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Acoustic heating produced in the thermoviscous flow of a Bingham plastic

Research Article

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Abstract: This study is devoted to the instantaneous acoustic heating of a Bingham plastic. The model of the Bingham plastic's viscous stress tensor includes the yield stress along with the shear viscosity, which differentiates a Bingham plastic from a viscous Newtonian fluid. A special linear combination of the conservation equations in differential form makes it possible to reduce all acoustic terms in the linear part of of the final equation governing acoustic heating, and to retain those belonging to the thermal mode. The nonlinear terms of the final equation are a result of interaction between sounds and the thermal mode. In the field of intense sound, the resulting nonlinear acoustic terms form a driving force for the heating. The final governing dynamic equation of the thermal mode is valid in a weakly nonlinear flow. It is instantaneous, and does not imply that sounds be periodic. The equations governing the dynamics of both sounds and the thermal mode depend on sign of the shear rate. An example of the propagation of a bipolar initially acoustic pulse and the evolution of the heating induced by it is illustrated and discussed.

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1. Introduction

A Bingham plastic is a viscoplastic material that behaves as a rigid body at low stresses but flows as a viscous fluid at high stress. The mathematical model of such a non-newtonian liquid was first proposed by Bingham [1]. A Bingham plastic does not exhibit any shear rate (no flow and thus no velocity) until a certain stress is achieved. For the Newtonian fluid the shear stress linearly depends on shear rate, and the coefficient of proportionality between these two values is the viscosity. By contrast the Bingham Plastic requires two parameters, the yield stress and the plastic viscosity. The physical reason for the non-Newtonian behavior is that the plastic liquid contains particles (e.g. clay) or large molecules (e.g. polymers) which have some kind of interaction, creating a weak solid structure. A certain amount of stress is required to break this structure [2, 3]. Once the structure has been broken, the liquid particles move under viscous forces. If the stress is removed, the particles associate again. A Bingham plastic is used, among other applications, as a mathematical model of mud flow in offshore engineering, and in the handling of slurries. Common examples include toothpaste, which will not be extruded until a certain hy-

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drostatic pressure is used on the tube, as well as paints. It is well known that sound attenuates linearly in the standard thermoviscous flow of a fluid. The acoustic heating is an increase of the ambient fluid temperature caused by *nonlinear* losses in acoustic energy. It is not an acoustic quantity but a value referred to the entropy, or thermal mode. The acoustic heating in the standard thermoviscous fluid flows is well-studied theoretically and experimentally regarding periodic sound as the origin of heating [4, 5]. Interest in acoustic heating in non-Newtonian fluids has grown in recent years in connection with biomedical applications. These require accurate estimations of heating during medical therapies which apply sounds of different kinds including non-periodic ones, particularly impulses [6, 7].

This study is devoted to nonlinear dissipation of sound energy in a Bingham plastic fluid. The mathematical technique has been worked out and applied previously by the author to some problems of thermoviscous nonlinear flow. It allows separation of the individual equations governing sound, vorticity and entropy modes in Newtonian and non-Newtonian (relaxing) fluids [8–10]. The method and results based upon it concerning flow over a Bingham plastic are described in Sections 3, 4. The important feature of the equations governing sound and corresponding acoustic heating is shown: they dependent on the sign of the shear rate. The dynamics of a bipolar pulse is illustrated as an example (Sec. 5).

2. Dynamic equations in a Bingham plastic fluid

The continuity, momentum and energy equations describing a thermoviscous fluid flow without external forces read:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \\ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \frac{1}{\rho}\left(-\vec{\nabla} p + Div\, \mathbf{P}\right), \\ \frac{\partial e}{\partial t} + (\vec{v} \cdot \vec{\nabla})e \\ = \frac{1}{\rho}\left(-p(\vec{\nabla} \cdot \vec{v}) + \chi \Delta T + \mathbf{P} : Grad\, \vec{v}\right). \end{aligned} \tag{1}$$

Here, $\vec{v}$ denotes the velocity of the fluid, ρ, p are the density and pressure, respectively; e, T denote the internal energy per unit mass and the temperature, respectively; χ is the thermal conductivity, and x_i, t are the spatial coordinates and time. The operator Div denotes the divergence of a tensor and $Grad$ is a dyad gradient. $\mathbf{P}$ is the viscous stress tensor. In the model of a Bingham plastic, the viscous stress tensor relates to the shear rate in the following manner:

$$\mathbf{P}_{i,k} = \begin{cases} P_0 + \frac{\eta}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right), & \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) > 0, \\ -P_0 + \frac{\eta}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right), & \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) < 0. \end{cases} \tag{2}$$

