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Note

A note on the strength and minimum color sum of bipartite graphs

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ABSTRACT

The *strength* of a graph G is the smallest integer s such that there exists a minimum sum coloring of G using integers $\{1, \ldots, s\}$, only. For bipartite graphs of maximum degree Δ we show the following simple bound: $s \leq \lceil \Delta/2 \rceil + 1$. As a consequence, there exists a quadratic time algorithm for determining the strength and minimum color sum of bipartite graphs of maximum degree $\Delta \leq 4$.

1. Introduction

For a simple undirected graph $G = (V, E)$, a (*proper*) *vertex coloring* c is an assignment $c : V \to \mathbb{N}$ such that for all edges $\{u, v\} \in E$, $c(u) \neq c(v)$. Given a coloring c, we define its *color sum* $\Sigma_c = \sum_{v \in V} c(v)$, and its *span* $\chi_c = \max_{v \in V} c(v)$. The *minimum color sum* $\Sigma(G)$ is the minimum value of the color sum taken over all colorings of G, the *chromatic number* $\chi(G)$ is the minimum value of span taken over all colorings of G, and the *strength* $s(G)$ is the minimum value of span taken over those colorings of G which have a color sum equal to $\Sigma(G)$. The maximum vertex degree in G is denoted by $\Delta(G)$, whereas the minimum vertex degree is denoted by $\delta(G)$.

The problem of bounding or determining the exact values of $\Sigma(G)$ and $s(G)$ for different graph classes has been given a lot of attention due to the importance of the sum coloring problem in task scheduling (see e.g. [4] for a nice survey of results). The following upper bound on $s(G)$ was shown in [2] and holds for all graphs:

$$s(G) \leq \left\lceil \frac{\Delta(G) + \operatorname{col}(G)}{2} \right\rceil, \tag{1}$$

where $\operatorname{col}(G) = 1 + \max_{H \subseteq G} \delta(H)$ is the so-called *coloring number* of G. It is known that $\chi(G) \leq \operatorname{col}(G) \leq \Delta(G) + 1$, and the authors of [2] have conjectured that bound (1) can in fact be strengthened as follows:

Conjecture 1 (*Mehrabadi's Conjecture [2,1]*). *For any graph G, $s(G) \leq \left\lceil \frac{\Delta(G)+\chi(G)}{2} \right\rceil$.*

The bound in Mehrabadi's conjecture has been proved to hold and be tight for the class of trees [3]. In this note we point out that the conjecture is in fact true for all bipartite graphs (i.e. whenever $\chi(G) = 2$), and remark on some algorithmic consequences of this observation.

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2. A proof of Mehrabadi's conjecture for bipartite graphs

Theorem 1. *For any bipartite graph G,* $s(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$.

Proof. Let G be a bipartite graph with bipartite partitions $V = V_1 \cup V_2$, and let c be a coloring of G with $\Sigma_c = \Sigma(G)$. To complete the proof it is enough to show a procedure which constructs a proper coloring c' of G such that $\Sigma_{c'} \leq \Sigma_c$ and $\chi_{c'} \leq \lceil \frac{\Delta(G)}{2} \rceil + 1$. Initially, for all $v \in V$, we put $c'(v) := \min\{c(v), \lceil \frac{\Delta(G)}{2} \rceil + 1\}$. At this point coloring c' may be improper due to the existence of neighboring vertices sharing color $\lceil \frac{\Delta(G)}{2} \rceil + 1$. We will proceed to modify coloring c' to eliminate these conflicts, in such a way that at every step the color sum of c' does not increase, and that c' restricted to vertices having colors $\{1, \ldots, \lceil \frac{\Delta(G)}{2} \rceil\}$ always remains proper. The condition $\chi_{c'} \leq \lceil \frac{\Delta(G)}{2} \rceil + 1$ will be fulfilled throughout the process.

