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Research Paper

About Unusual Diffraction and Thermal Self-Action of Magnetosonic Beam

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The dynamics of slightly diverging two-dimensional beams whose direction forms a constant angle θ with the equilibrium straight magnetic strength is considered. The approximate dispersion relations and corresponding links which specify hydrodynamic perturbations in confined beams are derived. The study is dedicated to the diffraction of a magnetosonic beam and nonlinear thermal self-action of a beam in a thermoconducting gaseous plasma. It is shown that the divergence of a beam and its thermal self-action is unusual in some particular cases of parallel propagation ($\theta = 0$) and has no analogues in the dynamics of the Newtonian beams. The nonlinear attenuation of Newtonian beams leads to their defocusing in gases, whereas the unusual cases correspond to the focusing in a presence of magnetic field. The examples of numerical calculations of thermal self-action of magnetoacoustic beams with shock fronts are considered in the usual and unusual cases of diffraction concerning stationary and non-stationary self-action. It is discovered that the diffraction is more ($\theta = 0$) or less ($\theta = \pi/2$) manifested as compared to that of the Newtonian beams. The beams which propagate oblique to the magnetic field do not reveal diffraction. The special case, when the sound and Alfvénic speeds are equal, is discussed. This magnetosonic beams incorporate acoustic and Alfvénic properties and do not undergo diffraction in this particular case.

Keywords: non-linear magnetoacoustics; diffraction of beams; acoustic thermal self-action; magnetohydrodynamics.

1. Introduction

The divergence is inherent to the flows exceeding one dimension. Two- and three-dimensional beams with initially planar fronts propagate usually with increasing of their characteristic width. Thus, a beam's energy and impulse spread over larger cross-sections. Diffraction is more pronounced in the cases when a ratio of the transducer characteristic length to the wave length is small. Hence, it is of great importance in medical and technical applications, where accurate evaluation of the actual focus and excess temperature in the focal zone is the key issue (Yufeng, 2015; Duck, 2002). Position of the actual focus differs from the geometric one due to diffraction and nonlinear effects following a beam. The nonlinear shift of the focus has been explained and described for the first time in (Rudenko, Sapozhnikov, 2004). The nonlinear phenomena are also associated with inhomogeneous distribution of magnitude of wave perturbations across the axis of a beam's propagation because they strengthen proportionally to the squared magnitude. In particular, the nonlinear transfer of energy into the entropy mode is stronger in the par-axial area. That leads to the larger heating in the par-axial area and formation of thermal lenses. In turn, thermal lenses have impact on the propagation of a beam since the local sound speed depends on the equilibrium temperature. The MHD (magnetohydrodynamic) beams and the related nonlinear phenomena reveal a wide variety of behaviour in view of strong dependence on the direction of the magnetic field, diversity of wave modes and equilibrium parameters of a plasma. Particularly, the parameter of nonlinearity and damping coefficient due to thermal conduction are variable (Chin *et al.*, 2010; Nakariakov *et al.*, 2000).

This study considers diffraction of a magnetosonic beam in dependence of orientation of the axis of propagation and the equilibrium parameters of a plasma. It turns out that the obliquely propagating beam almost does not diverge. As for the case of parallel propagation, the diffraction term may take an unusual sign. These features probably have no analogues in the wave theory of confined beams. As for the perpendicular propagation, it occurs commonly but with weaker divergence compared to that in the Newtonian beams.

2. Equations of MHD flow

The dynamics of a gas is governed by the ideal MHD equations. That means that the spatial and temporal scales of perturbations in a flow must be much larger than gyro-kinetic scales. The model imposes equal temperature of ions and electrons and makes use of one-fluid perfectly electrically conducting gas. The set of initial partial derivatives' equations consists of the continuity equation, momentum equation, energy balance equation, and electrodynamic equations (Freidberg, 1987; Krall, Trivelpiece, 1973; Callen, 2003):

$$\begin{gathered}\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0,\\ \rho\left(\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right)=-\nabla p+\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B},\\ \frac{\partial p}{\partial t}+(\mathbf{v}\cdot\nabla)p+\gamma p(\nabla\cdot\mathbf{v})=0,\\ \frac{\partial \mathbf{B}}{\partial t}=\nabla\times(\mathbf{v}\times\mathbf{B}),\\ \nabla\cdot\mathbf{B}=0,\end{gathered}\tag{1}$$