A liquid is therefore rigid for shear stress $|P_{i,k}|$, less than a critical value P_0. Two thermodynamic functions $e(p,\rho)$, $T(p,\rho)$ complement the system (1). They may be written as series for the excess internal energy $e' = e - e_0$ and temperature $T' = T - T_0$ in powers of excess pressure and density $p' = p - p_0$, $\rho' = \rho - \rho_0$ (ambient quantities are marked by index 0):

$$\begin{aligned} e' = \frac{E_1}{\rho_0}p' + \frac{E_2 p_0}{\rho_0^2}\rho' + \frac{E_3}{p_0\rho_0}p'^2 + \frac{E_4 p_0}{\rho_0^3}\rho'^2 \\ + \frac{E_5}{\rho_0^2}\rho' p', \\ T' = \frac{\Theta_1}{\rho_0 C_v}p' + \frac{\Theta_2 p_0}{\rho_0^2 C_v}\rho' + \frac{\Theta_3}{p_0\rho_0 C_v}p'^2 + \frac{\Theta_4 p_0}{\rho_0^3 C_v}\rho'^2 \\ + \frac{\Theta_5}{\rho_0^2 C_v}\rho' p', \end{aligned} \tag{3}$$

where $E_1, \ldots \Theta_5$ are dimensionless coefficients and C_v marks the heat capacity per unit mass under constant volume. The series (3) allows consideration of a wide variety of fluids in the general form: discrepancies are manifested by the different coefficients for different fluids. The common practice in nonlinear acoustics is to focus on the equations of the second order in the acoustic Mach number $M = v_0/c_0$, where v_0 is the magnitude of particle velocity and $c_0 = \sqrt{\frac{(1-E_2)p_0}{E_1\rho_0}}$ is an infinitesimal signal velocity in the Newtonian liquid (when $P_0 = 0$), respectively. The present study is also confined to consideration of nonlinearity of the second order, so that in the series (3) only terms up to the second order are kept. The expressions for the coefficients E_1 and E_2 in terms of the compressibility, κ, and the thermal expansion, β, are as follows:

$$E_1 = \frac{\rho_0 C_v \kappa}{\beta}, \quad E_2 = -\frac{C_p \rho_0}{\beta p_0} + 1. \tag{4}$$

C_p denotes the heat capacity per unit mass under constant pressure,

$$\begin{aligned} \kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T, \\ \beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p. \end{aligned} \tag{5}$$

A small variation in entropy is a total differential that provides a relationship between the first coefficients in the series (3):

$$\Theta_2 = \frac{C_v \rho_0 T_0}{E_1 p_0} - \frac{(1-E_2)\Theta_1}{E_1}. \tag{6}$$

The following small dimensionless parameters will be those responsible for viscoplastic viscosity, $\mu = \eta\Omega/(\rho_0 c_0^2)$ and the yield stress, $\Phi_0 = \frac{P_0}{E_1\rho_0 c_0^2}$ (Ω is the characteristic sound frequency). We consider also the small thermal conductivity: $\delta = \frac{\chi T_0 \Omega}{c_0^4 E_1^2 \rho_0}$, which is supposed to be of the same order as μ and Φ_0. We shall consider weakly nonlinear flows discarding $\mathcal{O}(M^3)$ terms in all expansions and confining of terms to be considered to include μ^0 or μ^1. The resulting model equations will account therefore for the combined effects of nonlinearity, attenuation, and thermal conductivity.

