pp. 72-81).

Let us recall that a linear operator $S : C(K) \to C(K)$ is called Markov if $Sf \geq 0$ for all (real valued) nonnegative $f \in C(K)_+$ and $S\mathbf{1} = \mathbf{1}$. Clearly, any Markov operator has norm 1, in particular it is a contraction. Given a semitopological semigroup G, a (bounded) representation of G on $C(K)$ is the semigroup of operators $\mathcal{S} = \{S_g : g \in G\}$ such that $S_{g_1 g_2} = S_{g_2} S_{g_1}$ and $G \ni g \to S_g f \in C(K)$ is norm continuous for every $f \in C(K)$ and $\sup_{g\in G} \|S_g\| < \infty$. If all S_g are Markovian, then the representation is called Markovian. A (complex) function $\chi : G \to \{z \in \mathbb{C} : |z| = 1\} = \mathbb{K}$ is called a semigroup character if it is continuous and $\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$ for all $g_1, g_2 \in G$. A semitopological semigroup G is called right amenable if the Banach lattice $(C_b(G), \|\cdot\|_{\sup})$ has a right invariant mean (i.e. there exists a positive linear functional m on $C_b(G)$ such that $\langle \mathbf{1}, m\rangle = 1$, and $\langle f, m\rangle = \langle f(\cdot g), m\rangle$ for all $g \in G$ and all $f \in C_b(G)$; cf. [3], [7]). In this paper we shall assume that considered semitopological semigroups G are right amenable.

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Extending the notion of Cesaro averages (cf. [2], [5], [6]) we say that a net $(A_\alpha^{\mathcal{S}})_\alpha$ of contraction operators on $C(K)$ is strong right $\mathcal{S}$-ergodic if $A_\alpha^{\mathcal{S}} \in \overline{\operatorname{conv}\mathcal{S}}^{\text{s.o.t}}$ (the closure is taken with repsect to the strong operator topology) and $\lim_\alpha \|A_\alpha^{\mathcal{S}} f - A_\alpha^{\mathcal{S}} S_g f\|_{\sup} = 0$ for all $g \in G$ and $f \in C(K)$. It is known (cf. [11], Proposition 1.3 and Theorem 1.4) that as long as the semigroup $\mathcal{S}$ is right amenable then its bounded representations admit strong right ergodic nets. The semigroup $\mathcal{S}$ is called mean ergodic if $\overline{\operatorname{conv}\mathcal{S}}^{s.o.t.}$ contains a (Markovian $\mathcal{S}$-absorbing projection) zero element Q (cf. [5], [8], pp. 80-81). We denote $\operatorname{Fix}(\mathcal{S}) = \{f \in C(K) : S_g f = f \text{ for all } g \in G\}$ and similarly $\operatorname{Fix}(\mathcal{S}') = \{\nu \in C(K)' : S'_g \nu = \nu \text{ for all } g \in G\}$. If for every non-zero $\nu \in \operatorname{Fix}(\mathcal{S}')$ there exists $f \in \operatorname{Fix}(\mathcal{S})$ such that $\langle f, \nu \rangle \neq 0$ then we say that $\operatorname{Fix}(\mathcal{S})$ separates $\operatorname{Fix}(\mathcal{S}')$. Let us recall a characterization of strong mean ergodicity for contraction (linear) semigroups (cf. [11] Theorem 1.7 and Corollary 1.8).

Proposition 1.1. *Let G be represented on $C(K)$ by a right amenable semigroup of contractions $\mathcal{S} = \{S_g : g \in G\}$. Then the following conditions are equivalent:*

(1) *$\mathcal{S}$ is mean ergodic with mean ergodic projection P,*
(2) $\operatorname{Fix}(\mathcal{S})$ *separates* $\operatorname{Fix}(\mathcal{S}')$*,*
(3) $C(K) = \operatorname{Fix}(\mathcal{S}) \oplus \overline{\operatorname{lin}\ \operatorname{rg}(I - \mathcal{S})}$*,*
(4) *$A_\alpha^{\mathcal{S}} f$ converges strongly (equivalently: weakly) to Qf for some/every strong right $\mathcal{S}$-ergodic net $A_\alpha^{\mathcal{S}}$ and all $f \in C(K)$.*