where p, ρ, $\mathbf{v}$, $\mathbf{B}$, are hydrodynamic pressure and mass density of a plasma, its velocity, the magnetic field, μ_0 is the magnetic permeability of a free space, and γ denotes the ratio of specific isobaric heat capacity and specific isochoric heat capacity, $\gamma = C_P/C_V$. The third equation in fact incorporates the continuity and the energy balance equation and relies on the internal energy for an ideal gas, $\frac{p}{\rho(\gamma-1)}$. The fourth equation in the set is the ideal induction equation, and the fifth one ensures the solenoidal character of $\mathbf{B}$ (the Maxwell equation). Many examples of plasmas such as Solar atmosphere, Earth's magnetosphere, neutron star magnetospheres are described reasonably well by the MHD system of equations. The exceptions are the problems which relate to kinetic effects, magnetic reconnection, some laboratory plasmas, weakly ionised plasmas, cosmic rays, molecular clouds, and some other cases. The equation of state for an ideal gas is valid besides applications dealing with very cool and dense plasmas.

The two-dimensional geometry of a flow is considered following Botha *et al.* (2000), McLaughlin *et al.* (2011). This assumes dependence of all perturbations on x and z. All equilibrium quantities are treated as constants and subscripted by 0, and the MHD perturbations are superscripted by $'$. The constant equilibrium magnetic field is aligned along axis z, so that $\mathbf{B}_0 = (0, 0, B_0)$. The bulk flow is absent, $\mathbf{v}_0 = 0$. The MHD equations may be written in the following form (McLaughlin *et al.*, 2011):

$$\begin{aligned}\frac{\partial \rho'}{\partial t}+\rho_0\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_z}{\partial z}\right)&=N_1,\\ \frac{\partial v_x}{\partial t}+\frac{1}{\rho_0}\frac{\partial p}{\partial x}-\frac{B_0}{\rho_0\mu_0}\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)&=N_2,\\ \frac{\partial v_y}{\partial t}-\frac{B_0}{\rho_0\mu_0}\frac{\partial B_y}{\partial z}&=N_3,\\ \frac{\partial v_z}{\partial t}+\frac{1}{\rho_0}\frac{\partial p'}{\partial z}&=N_4,\\ \frac{\partial p'}{\partial t}+\gamma p_0\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_z}{\partial z}\right)&=N_5,\\ \frac{\partial B_x}{\partial t}-B_0\frac{\partial v_x}{\partial z}&=N_6,\\ \frac{\partial B_y}{\partial t}-B_0\frac{\partial v_y}{\partial z}&=N_7,\\ \frac{\partial B_z}{\partial t}+B_0\frac{\partial v_x}{\partial x}&=N_8,\end{aligned}\tag{2}$$

where $\widetilde{N} = (N_1, ..., N_8)^T$ is a vector which consists of quadratically nonlinear terms. Equations (2) contain the terms of order M^1 (the left-hand side), M^2 (the right-hand side) in the series expansion in a small parameter, the magnetosonic Mach number M (M equals the ratio of characteristic magnitude of velocity to the speed of magnetosonic perturbations). The dispersion relations specify all kinds of wave and non-wave motion of small magnitudes in a plasma, that is, in the case of insignificant nonlinearity. They follow from the linearised version of Eqs (2), if one assumes that all perturbations are proportional to $\exp(i\omega t - ik_x x - ik_z z)$ ($\mathbf{k} = (k_x, 0, k_z)$ is the wave vector). They are in fact roots of the dispersion equation:

$$\begin{aligned}\omega^2(\omega^2-C_A^2\cos^2(\theta)k_z^2)\left(c_0^2(k_x^2+k_z^2)(C_A^2k_z^2-\omega^2)\right.\\ \left.+\,\omega^2(\omega^2-C_A^2(k_x^2+k_z^2))\right)=0,\end{aligned}\tag{3}$$

where

$$c_0=\sqrt{\frac{\gamma p_0}{\rho_0}}=\sqrt{(\gamma-1)C_PT_0}$$

denotes the acoustic speed in unmagnetised gas in equilibrium, and

$$C_A=\frac{B_0}{\sqrt{\rho_0\mu_0}}$$

is the Alfvén speed. Among roots of Eq. (3), there are two Alfvén (non-acoustic) modes of different directions of propagation ($\omega^2 - C_A^2\cos^2(\theta)k_z^2 = 0$) and two stationary modes ($\omega^2 = 0$). Four roots of Eq. (3) describe the magnetosonic modes, fast and slow, of different direction of propagation. The magnetosonic beams are of interest in this study. The dynamic equation for any non-zero wave perturbation $\varphi(x,z,t)$ follows from Eq. (3):

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2\varphi}{\partial t^2} - C_A^2\Delta\varphi\right) - c_0^2\Delta\left(\frac{\partial^2\varphi}{\partial t^2} - C_A^2\frac{\partial^2\varphi}{\partial z^2}\right) = 0, \quad (4)$$

where

$$\Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right).$$

3. Quasi-planar dynamics

Let us consider weak diffraction of small magnitude perturbations in a beam which is reasonably directed along axis x_1 and introduce the small parameter m responsible for its divergence. Figure 1 explains the geometry of a flow.