Let $V_C \subseteq V_2$ be the subset of nodes $v \in V_2$ such that $c'(v) = \lceil \frac{\Delta(G)}{2} \rceil + 1$ and v has at least one neighbor in V_1 colored with the same color as v. As long as V_C is non-empty, at each step we arbitrarily choose a vertex $v \in V_C$. Since v has at least one neighbor in V_1 also colored with color $\lceil \frac{\Delta(G)}{2} \rceil + 1$, v can have at most $(\Delta - 1)$ neighbors colored with colors from the range $\{1, \ldots, \lceil \frac{\Delta(G)}{2} \rceil\}$, and by the pigeon-hole principle there must exist a color value $a \in \{1, \ldots, \lceil \frac{\Delta(G)}{2} \rceil\}$ such that v has at most one neighbor colored with color a. If v has no neighbor colored with color a, we simply put $c'(v) := a$, thus decreasing the color sum of c' without creating any new conflicts. Otherwise, let $u \in V_1$ be the unique neighbor of v such that $c'(u) = a$. We now modify coloring c' by switching the color values of u and v, i.e. $c'(u) := \lceil \frac{\Delta(G)}{2} + 1 \rceil$ and $c'(v) = a$. This does not change the color sum of c', and moreover c' restricted to vertices having colors $\{1, \ldots, \lceil \frac{\Delta(G)}{2} \rceil\}$ remains proper since u was the unique neighbor of v originally having color a.

The above procedure is iterated until set V_C is empty. It terminates after at most a linear number of steps because at each step the number of vertices in V_2 having color $\lceil \frac{\Delta(G)}{2} \rceil + 1$ decreases by exactly 1. (In some steps the size of set V_C may increase, but this is irrelevant.) When the procedure terminates, since set V_C is empty and the graph is bipartite, c' is a proper coloring. Recalling that $\Sigma_{c'} \leq \Sigma_c$ and $\chi_{c'} \leq \lceil \frac{\Delta(G)}{2} \rceil + 1$ completes the proof. □

3. Sum coloring of bipartite graphs with $\Delta \leq 4$

The problem of determining the color sum $\Sigma(G)$ and strength $s(G)$ of a graph is known to be computationally hard even when restricted to special graph classes. For example, the problem "is $s(G) \leq 2$?" is coNP-complete even for bipartite graphs [6], whereas determining the exact value of $\Sigma(G)$ is NP-hard for bipartite graphs for any value of maximum degree $\Delta(G) \geq 5$ [5]. On the other hand, it was shown in [5] that it is possible to determine $\Sigma(G)$ precisely in polynomial time for bipartite graphs with $\Delta(G) \leq 3$, while the question of the complexity of determining $\Sigma(G)$ for bipartite graphs of maximum degree $\Delta(G) = 4$ was posed as the main open problem. Taking into account the proof of Theorem 1, we can now provide a positive answer to this question.

Theorem 2. *For any bipartite graph G of maximum degree* $\Delta(G) \leq 4$*, the values of* $\Sigma(G)$ *and* $s(G)$ *can be exactly determined in* $O(|V|^2)$ *time.*

Proof. In order to find $\Sigma(G)$, we take advantage of an advanced routine from [5, Thm. 3], which finds in $O(|V||E|)$ time an improper coloring c of any bipartite graph with colors $\{1, 2, 3\}$, such that c restricted to colors $\{1, 2\}$ is proper (though vertices having color 3 can be adjacent), and moreover $\Sigma_c \leq \Sigma(G)$. Observing that for $\Delta \leq 4$ we have $\lceil \frac{\Delta(G)}{2} \rceil + 1 \leq 3$, by applying the procedure from the proof of Theorem 1 to modify coloring c, we obtain in $O(|V||E|) = O(|V|^2)$ time a proper coloring c' of G such that $\Sigma_{c'} \leq \Sigma_c \leq \Sigma(G)$. Obviously, c' is an optimal sum coloring of G and $\Sigma(G) = \Sigma_{c'}$.

In order to determine $s(G)$, we note that by Theorem 1 for $\Delta \leq 4$, $s(G) \leq \lceil \frac{\Delta(G)}{2} \rceil + 1 \leq 3$. Assuming that G is non-empty, this means that either $s(G) = 2$, or $s(G) = 3$. So, it suffices to check whether $s(G) = 2$, and this holds if and only if for each connected component H of G we have $\Sigma(H) = \min\{\Sigma_{c_1}, \Sigma_{c_2}\}$, where c_1 and c_2 are the only two distinct colorings of bipartite graph H using 2 colors. Since the parameters $\Sigma(H)$, Σ_{c_1}, Σ_{c_2} can be determined in $O(|V|^2)$ time, this completes the proof. □

It is interesting to ask whether any of the simple techniques presented here, especially the proof of Theorem 1, can be generalized to non-bipartite graphs. A direct application of the proposed construction only removes color conflicts with respect to one independent set of the graph.

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