Let us return to the classical situation, when $G = \{0, 1, \dots\}$ is the semigroup of nonnegative integers and $S : C(K) \to C(K)$ is a Markov operator. If S is mean ergodic the following natural question arises (see [9]):

$$\text{Do the modified averages } \frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi itk} S^k f(x) \text{ converge uniformly on } K?$$

The answer was not immediate. The above problem was an inspiration for a new research topic (cf. [1], [9], [13]), the so-called Wiener-Wintner ergodic theorems. To be precise, let us emphasize that its roots lay in the much earlier paper [14] due to Wiener and Wintner. The question was originally raised in the context of measure preserving ergodic transformations. Recent works in the area deal with general semigroups and their representations. The weights $e^{2\pi itk}$, where $t \in \mathbb{R}$, are replaced by abstract characters. Very recently further generalizations (i.e. polynomial or multiple extensions) of the Wiener-Wintner theorem are intensively studied with several problems remaining open.

Given a semigroup character $\chi : G \to \mathbb{K}$ let ${}_\chi\mathcal{S}$ denote the semigroup $\{\chi(g)S_g : g \in G\}$. We repeat the question whether mean ergodicity of $\mathcal{S}$ is preserved when we pass to the distorted semigroup ${}_\chi\mathcal{S}$ (it is originally addressed in recent papers [10] and [11]). Let us emphasize that if G is right amenable, then there exist strong right ${}_\chi\mathcal{S}$-ergodic nets, as naturally ${}_\chi\mathcal{S}$ is a bounded representation of G. In particular, the question on mean ergodicity of ${}_\chi\mathcal{S}$ is well posed. Clearly, these semigroups are not Markovian in general.

A Markovian semigroup $\mathcal{S}$ is called uniquely ergodic if $\dim(\operatorname{Fix}(\mathcal{S}')) = 1$ (c.f. [2]). In this case there exists a unique probability measure $\mu \in C(K)'$ such that $S'_g \mu = \mu$ for all $g \in G$. Clearly unique ergodicity implies (cf. [10] Proposition 2.2 and [2]) that $\mathcal{S}$ is mean ergodic and $\operatorname{Fix}(\mathcal{S}) = \mathbb{C}\mathbf{1}$. Even for a Markovian

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representation $\mathcal{S}$ which is uniquely ergodic, it may happen that for some characters χ the semigroups ${}_\chi\mathcal{S}$ are not mean ergodic (cf. [9], [10], [13]). On the other hand, a necessary and sufficient condition guaranteeing mean ergodicity of ${}_\chi\mathcal{S}$ is formulated in [10] in terms of yet another semigroup ${}_\chi\mathcal{S}_2$.

It is well known that the domain of any Markov operator S may be extended by $(Sg(x) = \int g(y)S'\delta_x(dy))$ to all bounded and Borel measurable functions. If μ is an S'-invariant probability, then this canonical extension appears to be a positive linear contraction once acting on $L^2(\mu)$. Following [10] let $\mathcal{S}_2$ denote the positive semigroup of linear contractions S_g which are extended to $L^2(\mu)$. Similarly ${}_\chi\mathcal{S}_2$ stands for the semigroup $\{\chi(g)S_g : g \in G\}$. If we deal with several different $\mathcal{S}$-invariant measures μ, then we distinguish these semigroups by writing $S_{2,\mu}$ and ${}_\chi\mathcal{S}_{2,\mu}$ respectively. Clearly they are all contraction semigroups on $L^2(\mu)$.

By $P(K)$ we denote the convex and weakly* compact set of all probability (regular, Borel) measures on K. We set $\mathbb{P}_\mathcal{S} = \{\mu \in P(K) : S'_g\mu = \mu \text{ for all } g \in G\}$. Clearly this set in nonempty whenever there exists at least one right ergodic net $(A_\alpha^\mathcal{S})_\alpha$. In fact, any weak* cluster point of a right ergodic net $(A_\alpha^\mathcal{S})'\mu$, where $\mu \in P(K)$, is $\mathcal{S}'_g$-invariant (for all $g \in G$). Applying the classical Hahn-Banach separation theorem and assuming existence of right invariant nets we can prove (using a similar trick) that $\mathcal{S}'$-invariant measures separate $\mathcal{S}$-invariant functions. Moreover $\mathbb{P}_\mathcal{S}$ is weakly* compact. By the Krein-Milman theorem $\mathbb{P}_\mathcal{S}$ is the weak* closure of the convex hull of $\operatorname{ex}\mathbb{P}_\mathcal{S}$, the set of extreme invariant measures.