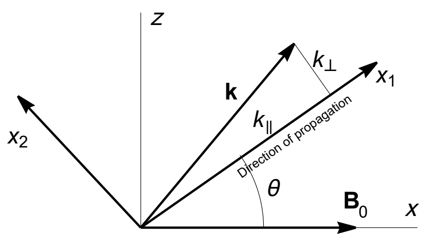

Fig. 1. Geometry of a flow.

The component of the wave vector **k** parallel to the direction of propagation is $k_{||}$ (it is of order 1), and the perpendicular component is $k_\perp$ (it is of order $\sqrt{m}$), so as:

$$k_z = k_{||}\cos(\theta) - k_\perp\sin(\theta), \quad k_x = k_{||}\sin(\theta) + k_\perp\cos(\theta).$$

All subsequent formulas are leading order; they include the terms up to m^1 in the power series. By expanding Eq. (3) in the series and collecting the leading order terms, one arrives at the dispersion relation:

$$\omega = \pm\left(Ck_{||} + Gk_\perp + \frac{D^2C}{2}k_\perp^2\right), \quad (5)$$

where C is the positive root of the equation:

$$C^4 - C^2(c_0^2 + C_A^2) + c_0^2C_A^2\cos^2(\theta) = 0. \quad (6)$$

C is a speed of propagation of both fast or slow modes, and

$$G = \frac{c_0^2C_A^2C\cos(\theta)\sin(\theta)}{C^4 - c_0^2C_A^2\cos^2(\theta)},$$

$$D^2 = \frac{2\left(c_0^2 + C_A^2 + \frac{a^*}{(C^4 - c_0^2C_A^2\cos^2(\theta))^3}C^6\right) - \frac{C^2c_0^4C_A^4\sin^2(2\theta)}{(C^4 - c_0^2C_A^2\cos^2(\theta))^2}}{4C^2k_{||}},$$

where

$$a^* = c_0^8 + c_0^4C_A^4 + C_A^8 - 2c_0^2C_A^2(c_0^4 + C_A^4)\cos(2\theta) + c_0^4C_A^4\cos(4\theta).$$

The case of the parallel propagation ($\theta = 0$, $k_{||} = k_z$, $k_\perp = k_x$) imposes two roots of Eq. (6), $C = C_A$ and $C = c_0$. If $C_A = c_0$, the roots of dispersion relation (3) are degenerate and the denominator $C^4 - c_0^2C_A^2$ equals zero. This case should be considered individually. We start with the cases of wave vector parallel or perpendicular to the equilibrium magnetic field. These cases impose zero G.

3.1. Parallel wave vector and the equilibrium magnetic field

This is the case of $\theta = 0$. The leading order dispersion relations for the magnetosonic modes are shown in Table 1.

Table 1. Leading order dispersion relations.

$c_0 \neq C_A$	$\pm\left(C_Ak_z + \frac{C_A^3k_x^2}{2(C_A^2-c_0^2)k_z}\right)$	$\pm\left(c_0k_z + \frac{c_0^3k_x^2}{2(c_0^2-C_A^2)k_z}\right)$
$c_0 = C_A$	$\pm\left(c_0k_z - \frac{c_0k_x}{2} + \frac{3c_0k_x^2}{8k_z}\right)$	$\pm\left(c_0k_z + \frac{c_0k_x}{2} + \frac{3c_0k_x^2}{8k_z}\right)$

The modes differ by the relations between specific perturbations. These relations unambiguously determine modes along with the dispersion relations. For all these modes, $v_y = 0$, $B'_y = 0$.

3.1.1. Case $c_0 \neq C_A$

In particular, the links inherent to the dispersion relations for acoustic beams,

$$\omega = \pm\left(c_0k_z + \frac{c_0^3k_x^2}{2(c_0^2 - C_A^2)k_z}\right),$$

take the form:

$$v_x = \pm\frac{c_0^3}{(c_0^2 - C_A^2)\rho_0}\int dz\frac{\partial\rho'}{\partial x},$$

$$v_z = \pm\left(\frac{c_0\rho'}{\rho_0} - \frac{c_0^3}{2(c_0^2 - C_A^2)\rho_0}\int dz\int dz\frac{\partial^2\rho'}{\partial x^2}\right),$$

$$p' = c_0^2\rho', \quad (7)$$

$$B'_x = -\frac{c_0^2B_0}{(c_0^2 - C_A^2)\rho_0}\int dz\frac{\partial\rho'}{\partial x},$$

$$B'_z = \frac{c_0^2B_0}{(c_0^2 - C_A^2)\rho_0}\int dz\int dz\frac{\partial^2\rho'}{\partial x^2}.$$