3. Definition of modes in planar flow of infinitesimal amplitude

We consider one-dimensional flow along the Ox axis. The dispersion equation follows from the linearized version of Eq. (1). Its roots determine three independent modes (or types of linear flow) of infinitesimal signal disturbances in an unbounded fluid. In one dimension, there exist the acoustic (two branches), and the thermal (or entropy) modes. In general, every perturbation of the field variables contains contributions from any of the three modes, for example, $\rho' = \rho'_{a,1} + \rho'_{a,2} + \rho'_e$. That allows separation in the linear part of the governing equations using the specific properties of the modes, namely the relationships between components of velocity and excess quantities of two thermodynamic functions, for example pressure and density. The method developed by the author in [8–10] provides the possibility of consequent decoupling of the initial system (1) into specific dynamic equations for every mode based upon the specific properties of each mode in a weakly nonlinear and standard thermoviscous flow as well. All formulae everywhere below in the text, including links between modes and the governing equations, are written in leading order with respect to powers of the small parameters M, μ, Φ_0, and δ.
It is convenient to rearrange formulae in the dimensionless quantities in the following way:

$$p^* = \frac{p'}{c_0^2 \cdot \rho_0},\ \rho^* = \frac{\rho'}{\rho_0},\ v^* = \frac{v}{c_0},\ x^* = \frac{\Omega x}{c_0},\ t^* = \Omega t. \tag{7}$$

Starting from Eq. (8), the upper indexes (asterisks) denoting the dimensionless quantities will be omitted everywhere in the text. In the dimensionless quantities, Eq. (1) accounting for Eqs. (2, 3) reads:

$$\begin{aligned}
\frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} - \mu\frac{\partial^2 v}{\partial x^2} &= -v\frac{\partial v}{\partial x} + \rho\frac{\partial p}{\partial x} - \mu\rho\frac{\partial^2 v}{\partial x^2}, \\
\frac{\partial p}{\partial t} + (1+\Phi)\frac{\partial v}{\partial x} - \delta_1\frac{\partial^2 p}{\partial x^2} - \delta_2\frac{\partial^2 \rho}{\partial x^2} &= -v\frac{\partial p}{\partial x} + (D_1 p + D_2\rho)\frac{\partial v}{\partial x} + \frac{\mu}{E_1}\left(\frac{\partial v}{\partial x}\right)^2 \\
+\delta_3\frac{\partial^2 p^2}{\partial x^2} + \delta_4\frac{\partial^2 \rho^2}{\partial x^2} + \delta_5\frac{\partial^2(\rho p)}{\partial x^2} + \Phi\rho\left(\frac{\partial v}{\partial x}\right), \\
\frac{\partial \rho}{\partial t} + \frac{\partial v}{\partial x} &= -v\frac{\partial \rho}{\partial x} - \rho\frac{\partial v}{\partial x},
\end{aligned} \tag{8}$$

where

$$\Phi\left(\operatorname{sgn}\left(\frac{\partial v}{\partial x}\right)\right) = \begin{cases} -\frac{P_0}{E_1\rho_0 c_0^2} \equiv -\Phi_0, & \left(\frac{\partial v}{\partial x}\right) > 0, \\ \frac{P_0}{E_1\rho_0 c_0^2} \equiv \Phi_0, & \left(\frac{\partial v}{\partial x}\right) < 0. \end{cases} \tag{9}$$

originates from the yield stress. The main nonlinear terms of order M^2 form the right-hand side of the set (8). The dynamic equations in the rearranged form include the dimensionless quantities

$$\begin{aligned}
\delta_1 = \frac{\chi\Theta_1\Omega}{\rho_0 c_0^2 C_v E_1}, \delta_2 = \frac{\chi\Theta_2\Omega}{\rho_0 c_0^2 C_v (1-E_2)}, \\
\delta_3 = \frac{\Theta_3\chi\Omega}{E_1\rho_0 c_0^2 C_v}\frac{1-E_2}{E_1}, \delta_4 = \frac{\Theta_4\chi\Omega}{(1-E_2)\rho_0 c_0^2 \lambda C_v}, \\
\delta_5 = \frac{\Theta_5\chi\Omega}{E_1\rho_0 c_0^2 C_v}, \\
D_1 = \frac{1}{E_1}\left(-1 + 2\frac{1-E_2}{E_1}E_3 + E_5\right), \\
D_2 = \frac{1}{1-E_2}\left(1 + E_2 + 2E_4 + \frac{1-E_2}{E_1}E_5\right).
\end{aligned} \tag{10}$$

The sum of two first coefficients is the linear attenuation due to the thermal conductivity, $\delta = \delta_1 + \delta_2$. The linearized version of Eq. (8) describes a flow of infinitesimal magnitude, when $M \to 0$:

$$\begin{aligned}
\frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} - \mu\frac{\partial^2 v}{\partial x^2} &= 0, \\
\frac{\partial p}{\partial t} + (1+\Phi)\frac{\partial v}{\partial x} - \delta_1\frac{\partial^2 p}{\partial x^2} - \delta_2\frac{\partial^2 \rho}{\partial x^2} &= 0, \\
\frac{\partial \rho}{\partial t} + \frac{\partial v}{\partial x} &= 0.
\end{aligned} \tag{11}$$