In the sequel we will need a few more basic facts on the ergodic structure of Markov semigroups (cf. [2], or in a case of a single Markov operator the reader is referred to [12], Theorems 1.3, 1.7, and Corollary 2.3). As mentioned, the convex set $\mathbb{P}_\mathcal{S}$ of $\mathcal{S}$-invariant probability measures is nonempty and weak* compact as long as we deal with right amenable semigroups. For every $\mu \in \mathbb{P}_\mathcal{S}$ its topological support $\operatorname{supp}(\mu)$ is an invariant set (i.e. it is closed and $S'_g\delta_x(\operatorname{supp}(\mu)) = 1$ for all $x \in \operatorname{supp}(\mu)$ and all $g \in G$). If $\mu \in \operatorname{ex}\mathbb{P}_\mathcal{S}$, then every $\mathcal{S}$-invariant function $f \in \operatorname{Fix}(\mathcal{S})$ is constant of $\operatorname{supp}(\mu)$. In fact, it may be easily proved (assume for a while that f is real valued) that for every $\alpha \in \mathbb{R}$ characteristic functions $\mathbf{1}_{\{f\le\alpha\}}$, $\mathbf{1}_{\{f>\alpha\}} \in \operatorname{Fix}(\mathcal{S}_{2,\mu})$. It follows that $\mu(\cdot \cap \{f \le \alpha\}), \mu(\cdot \cap \{f > \alpha\}) \in \operatorname{Fix}(\mathcal{S}')$. Clearly $\mu = \mu(\cdot\cap\{f \le \alpha\}) + \mu(\cdot\cap\{f > \alpha\})$ for every $\alpha \in \mathbb{R}$. Hence $\mu(\{f \le \alpha\}) = 1$ or 0 as μ is extremal. We obtain $f = \text{const}$ on $\operatorname{supp}(\mu)$ as f is continuous. In order to end consider separately real and imaginary parts of f.

Now let us recall from the recent paper [10] the following characterization of mean ergodicity of distorted semigroups ${}_\chi\mathcal{S}$ in the case when the leading Markov semigroup $\mathcal{S}$ is uniquely ergodic.

Theorem 1.2 (M. Schreiber). *Let $\mathcal{S} = \{S_g : g \in G\}$ be a representation of a right amenable semigroup G as Markov operators on $C(K)$ and assume that $\mathcal{S}$ is uniquely ergodic with invariant probability measure μ. Then for a continuous character χ on G the following conditions are equivalent:*

(1) $\operatorname{Fix}({}_\chi\mathcal{S}_2) \subseteq \operatorname{Fix}({}_\chi S)$,
(2) ${}_\chi\mathcal{S}$ *is mean ergodic with mean ergodic projection* P_χ,
(3) $\operatorname{Fix}({}_\chi\mathcal{S})$ *separates* $\operatorname{Fix}({}_\chi\mathcal{S}')$,
(3) $C(K) = \operatorname{Fix}({}_\chi\mathcal{S}) \oplus \overline{\operatorname{lin}\ \operatorname{rg}(I - {}_\chi\mathcal{S})}$,
(4) $A_\alpha^{{}_\chi\mathcal{S}} f$ *converges strongly (equivalently weakly) for some/every strong right* ${}_\chi\mathcal{S}$*-ergodic net* $A_\alpha^{{}_\chi\mathcal{S}}$ *and all* $f \in C(K)$.

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2. Result

In this section we generalize the above result and obtain the version of the Wiener-Wintner ergodic theorem for general Markov semigroups.