The terms including partial derivative with respect to x are of order $\sqrt{m}$, and these including the second order derivative are of order m^1. This case is purely acoustic in the absence of the magnetic field with the non-zero leading order perturbations as follows:

$$p' = c_0^2 \rho' = c_0 \rho_0 v_z.$$

The relations of perturbations specify modes uniquely. They reveal the difference between modes and may indicate the type of a motion on a pair of the dispersion relations. To be specific, a wave propagating in the positive direction of axis z is considered. Any non-zero perturbation inherent to this mode is supposed to be a function of the retarded time $\tau = t - z/c_0$, mz, $\sqrt{m}x$. This choice of slow scale suggests that the spatial variations occur more slowly along the axis z of a beam than across the beam from the point of view of an observer who moves at the speed c_0 along the axis of a beam (Rudenko, Soluyan, 1977; Hamilton, Blackstock, 1988). Assuming the Mach number M and m of the same order, taking into account nonlinear terms $N_1, ..., N_8$, discarding terms of the high order in smallness and transforming equation from the slow scale back to x, z yields an equation governing the magnetosonic pressure:

$$\frac{\partial}{\partial \tau}\left(\frac{\partial p'}{\partial z} - \frac{\varepsilon_{\|}}{\rho_0 c_0^3} p' \frac{\partial p'}{\partial \tau}\right) = \begin{cases} \dfrac{D_{\|,0}^2 c_0}{2} \dfrac{\partial^2 p'}{\partial x^2}, & c_0 > C_A, \\ -\dfrac{D_{\|,0}^2 c_0}{2} \dfrac{\partial^2 p'}{\partial x^2}, & c_0 < C_A, \end{cases} \tag{8}$$

where

$$\varepsilon_{\|} = \frac{\gamma + 1}{2}$$

is the parameter of nonlinearity,

$$D_{\|,0}^2 = \frac{c_0^2}{|c_0^2 - C_A^2|} = \frac{\beta\gamma}{|\beta\gamma - 2|}$$

is the squared parameter responsible for diffraction, and

$$\beta = \frac{2}{\gamma} \frac{c_0^2}{C_A^2}$$

is the plasma-β which reflects the ratio of hydrostatic and magnetic pressures. Equation (8) recalls the famous Khokhlov-Zabolotskaya (KZ) equation for an acoustic pressure in a slightly diverging beam in an ideal gas (Kuznetsov, 1971):

$$\frac{\partial}{\partial \tau}\left(\frac{\partial p'}{\partial z} - \frac{\varepsilon_{\|}}{\rho_0 c_0^3} p' \frac{\partial p'}{\partial \tau}\right) = \frac{c_0}{2} \frac{\partial^2 p'}{\partial x^2}. \tag{9}$$

Equation in the upper row of set (8) may be rearranged into the KZ equation by the substitution $X = \frac{x}{D_{\|,0}}$. Since $D_{\|,0} > 1$, the divergence is more pronounced in the case of a plasma affected by a magnetic field. In the planar case ($\frac{\partial}{\partial x} \equiv 0$), the links

$$\omega = \pm\left(C_A k_z + \frac{C_A^3 k_x^2}{2(C_A^2 - c_0^2) k_z}\right)$$

transform to the only link recalling that for the Alfvén modes ($B_y' = \mp \frac{B_0}{C_A} v_y$):

$$B_x' = \mp \frac{B_0}{C_A} v_x.$$

It has a property which differentiates it from the Alfvén modes. Namely, this mode reveals divergence while the Alfvén mode refers to exact dispersion relations $\omega = \pm C_A k_z$ and behaves like a planar wave.

The non-acoustic wave may be extracted from Eq. (4) by means of making use of the set of variables $\tau = t - z/C_A$, mz, $\sqrt{m}x$. One arrives at the dynamic equation:

$$\frac{\partial}{\partial \tau}\left(\frac{\partial v_x}{\partial z} - \frac{3}{2C_A^2} v_x \frac{\partial v_x}{\partial \tau}\right) = \begin{cases} \dfrac{D_{\|,A}^2 C_A}{2} \dfrac{\partial^2 v_x}{\partial x^2}, & c_0 < C_A, \\ -\dfrac{D_{\|,A}^2 C_A}{2} \dfrac{\partial^2 p'}{\partial x^2}, & c_0 > C_A, \end{cases} \tag{10}$$

where

$$D_{\|,A}^2 = \frac{C_A^2}{|c_0^2 - C_A^2|} = \frac{2}{|\beta\gamma - 2|}.$$

Hence, the divergence may be anomalous in dependence on the ratio of C_A and c_0 and the kind of a wave mode. This relates to the signs minus in the right-hand sides of Eqs (8) and (10) which reflect diffraction.