The linear hydrodynamic field is represented by acoustic modes, propagating in the positive or negative direction of the Ox axis, respectively, and the non-wave thermal, or entropy, mode. Every type of motion is determined in

fact by one of the roots of the dispersion relation of the linear flow, $\omega(k)$ (k is the wave number) [4, 5, 11] and fixes the relations between hydrodynamic perturbations, which are independent of time [8–10]. The dispersion relations for sound propagating in the positive Ox axis direction (marked by index 1), sound propagating in the negative Ox axis direction (marked by index 2), and the entropy mode (marked by index 3) in a Bingham plastic are as follows:

$$\begin{aligned} \omega_{a,1} &= k\left(1+\frac{\Phi}{2}\right)+\frac{ik^2}{2}(\mu+\delta),\\ \omega_{a,2} &= -k\left(1+\frac{\Phi}{2}\right)+\frac{ik^2}{2}(\mu+\delta),\\ \omega_e &= -ik^2\delta_2. \end{aligned} \tag{12}$$

They uniquely determine relations between velocity, excess density and pressure attributable to every mode, which are valid at any time of a hydrodynamic field evolution:

$$\begin{aligned} \psi_{a,1} &= \begin{pmatrix} v_{a,1}\\ p_{a,1}\\ \rho_{a,1}\end{pmatrix} = \begin{pmatrix} 1\\ \left(1+\frac{\Phi}{2}\right)+\frac{\mu-\delta}{2}\frac{\partial}{\partial x}\\ \left(1-\frac{\Phi}{2}\right)+\frac{\mu+\delta}{2}\frac{\partial}{\partial x}\end{pmatrix} v_{a,1},\\ \psi_{a,2} &= \begin{pmatrix} 1\\ -\left(1+\frac{\Phi}{2}\right)+\frac{\mu-\delta}{2}\frac{\partial}{\partial x}\\ -\left(1-\frac{\Phi}{2}\right)+\frac{\mu+\delta}{2}\frac{\partial}{\partial x}\end{pmatrix} v_{a,2},\\ \psi_e &= \begin{pmatrix} \delta_2\frac{\partial}{\partial x}\\ 0\\ 1\end{pmatrix}\rho_e. \end{aligned} \tag{13}$$

The important and unusual property of both acoustic roots of the dispersion relation and the modes correspondent to them is that they depend (by means of Φ) on the sign of the velocity gradient. The linear equation describing the fluid velocity of an acoustic wave propagating in the positive Ox axis direction agrees with $\omega_{a,1}$ from Eq. (12) and takes the form:

$$\frac{\partial v_{a,1}}{\partial t}+\left(1+\frac{\Phi}{2}\right)\frac{\partial v_{a,1}}{\partial x}-\frac{\mu+\delta}{2}\frac{\partial^2 v_{a,1}}{\partial x^2}=0, \tag{14}$$

which describes differently the cases of positive and negative shear rates:

$$\begin{aligned} &\frac{\partial v_{a,1}}{\partial t}+\left(1-\frac{\Phi_0}{2}\right)\frac{\partial v_{a,1}}{\partial x}-\frac{\mu+\delta}{2}\frac{\partial^2 v_{a,1}}{\partial x^2}=0,\\ &\qquad\qquad\qquad\qquad\qquad\qquad\frac{\partial v_{a,1}}{\partial x}>0,\\ &\frac{\partial v_{a,1}}{\partial t}+\left(1+\frac{\Phi_0}{2}\right)\frac{\partial v_{a,1}}{\partial x}-\frac{\mu+\delta}{2}\frac{\partial^2 v_{a,1}}{\partial x^2}=0,\\ &\qquad\qquad\qquad\qquad\qquad\qquad\frac{\partial v_{a,1}}{\partial x}<0. \end{aligned} \tag{15}$$

The density perturbation in the entropy motion satisfies the diffusion equation:

$$\frac{\partial \rho_e}{\partial t}+\delta_2\frac{\partial^2\rho_e}{\partial x^2}=0. \tag{16}$$

Equations for every type of motion may be also extracted from the system (8) in accordance to relationships specific for each mode. That can proceed formally by means of projecting of the equations into specific sub-spaces. The linear dynamic equations are obviously independent.