Theorem 2.1. *Let $\mathcal{S} = \{S_g : g \in G\}$ be a representation of a right amenable semigroup G as Markov operators on $C(K)$ .Then for every continuous character χ on G the following conditions are equivalent:*

(1) $\overline{\mathrm{Fix}({}_\chi\mathcal{S})}^{L^2(\mu)} = \mathrm{Fix}({}_\chi\mathcal{S}_{2,\mu})$ *for every* $\mu \in \mathbb{P}_\mathcal{S}$,
(2) ${}_\chi\mathcal{S}$ *is mean ergodic with mean ergodic projection* Q_χ,
(3) $\mathrm{Fix}({}_\chi\mathcal{S})$ *separates* $\mathrm{Fix}({}_\chi\mathcal{S}')$,
(4) $C(K) = \mathrm{Fix}({}_\chi\mathcal{S}) \oplus \overline{\mathrm{lin}\ \mathrm{rg}(I - {}_\chi\mathcal{S})}$,
(5) $A_\alpha^{\chi\mathcal{S}} f$ *converges strongly (equivalently weakly) to* ${}_\chi Q$ *for some/every strong right* ${}_\chi\mathcal{S}$*-ergodic net* $A_\alpha^{\chi\mathcal{S}}$ *and all* $f \in C(K)$.

Proof: It follows from the general abstract operator ergodic theorem (see Proposition 1.1) that it is sufficient to prove equivalence of (1) and (3).

(1) $\Rightarrow$ (3). Let $\nu \in \mathrm{Fix}({}_\chi\mathcal{S}')$ be nonzero (if $\mathrm{Fix}({}_\chi\mathcal{S}') = \{0\}$, then obviously $\mathrm{Fix}({}_\chi\mathcal{S}) = \{0\}$ and (3) is trivially satisfied). We have ${}_\chi S_g'\nu = \nu$ or equivalently $S_g'\nu = \overline{\chi(g)}\nu$ for all $g \in G$. Since S_g are positive linear contractions on the (complex) Banach lattice $C(K)' = M(K)$ of regular finite (complex) measures on K it follows that $S_g'|\nu| \geq |S_g'\nu| = |\overline{\chi(g)}\nu| = |\nu|$, where $|\cdot|$ denotes the lattice modulus in $M(K)$. Hence $S_g'|\nu| = |\nu|$ as $\|S_g'\| = 1$. Without loss of generality we may assume that $|\nu| \in \mathbb{P}_\mathcal{S}$. Clearly $\nu = \overline{h}|\nu|$ for some modulus 1 function h. Now following arguments used in the proof of Lemma 2.5 in [10] we find that $h \in \mathrm{Fix}({}_\chi\mathcal{S}_{2,|\nu|})$ (unique ergodicity is not necessary here). By (1) we find a sequence $h_n \in \mathrm{Fix}({}_\chi\mathcal{S})$ such that $\|h_n - h\|_{L^2(|\nu|)} \to 0$. Now $\langle h_n, \nu\rangle = \int h_n\overline{h}d|\nu| \to \int h\overline{h}d|\nu| = 1$. Hence $\langle h_n, \nu\rangle \neq 0$ for some n. It follows that $\mathrm{Fix}({}_\chi\mathcal{S})$ separates $\mathrm{Fix}({}_\chi\mathcal{S}')$.

(3) $\Rightarrow$ (1). Suppose that there exists $\mu \in \mathbb{P}_\mathcal{S}$ such that (1) fails. Then there exists nonzero $f \in \mathrm{Fix}({}_\chi\mathcal{S}_{2,\mu})$ such that $f \perp \mathrm{Fix}({}_\chi\mathcal{S})$ in $L^2(\mu)$. Applying once again Lemma 2.5 from [10] we have $\overline{f}\mu \in \mathrm{Fix}({}_\chi\mathcal{S}')$. By (3) there exists $q \in \mathrm{Fix}({}_\chi\mathcal{S})$ such that $0 \neq \langle q, \overline{f}\mu\rangle = \int_K q\overline{f}d\mu = \langle q, f\rangle_{L^2(\mu)}$, a contradiction.

■

If $\mathcal{S}$ is uniquely ergodic then by Lemma 2.6 in [10] $\dim(\mathrm{Fix}({}_\chi\mathcal{S})) \leq 1$ in $L^2(\mu)$, and therefore the closure operation in condition (1) is redundant. Now we extend Theorem 2.7 from [10] (simultaneously giving a more elementary proof).