3.1.2. Case $c_0 = C_A$

The links of perturbations inherent to the dispersion relation $\omega = \pm\left(c_0 k_z - \frac{c_0 k_x}{2} + \frac{3 c_0 k_x^2}{8 k_z}\right)$ (this is the case $C_A = c_0$), take the forms:

$$\begin{aligned} v_x &= \pm\left(-\frac{c_0 \rho'}{\rho_0} + \frac{c_0}{2\rho_0} \int \mathrm{d}z \frac{\partial \rho'}{\partial x} + \frac{c_0}{4\rho_0} \int \mathrm{d}z \int \mathrm{d}z \frac{\partial^2 \rho'}{\partial x^2}\right), \\ v_z &= \pm\left(\frac{c_0 \rho'}{\rho_0} + \frac{c_0}{2\rho_0} \int \mathrm{d}z \frac{\partial \rho'}{\partial x} - \frac{c_0}{8\rho_0} \int \mathrm{d}z \int \mathrm{d}z \frac{\partial^2 \rho'}{\partial x^2}\right), \\ p' &= c_0^2 \rho', \\ B_x' &= \frac{B_0}{\rho_0} \rho' - \frac{5 B_0}{8 \rho_0} \int \mathrm{d}z \int \mathrm{d}z \frac{\partial^2 \rho'}{\partial x^2}, \\ B_z' &= -\frac{B_0}{\rho_0} \int \mathrm{d}z \frac{\partial \rho'}{\partial x}. \end{aligned} \tag{11}$$

The case $c_0 = C_A$ is especial, it is not a limit of the general case and represents the type of motion combining Alfvén and acoustic properties. In particular, v_x and v_z are of the same order. Operating on the variables $\tau = t - z/c_0$, mz, $\sqrt{m}x$, one concludes that there are no physically meaningful perturbations in this form. Any perturbation of the form $\varphi(\tau, mz, mx)$ is a solution to Eq. (4). The conclusion is that there is no leading order diffraction in this case and the dynamic equation for the v_x is as follows:

$$\frac{\partial v_x}{\partial z} - \frac{\gamma}{2c_0^2} v_x \frac{\partial v_x}{\partial \tau} = 0. \tag{12}$$

It describes a perturbation in a planar wave and has unusual parameter of nonlinearity different from that in the pure acoustic case.

3.2. Perpendicular wave vector and the equilibrium magnetic field

In the case $\theta = \pi/2$, $k_{\|} = k_x$, $k_\perp = k_z$, one arrives at the leading order dispersion relations:

$$\omega = \pm\left(C_\perp k_x + \frac{(C_\perp^4 - c_0^2 C_A^2)k_z^2}{2C_\perp^3 k_x}\right), \qquad C_\perp = \sqrt{c_0^2 + C_A^2}. \tag{13}$$

A perturbation in pressure in the mode propagating in the positive direction of axis x is described by the dynamic equation (we seek perturbations as functions of $\tau = t - x/C_\perp$, mx, $\sqrt{m}z$ and account for nonlinear distortion of a wave):

$$\frac{\partial}{\partial\tau}\left(\frac{\partial p'}{\partial x} - \frac{\varepsilon_\perp}{\rho_0 C_\perp^3} p' \frac{\partial p'}{\partial \tau}\right) = \frac{D_\perp^2 C_\perp}{2}\frac{\partial^2 p'}{\partial z^2}, \tag{14}$$

where

$$D_\perp^2 = \frac{c_0^4 + C_A^4 + c_0^2 C_A^2}{(c_0^4 + C_A^4)^2} = 1 - \frac{2\beta\gamma}{(\beta\gamma + 2)^2}.$$

The parameter of nonlinearity $\varepsilon_\perp$ depends on the equilibrium parameters of a plasma (Chin *et al.*, 2010; Nakariakov *et al.*, 2000):

$$\varepsilon_\perp = \frac{3C_A^2 + (\gamma+1)c_0^2}{2C_\perp^2} = \frac{9 + 4\gamma\beta}{8 + 4\gamma\beta}\varepsilon_{\|}.$$

Equation (14) may be rearranged into the KZ equation by substitution $X = \frac{x}{D_\perp}$. Since $\frac{3}{4} < D_\perp^2 < 1$, the divergence is usual but less manifested in the case of a plasma affected by a magnetic field. All conclusions concerning linear and nonlinear dynamics of perturbations in an acoustic beam can be generalised in the case of a plasma affected by a magnetic field in view of the transversal scaling.