4. Dynamic equations in a weakly nonlinear flow

4.1. Weakly nonlinear dynamic equation of sound

The nonlinear terms in every equation from the right-hand side of system (8) include in general parts attributable to every mode. We fix relations determining every mode in a linear flow and consider every excess quantity as a sum of the specific excess quantities of all modes. The consequent decomposition of the governing equations for both branches of sound and the thermal mode may be still done by means of linear projection, for details see [10]. Projection points out a method of linearly combining the equations in order to keep terms belonging to the appropriate mode in the linear part and to reduce all other terms there. Keeping only terms corresponding to the acoustic rightwards propagating wave in the nonlinear part, and expressing all acoustic quantities in terms of velocity by use of links ($\psi_{a,1}$ from Eqs. (13)), one can easily obtain an equation analogous to the well-known Burgers' equation:

$$\begin{aligned} &\frac{\partial v_{a,1}}{\partial t}+\left(1+\frac{\Phi}{2}\right)\frac{\partial v_{a,1}}{\partial x}-\frac{\mu+\delta}{2}\frac{\partial^2 v_{a,1}}{\partial x^2}\\ &\qquad=-\frac{1-D_1-D_2}{2}v_{a,1}\frac{\partial}{\partial x}v_{a,1}. \end{aligned} \tag{17}$$

The nonlinear term in the right-hand side of Eq. (17) may be considered as a result of self-action of sound which corrects the linear equation governing sound (14) by nonlinear terms.

4.2. The thermal mode in a sound-dominant field. Acoustic heating.

In the context of acoustic heating, the magnitude of excess density specific for the entropy mode is small compared to that for sound. We consider the ratio of the characteristic

amplitudes of the excess densities specifying entropy motion and sound to be of order M. The modes (13) satisfy in leading order (up to terms of order δ^2) the equality as follows:

$$\begin{pmatrix} -\delta\frac{\partial}{\partial x} & -1+\Phi & 1 \end{pmatrix} \begin{pmatrix} v_{a,1}+v_{a,2}+v_e \\ p_{a,1}+p_{a,2}+p_e \\ \rho_{a,1}+\rho_{a,2}+\rho_e \end{pmatrix} = \rho_e, \tag{18}$$

which points out a way of combining equations (8) in order to reduce all acoustic quantities in the linear part of the final equations. The important property of the projection is not only to decompose specific perturbations in the linear part of the equations, but to distribute nonlinear terms correctly between the different dynamic equations. The links within the sound equations should be supplemented by nonlinear quadratic terms making sound isentropic to leading order. These corrections in the Bingham plastic are similar to these, specific for the Riemann wave in an ideal gas [12]:

$$\begin{aligned} p_{a,1} &= \left(1+\frac{\Phi}{2}\right) v_{a,1} + \frac{\mu-\delta}{2}\frac{\partial}{\partial x} v_{a,1} \\ &\quad + \frac{1}{4}(1-D_1-D_2)v_{a,1}^2, \\ \rho_{a,1} &= \left(1-\frac{\Phi}{2}\right) v_{a,1} + \frac{\mu+\delta}{2}\frac{\partial}{\partial x} v_{a,1} \\ &\quad + \frac{1}{4}(3+D_1+D_2)v_{a,1}^2. \end{aligned} \tag{19}$$

but involve additional terms proportional to Φ, μ and δ. The nonlinear corrections of second and higher order depend on equation of state.

For simplicity, let a sound be associated only with wave propagation in the positive Ox axis direction: $p_a = p_{a,1}$, $\rho_a = \rho_{a,1}$ $v_a = v_{a,1}$. The linear combination of the left-hand sides of the equations in (8) in accordance with (18) and (19) results in the equality:

$$\begin{aligned} \frac{\partial}{\partial t}\left(-\delta\frac{\partial}{\partial x}v-(1-\Phi)p+\rho\right) - \delta\frac{\partial^2}{\partial x^2}p + \delta_1\frac{\partial^2 p}{\partial x^2} + \delta_2\frac{\partial^2\rho}{\partial x^2} \approx \frac{\partial}{\partial t}\rho_e + \delta_2\frac{\partial^2\rho_e}{\partial x^2} - \delta\frac{(D_1+D_2-1)}{4}\frac{\partial^2}{\partial x^2}v_a^2 \\ + \frac{(1+D_1+D_2)}{2}\left(-\frac{\partial}{\partial x}v_a^2 + \delta_2\frac{\partial^2}{\partial x^2}v_a^2 + (\mu+\delta)v_a\frac{\partial^2}{\partial x^2}v_a\right) - \frac{\Phi}{2}\frac{\partial}{\partial x}v_a^2. \end{aligned} \tag{20}$$