Theorem 2.2. *Let $\mathcal{S} = \{S_g : g \in G\}$ be a representation of a right amenable semigroup G as Markov operators on $C(K)$. If $\mathcal{S}$ is mean ergodic with finite dimensional ergodic projection Q then for any continuous character χ on G the following conditions are equivalent:*

(1) $\mathrm{Fix}({}_\chi\mathcal{S}) = \mathrm{Fix}({}_\chi\mathcal{S}_{2,\mu})$ *for every* $\mu \in \mathbb{P}_\mathcal{S}$,
(2) ${}_\chi\mathcal{S}$ *is mean ergodic with mean ergodic projection* Q_χ,
(3) $\mathrm{Fix}({}_\chi\mathcal{S})$ *separates* $\mathrm{Fix}({}_\chi\mathcal{S}')$,
(4) $C(K) = \mathrm{Fix}({}_\chi\mathcal{S}) \oplus \overline{\mathrm{lin}\ \mathrm{rg}(I - {}_\chi\mathcal{S})}$,
(5) $A_\alpha^{\chi\mathcal{S}} f$ *converges strongly (equivalently weakly) to* Q_χ *for some/every strong right* ${}_\chi\mathcal{S}$*-ergodic net* $A_\alpha^{\chi\mathcal{S}}$ *and all* $f \in C(K)$.

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Proof: Clearly condition (1) here implies condition (1) in Theorem 2.1. Hence it is sufficient to prove (3) $\to$ (1). For this we will show that $\mathrm{Fix}({}_\chi\mathcal{S})$ is finite dimensional, and next we again use Theorem 2.1.

Given a character χ on G we shall prove that $\dim \mathrm{Fix}({}_\chi\mathcal{S}) < \infty$ in $L^2(\mu)$ for every $\mu \in \mathbb{P}_\mathcal{S}$. By the mean ergodicity assumption $\mathrm{Fix}(\mathcal{S})$ separates $\mathrm{Fix}(\mathcal{S}')$. It follows that $\dim \mathrm{Fix}(\mathcal{S}') < \infty$. Mean ergodicity of $\mathcal{S}$ also implies that distinct extremal invariant probabilities $\mu_1, \mu_2 \in \mathrm{ex}\,\mathbb{P}_\mathcal{S}$ have disjoint (topological) supports. Indeed, let us take $f \in \mathrm{Fix}(\mathcal{S})$ such that $\langle f, \mu_1\rangle \neq \langle f, \mu_2\rangle$. This function f is constant both on $\mathrm{supp}(\mu_1)$ and $\mathrm{supp}(\mu_2)$. Hence $\mathrm{supp}(\mu_1) \cap \mathrm{supp}(\mu_2) = \emptyset$. It follows that (topological) supports of extremal invariant probabilities are closed, pairwise disjoint invariant sets (even minimal). Thus $\mathrm{ex}\,\mathbb{P}_\mathcal{S}$ is a linearly independent family, and by our assumption there are finitely many of them, i.e. $\mathrm{ex}\,\mathbb{P}_\mathcal{S} = \{\mu_1, \mu_2, \ldots, \mu_k\}$ for some natural k. Let $C_\mathcal{S}$ be the union $\bigcup_{j=1}^k \mathrm{supp}(\mu_j)$ and $\nu = \frac{1}{k}(\mu_1 + \ldots + \mu_k) \in \mathbb{P}_\mathcal{S}$ (clearly $C_\mathcal{S}$ is a closed invariant set).

If $f \in \mathrm{Fix}({}_\chi\mathcal{S}) \subseteq L^2(\nu)$ then $\chi(g)S_g f = f$ and therefore $S_g f = \overline{\chi(g)} f$. Considering S_g as a linear contraction on $L^2(\nu)$ we get $S_g|f| = |f|$ ν *a.e.*. By continuity, $S_g|f| = |f|$ on $C_\mathcal{S}$. Hence on each support $\mathrm{supp}(\mu_j)$, where $j = 1, \ldots, k$, the function $|f|$ is constant. Let us take arbitrary $x \in \mathrm{supp}(\mu_j)$ and $g \in G$. We have

$$f(x) = \chi(g) \int_K f(y) S'_g \delta_x(dy) = \chi(g) \int_{\mathrm{supp}(\mu_j)} f(y) S'_g \delta_x(dy) \ .$$