4. Thermal self-action of sound beams

The magnitude of a magnetosonic pressure is larger in the vicinity of a beam's axis than on the periphery. That leads to more effective nonlinear excitation of the entropy mode in this area, that is, to magnetoacoustic heating. Enlargement in temperature entails a change in the local speed of sound propagation. Some kind of a thermal lense forms in the vicinity of axis of a beam. In turn, these uneven variations distort the wave front. In order to study unusual features of perturbations which associate with the anomalous divergence, we focus on the parallel propagation with the speed c_0. Usually, a magnetosonic beam undergoes thermal self-defocusing specific for beams in the gases due to the positive thermal coefficient:

$$\delta = \frac{1}{c_0}\left.\frac{\partial c_0}{\partial T}\right|_p,$$

which equals $\frac{1}{2T_0}$ (T_0 designates the equilibrium temperature). Some damping mechanism along with nonlinearity is a necessary condition for nonlinear transfer of wave energy into the non-wave entropy mode. We consider exclusively thermal conduction χ among these mechanisms. The system of equations which describes nonlinear thermal self-action of a beam, is as follows:

$$\frac{\partial}{\partial\tau}\left(\frac{\partial p'}{\partial z} - \frac{\delta T'}{c_0}\frac{\partial p'}{\partial \tau} - \frac{\gamma+1}{2\rho_0 c_0^3} p'\frac{\partial p'}{\partial\tau} - \frac{(\gamma-1)\chi}{2C_P c_0^3 \rho_0}\frac{\partial^2 p'}{\partial \tau^2}\right) = \pm\frac{D_{\|,0}^2 c_0}{2}\frac{\partial^2 p'}{\partial x^2}, \tag{15}$$

$$\frac{\partial T'}{\partial t} - \frac{\chi}{\rho_0 C_P}\frac{\partial^2 T'}{\partial x^2} = \frac{c_0}{C_p}F_{ms}, \tag{16}$$

where F_{ms} designates the averaged over period magnetosonic force:

$$F_{ms} = \frac{(\gamma-1)\chi}{C_P c_0^5 \rho_0^3}\left\langle\left(\frac{\partial p'}{\partial\tau}\right)^2\right\rangle. \tag{17}$$

T' denotes perturbation of temperature which in the case of periodic excitation satisfies the inhomogeneous diffusion Eq. (16). The acoustic force which is valid for all kinds of exciters has been derived by (Leble, Perelomova, 2018). Equation (15) resembles Eq. (1) (Rudenko, Sapozhnikov, 2004) but differs by the right-hand part which may be negative and includes the scaling coefficient $D_{\|,0}^2$.

In the cases when nonlinear effects dominate over diffraction, the "fast time" may be eliminated in the frames of geometric acoustics. The subsequent analysis which shows the pecularities of anomalous divergence will be done for the example of cylindrically symmetric beam, since the theory is well developed in this particular case (Rudenko, Sapozhnikov, 2004). Introducing new variables $R = \frac{r}{D_{\|,0}}$ and the eikonal $\psi(x,r)$ (r is the transversal to the direction of a beam coordinate) and substituting:

$$p' = (x, r, \theta = \tau - \psi(x,r)/c_0),$$

one arrives at the limit of short wavelengths small in comparison with the scale of thermal inhomogeneties to the equation:

$$\frac{\partial}{\partial\theta}\left(\frac{\partial p'}{\partial z} - \frac{\gamma+1}{2\rho_0 c_0^3}p'\frac{\partial p'}{\partial\theta} - \frac{(\gamma-1)\chi}{2C_P c_0^3\rho_0}\frac{\partial^2 p'}{\partial\theta^2} \pm \frac{\partial p'}{\partial R}\frac{\partial\psi}{\partial R} \pm p'\frac{\partial^2\psi}{\partial R^2} - \frac{1}{c_0}\frac{\partial p'}{\partial\theta}\left(\frac{\partial\psi}{\partial z} \pm \frac{1}{2}\left(\frac{\partial\psi}{\partial R}\right)^2 + \delta T'\right)\right) = \pm c_0\frac{\partial^2 p'}{\partial R^2}. \tag{18}$$

In the frames of geometrical acoustics, one sets the right-hand side of Eq. (18) zero and arrives at the system:

$$\frac{\partial p'}{\partial z} - \frac{\gamma+1}{2\rho_0 c_0^3} p' \frac{\partial p'}{\partial \theta} - \frac{(\gamma-1)\chi}{2C_P c_0^3 \rho_0} \frac{\partial^2 p'}{\partial \theta^2} \pm \frac{\partial p'}{\partial R}\frac{\partial \psi}{\partial R} \pm p' \frac{\partial^2 \psi}{\partial R^2} = 0,$$
$$\frac{\partial \psi}{\partial z} \pm \frac{1}{2}\left(\frac{\partial \psi}{\partial R}\right)^2 + \delta T' = 0. \tag{19}$$

The upper sign in the previous and next formulas designates the normal diffraction, and the lower sign concerns the unusual case. Equation (19) describes among other formation and propagation of a sawtooth wave with a finite shock front with the magnitude A and the frequency ω, so as:

$$p'(z,R,\theta) = A(z,R) \cdot \left(-\frac{\omega\theta}{\pi} + \tanh\left(\frac{(\gamma+1)C_P}{2(\gamma-1)\chi} A(z,R)\theta\right)\right),$$
$$-\frac{\pi}{\omega} \le \theta \le \frac{\pi}{\omega}. \tag{20}$$

Substituting Eq. (20) into (19) and (17) and letting $\chi \to 0$ (this makes the wave saw-tooth shaped) results in equations:

$$\frac{\partial A}{\partial z} + \frac{A^2}{P_0 z_s} \pm \frac{\partial A}{\partial R}\frac{\partial \psi}{\partial R} \pm A\frac{\partial^2 \psi}{\partial R^2} = 0,$$
$$F_{ms} = \frac{(\gamma+1)\omega}{3\pi c_0^5 \rho_0^3} A^3, \tag{21}$$

where

$$z_s = \frac{2\pi c_0^3 \rho_0}{\omega P_0}$$

is the shock formation distance, and P_0 is the initial magnitude of magnetosonic pressure at the axis of a beam. Equation (21) may be resolved by assuming the parabolic wave form (where $f(z,t)$ is some function):

$$\psi(z,R,t) = \phi(z,t) \pm \frac{R^2}{2}\frac{\partial}{\partial z}\ln f(z,t)$$

with the solution:

$$A = \frac{P_0}{f(z,t)} \Phi\left(\frac{R}{a_0 f(z,t)}\right) \cdot \left(1 + \frac{1}{z_s}\Phi\left(\frac{R}{a_0 f(z,t)}\right)\int_0^z \frac{\mathrm{d}y}{f(y,t)}\right),$$

where a_0 is the initial beam radius, and Φ is responsible for the pressure transversal distribution, $A(z=0,R) = P_0\Phi(R/a_0)$. The eikonal equation takes the form:

$$\pm \frac{\partial^2 f}{\partial z^2} = \delta T_2 f, \tag{22}$$

where T_2 is the coefficient in the transversal expansion:

$$T' = T_0 - \frac{R^2}{2} T_2(z,t). \tag{23}$$

4.1. Non-stationary self-focusing

In the case of small thermal conductivity, when the diffusion term in Eq. (16) may be neglected, a variation in temperature takes the form:

$$\frac{\partial T'}{\partial t} = \frac{(\gamma+1)\omega}{3\pi c_0^4 \rho_0^2 C_P} A^3.$$

The function f follows from Eqs (22) and (23):

$$f^5 \left(1 + \int_0^Z \frac{\mathrm{d}y}{f(y,\Theta)}\right)^4 \frac{\partial}{\partial \Theta}\left(\frac{1}{f}\frac{\partial^2 f}{\partial Z^2}\right) = \pm 1, \tag{24}$$

where the following dimensionless quantities are introduced:

$$Z = \frac{z}{z_s}, \qquad \Theta = \frac{t}{t_0}, \qquad t_0 = \frac{(\gamma+1)\omega\rho_0 C_P a_0^2 T_0}{4\pi c_0^2 P_0}.$$

The boundary and initial conditions for initially non-focused beam are as follows:

$$f(Z=0,\Theta) = f(Z,\Theta=0) = 1, \qquad \frac{\partial f}{\partial Z}(Z=0,\Theta) = 0.$$

Equation (24) recalls the dynamic equation (Eq. (22) in Rudenko, Sapozhnikov, 2004) but has a different meaning. The sign plus corresponds to the normal defocusing in gases, and the sign minus corresponds to anomalous divergence. Hence, the unusual divergence corresponds to the focusing which specifies majority of liquids (apart from water).