Hence $f(y) = \overline{\chi(g)} f(x)$ for $y \in \mathrm{supp}(S'_g\delta_x)$. It follows that for any $f_1, f_2 \in \mathrm{Fix}({}_\chi\mathcal{S})$ and $f_2 \neq 0$ on $\mathrm{supp}(\mu_j)$, then for all $x \in \mathrm{supp}(\mu_j)$ and all $g \in G$ we have

$$S_g\left(\frac{f_1}{f_2}\right)(x) = \int_{\mathrm{supp}(\mu_j)} \frac{f_1(y)}{f_2(y)} S'_g\delta_x(dy) = \frac{\overline{\chi(g)} f_1(x)}{\overline{\chi(g)} f_2(x)} = \frac{f_1}{f_2}(x).$$

Since $\mathcal{S}$-invariant functions are constant on supports of extremal invariant probabilities, $\frac{f_1}{f_2} = c$ on $\mathrm{supp}(\mu_j)$. In other words if $f_1, f_2 \in \mathrm{Fix}({}_\chi\mathcal{S})$ then $f_1 = \alpha_j(f_1, f_2) f_2$ on $\mathrm{supp}(\mu_j)$ or simply $\dim \mathrm{Fix}({}_\chi\mathcal{S})\mathbf{1}_{\mathrm{supp}\,\mu_j} = 1$ for any $j = 1, ..., k$.

Let $w_j \in \mathrm{Fix}({}_\chi\mathcal{S})\mathbf{1}_{\mathrm{supp}\,\mu_j}$ be nonzero (as long as such a function exists). Then every $f \in \mathrm{Fix}({}_\chi\mathcal{S})$ may be represented in $L^2(\nu)$ as

$$f = \sum_{j=1}^k f\mathbf{1}_{\mathrm{supp}\,\mu_j} = \sum_{j=1}^k \alpha_j w_j \mathbf{1}_{\mathrm{supp}\,\mu_j}.$$

In particular, regardless of the choice of invariant $\mu \in \mathbb{P}_\mathcal{S}$, μ is a convex combination of $\mu_1, \mu_2, ..., \mu_k$, so the estimation $\dim \mathrm{Fix}({}_\chi\mathcal{S}) \leq k = \dim \mathrm{Fix}(\mathcal{S})$ in $L^2(\mu)$ holds true. Hence using Theorem 2.1 the condition (3) implies that

$$\mathrm{Fix}({}_\chi\mathcal{S}_{2,\mu}) = \overline{\mathrm{Fix}({}_\chi\mathcal{S})}^{L^2(\mu)} = \mathrm{Fix}({}_\chi\mathcal{S}) \ (\mu \ a.e.)$$

for all $\mu \in \mathbb{P}_\mathcal{S}$.

■

We can also characterize convergence of the ergodic net $A_\alpha^{\chi\mathcal{S}}$ on a single function $f \in C(K)$. Our proof is based on the proof of Theorem 2.8 in [10]. However, we do not assume here mean ergodicity. In particular, Schreiber's assumption on unique ergodicity is removed. For the sake of completeness of the paper and for the convenience of the reader, a full proof is included.

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Theorem 2.3. *Let $\mathcal{S} = \{S_g : g \in G\}$ be a representation of a right amenable semigroup G as Markov operators on $C(K)$. For any $f \in C(K)$ and continuous character χ on G the following conditions are equivalent:*

(1) *$P_{\mathrm{Fix}({}_\chi\mathcal{S}_{2,\mu})} f \in \mathrm{Fix}({}_\chi\mathcal{S})$ for every $\mu \in \mathbb{P}_{\mathcal{S}}$, where $P_{\mathrm{Fix}({}_\chi\mathcal{S}_{2,\mu})}$ denotes the orthogonal projection on the subspace $\mathrm{Fix}({}_\chi\mathcal{S}_{2,\mu})$ in $L^2(\mu)$,*

(2) *$A_\alpha^{\chi\mathcal{S}} f$ converges to a fixed point of ${}_\chi\mathcal{S}$ for some/every strong right $\mathcal{S}$-ergodic net $(A_\alpha^{\chi\mathcal{S}})_{\alpha\in\Lambda}$,*

(3) *$f \in \mathrm{Fix}({}_\chi\mathcal{S}) \oplus \overline{\mathrm{lin}}\ rg(I - {}_\chi\mathcal{S})$.*

Proof: It is known that conditions (2) and (3) are equivalent (cf. Proposition 1.11 in [11]).