4.2. Stationary self-focusing

We consider stationary temperature field in Eq. (16), so that $\frac{\partial T'}{\partial t} = 0$. An equation for the unknown function f takes the form:

$$\left(1 + \Pi \int_0^Z \frac{\mathrm{d}y}{f(y)}\right)^3 f^2 \frac{d^2 f}{\mathrm{d}Z^2} = \pm \Pi^3, \tag{25}$$

where

$$z_0 = \frac{\pi^2 c_0^5 \rho_0}{3T_0\chi(\gamma+1)^2\omega^2}, \qquad Z = \frac{z}{z_0}, \qquad \Pi = \frac{z_0}{z_s}.$$

The boundary conditions for an initially unfocused beam are:

$$f(Z=0) = 1, \qquad \frac{\mathrm{d}f}{\mathrm{d}Z}(Z=0) = 0.$$

Hence, the case of anomalous divergence may be considered as usual with the normal divergence but with the negative coefficient, $-\delta$ (Eq. (20) in (Rudenko, Sapozhnikov, 2004)). In this case, unusual focusing of a beam in a gaseous plasma occurs. This corresponds to the sign minus on the right of Eq. (25). For a Gaussian at $Z = 0$ beam, $\Phi(\xi) = \exp(-\xi^2)$, and the characteristic width of a beam is defined as the transversal distance at which the peak pressure is less e times its value at the axis:

$$\frac{a}{a_0} = f\sqrt{\log\left(e + (e-1)\int\limits_0^Z \frac{\mathrm{d}y}{f(y)}\right)}.$$

Figure 2 shows the characteristic width of a beam in the case of stationary and non-stationary thermal self-action in the normal (defocusing) and unusual cases (focusing). Numerical calculations of Eqs (24) and (25) with the appropriate initial and boundary conditions have been undertaken in *Mathematica*.

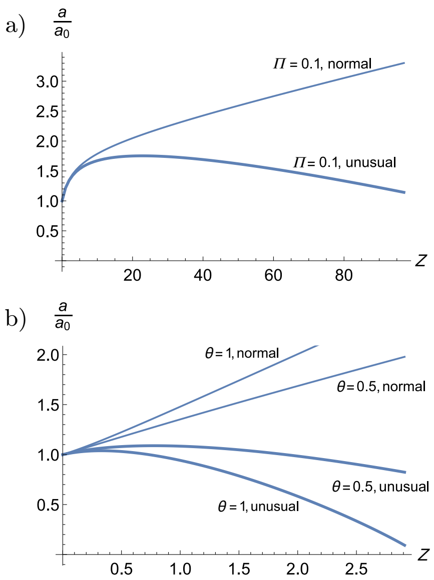

Fig. 2. Dimensionless width of a beam which is planar at a transducer: a) stationary self-action, b) non-stationary self-action.

5. Concluding remarks

The unusual diffraction behaviour of magnetohydrodynamic beams in a plasma is discovered. The beams oblique to the magnetic field do not reveal diffraction. The beams directed along the magnetic field (this is the case $\theta = 0$) may behave unusually in dependence to the ratio of the sound speed c_0 and the speed of Alfvénic mode C_A. In particular, the unusual diffraction (that means the minus sign by the diffraction term) specifies a sound beam which propagates with the speed c_0 if $C_A > c_0$. The discrepancy of a beam during parallel propagation is more manifested as compared to the Newtonian case. As for magnetosound beam propagating perpendicularly to the magnetic field with the speed $\sqrt{c_0^2 + C_A^2}$, it reveals normal but somewhat smaller diffraction than a Newtonian beam. Probably, the unusual cases do not have counterparts in the wave theory. The links of perturbations which specify every mode on par with dispersion relations in the cases of parallel or perpendicular propagation are derived. They depend on the equilibrium parameters of a plasma. In general, these links include integral operators and may be used as indicators of individual kinds of plasma motion. The magnetosonic modes are p-modes, that is, a perturbation in pressure is not zero for these modes. It is discovered that the case $C = c_0 = C_A$ is very special. The links of perturbations combine properties of acoustic and magnetic modes, Eqs (11). A beam does not undergo diffraction and behaves as a planar wave with especial parameter of nonlinearity $\gamma/2$.

The nonlinear thermal self-action is also adjusted by transversal scaling. One may suppose that the thermal self-action may occur unusually if a beam diverges unusually. It turns out that the anomalous diffraction operates in such a way that it changes the sign of thermal coefficient δ. This corresponds to the thermal focusing of a beam instead of defocusing which normally takes place in all gases, that is, to the negative thermal coefficient δ. The stationary and non-stationary self-action is considered. The only damping mechanism which is taken into account, is the thermal conduction of a plasma. The thermal self-action leads to formation of thermal lenses with the exception of the case $C = c_0 = C_A$ when it leads to formation of a heated flat layer. In this case, diffraction is not of importance in concerning to a beam itself and to the thermal effects in its field. The non-acoustic beams propagate with the speed C_A, which also depends on the temperature. But it is not a leading order coupling of these modes with the entropy mode. Hence, this is not a case of thermal self-action.

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