(2) $\Rightarrow$ (1). ${}_\chi\mathcal{S}_{2,\mu}$ is mean ergodic as a contraction semigroup on Hilbert space (cf. Corollary 1.9 in [11]). Its mean ergodic projection is the orthogonal projection $P_{\mathrm{Fix}({}_\chi\mathcal{S}_{2,\mu})}$. If $A_\alpha^{\chi\mathcal{S}}$ is a strong right ${}_\chi\mathcal{S}$-ergodic net, then $A_\alpha^{\chi\mathcal{S}}$ is also a strong right ${}_\chi\mathcal{S}_{2,\mu}$-ergodic net, so $A_\alpha^{\chi\mathcal{S}} f$ converges in $L^2(\mu)$ to $P_{\mathrm{Fix}({}_\chi\mathcal{S}_{2,\mu})} f$. By (2) we have that $A_\alpha^{\chi\mathcal{S}} f$ converges also in $C(X)$ to $h \in \mathrm{Fix}({}_\chi\mathcal{S})$, hence $P_{\mathrm{Fix}({}_\chi\mathcal{S}_{2,\mu})} f \in \mathrm{Fix}({}_\chi\mathcal{S})$.

(1) $\Rightarrow$ (3). Let $\mu \in C(K)'$ vanish on $\mathrm{Fix}({}_\chi\mathcal{S}) \oplus \overline{\mathrm{lin}}\,\mathrm{rg}(I - {}_\chi\mathcal{S})$. The Hahn-Banach theorem yields that it suffices to show that $\langle f, \mu\rangle = 0$, since $\mathrm{Fix}({}_\chi\mathcal{S}) \oplus \overline{\mathrm{lin}}\,\mathrm{rg}(I - {}_\chi\mathcal{S})$ is a closed subspace of $C(X)$. For every $h \in C(X),\ g \in G$ we have $\langle h - \chi(g)S_g h, \mu\rangle = 0$, so $\langle h, (\chi(g)S_g)'\mu\rangle = \langle \chi(g)S_g h, \mu\rangle = \langle h, \mu\rangle$, hence $\mu \in \mathrm{Fix}({}_\chi\mathcal{S})'$. We have $S_g'|\mu| \geq |S_g'\mu| = |\overline{\chi}(g)\mu| = |\mu|$, so $|\mu| \in \mathrm{Fix}(\mathcal{S}')$. We can assume that $|\mu| \in \mathbb{P}_{\mathcal{S}}$. There exists $h \in L^2(|\mu|)$ with $\mu = \overline{h}|\mu|$ and by Lemma 2.5 in [10] there is $h \in \mathrm{Fix}({}_\chi\mathcal{S}_{2,|\mu|})^*$. We have

$$\langle f, \mu\rangle = \int_X f d\mu = \int_X f\overline{h} d|\mu| = \langle f, h\rangle_{L^2(|\mu|)} =$$
$$= \langle f, (A_\alpha^{\chi\mathcal{S}_{2,|\mu|}})^* h\rangle_{L^2(|\mu|)} = \langle A_\alpha^{\chi\mathcal{S}_{2,|\mu|}} f, h\rangle_{L^2(|\mu|)}$$

for some strong right ${}_\chi\mathcal{S}_{2,|\mu|}$-ergodic net $A_\alpha^{\chi\mathcal{S}_{2,|\mu|}}$. Passing to the limit gives

$$\langle f, \mu\rangle = \langle P_{Fix({}_\chi\mathcal{S}_{2,|\mu|})} f, h\rangle_{L^2(|\mu|)} = \langle P_{\mathrm{Fix}({}_\chi\mathcal{S}_{2,|\mu|})} f, \mu\rangle = 0,$$

since $P_{\mathrm{Fix}({}_\chi\mathcal{S}_{2,|\mu|})} f \in \mathrm{Fix}({}_\chi\mathcal{S})$ by (1).

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