In the prior simple evaluations, the corrected links (19) are used, as well as the dynamic equation (17) to exclude the partial derivative with respect to time in the nonlinear terms. In the context of acoustic heating, the sound dominates, so that only acoustic quadratic terms should be kept. Combining in a similar way the right-hand sides of the equations from the set (8), and comparing the result with Eq. (20), one obtains the dynamic equation governing acoustic heating:

$$\begin{aligned} \frac{\partial}{\partial t}\rho_e + \delta_2\frac{\partial^2\rho_e}{\partial x^2} - \delta\frac{(D_1+D_2-1)}{4}\frac{\partial^2}{\partial x^2}v_a^2 + \frac{(1+D_1+D_2)}{2}\left(-\frac{\partial}{\partial x}v_a^2 + \delta_2\frac{\partial^2}{\partial x^2}v_a^2 + (\mu+\delta)v_a\frac{\partial^2}{\partial x^2}v_a\right) - \frac{\Phi}{2}\frac{\partial}{\partial x}v_a^2 \\ = -\frac{(1+D_1+D_2)}{2}\frac{\partial}{\partial x}v_a^2 - \frac{\mu}{E_1}\left(\frac{\partial v_a}{\partial x}\right)^2 + \frac{\delta}{2}(D_1-D_2-1)\left(\frac{\partial v_a}{\partial x}\right)^2 - \frac{\mu}{2}(1+D_1+D_2)\left(\frac{\partial v_a}{\partial x}\right)^2 \\ + \frac{\Phi(D_1+3D_2+1)}{4}\frac{\partial}{\partial x}v_a^2 - \delta v_a\frac{\partial^2}{\partial x^2}v_a, \end{aligned} \tag{21}$$

which becomes simpler after ordering:

$$\begin{aligned} \frac{\partial}{\partial t}\rho_e + \delta_2\frac{\partial^2\rho_e}{\partial x^2} = \delta\left(-2v_a\frac{\partial^2}{\partial x^2}v_a + (D_1-1)\left(\frac{\partial v_a}{\partial x}\right)^2\right) - \delta_2\frac{(1+D_1+D_2)}{2}\frac{\partial^2}{\partial x^2}v_a^2 \\ -\mu\left(\frac{(1+D_1+D_2)}{4}\frac{\partial^2}{\partial x^2}v_a^2 + \frac{1}{E_1}\left(\frac{\partial v_a}{\partial x}\right)^2\right) + \Phi\frac{(D_1+3D_2+3)}{4}\frac{\partial}{\partial x}v_a^2. \end{aligned} \tag{22}$$

It is remarkable that the dynamic equation for acoustic heating is a result of combining the energy and continuity equations in the absence of thermal conduction. Otherwise, it is a result of combining the energy, continuity,

and momentum equations in accordance with Eq. (18). The acoustic terms of the leftward propagating sound become completely reduced in the linear part of the final equation. Consideration in this chapter is restricted to an acoustic field represented by rightward propagating sound (i.e, in the positive direction of the Ox axis), though it may be easily expanded to leftward propagation or any mixture of the two acoustic branches.

5. Heating of a Bingham plastic by a bipolar impulse

Solving Eq. (22), governing the decrease in the ambient density ρ_e, is a fairly complex problem, because the excess acoustic density itself should satisfy Eq. (17). Both dynamic equations are nonlinear and account for attenuation due to thermal conduction and viscosity. The equation governing dynamics of ρ_e includes nonlinear acoustic terms standing by dissipative coefficients. They form the nonlinear source of acoustic heating and reflect the fact that the origins of the phenomenon are both nonlinearity and absorption. The diffusion equation (22) is instantaneous: it describes dynamics of the thermal mode in any time and does not require periodic sounds. Let us consider only terms originating from the plastic viscosity both in the governing equations of sound and for the excess density attributable to the entropy mode (Eqs. (17, 22)). We arrive at the following system of equations governing sound:

$$\frac{\partial v_a}{\partial t}+(1-\Phi_0)\frac{\partial v_a}{\partial x}-\frac{\mu}{2}\frac{\partial^2 v_a}{\partial x^2}=-\frac{1-D_1-D_2}{2}v_a\frac{\partial}{\partial x}v_a,\quad \frac{\partial v_a}{\partial x}>0,$$
$$\frac{\partial v_a}{\partial t}+(1+\Phi_0)\frac{\partial v_a}{\partial x}-\frac{\mu}{2}\frac{\partial^2 v_a}{\partial x^2}=-\frac{1-D_1-D_2}{2}v_a\frac{\partial}{\partial x}v_a,\quad \frac{\partial v_a}{\partial x}<0, \tag{23}$$

and the excess density of the entropy mode

$$\frac{\partial}{\partial t}\rho_e=-\mu\left(\frac{(1+D_1+D_2)}{2}\frac{\partial^2}{\partial x^2}v_a^2+\frac{1}{E_1}\left(\frac{\partial v_a}{\partial x}\right)^2\right)-\Phi_0\frac{(D_1+3D_2+3)}{4}\frac{\partial}{\partial x}v_a^2,\ \frac{\partial v_a}{\partial x}>0,$$
$$\frac{\partial}{\partial t}\rho_e=-\mu\left(\frac{(1+D_1+D_2)}{2}\frac{\partial^2}{\partial x^2}v_a^2+\frac{1}{E_1}\left(\frac{\partial v_a}{\partial x}\right)^2\right)+\Phi_0\frac{(D_1+3D_2+3)}{4}\frac{\partial}{\partial x}v_a^2,\ \frac{\partial v_a}{\partial x}<0. \tag{24}$$

Eqs. (24) describe differently domains of positive and negative velocity gradients. To simplify estimation, we will refer to an acoustic pulse satisfying the linear wave equations (15). The excess temperature $T_{e,0}$ (a non-wave quantity after a pulse passes), takes the following form in view of Eqs. (3):

$$T_{e,0}(x)=\frac{\Theta_2 p_0}{\rho_0 C_v}\rho_{e,0}(x)=-\frac{1}{\beta}\rho_{e,0}=-\frac{1}{\beta}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\rho_e dt$$
$$=\frac{\mu}{\beta E_1}S_1+\Phi_0\frac{(D_1+3D_2+3)}{4\beta}S_2+\mu\frac{(1+D_1+D_2)}{2\beta}S_3, \tag{25}$$

where

$$S_1(x)=\int_{-\infty}^{\infty}dt\left(\frac{\partial v_a}{\partial x}\right)^2,$$
$$S_2(x)=\begin{cases}\int_{-\infty}^{\infty}dt\frac{\partial}{\partial x}v_a^2, & \frac{\partial}{\partial x}v_a>0,\\ -\int_{-\infty}^{\infty}dt\frac{\partial}{\partial x}v_a^2, & \frac{\partial}{\partial x}v_a<0,\end{cases}$$
$$S_3(x)=\int_{-\infty}^{\infty}dt\frac{\partial^2}{\partial x^2}v_a^2.$$

It is remarkable that evaluation of S_2 is different for positive and negative velocity gradients. As example, we consider an initially bipolar symmetric pulse

$$\frac{1}{2M}v(t,x=0)=\begin{cases}t+1/\Phi_0, & -1/\Phi_0\le t<-1/2\Phi_0,\\ -t, & -1/2\Phi_0\le t\le 1/2\Phi_0,\\ t-1/\Phi_0, & 1/2\Phi_0<t\le 1/\Phi_0.\end{cases} \tag{26}$$

In order to consider the exclusive properties of heating in a Bingham plastic, we account for variations in the speed of the pulse caused by Φ. That means that only two first terms in the governing equations (23) are taken into account. The parts with positive velocity gradient move slower than those with negative velocity gradients. Fig. 1a shows the initial waveform as a function of the retarded time $t-x$ at $x=0$ and waveforms at $x=0.2/\Phi_0$ and $x=0.5/\Phi_0$ (marked by 1, 2 and 3, correspondingly). It is also remarkable that a bipolar pulse transforms into a unipolar one. A similar behavior for acoustic pulses has been observed experimentally and explained theoretically in continuum solid media with hysteresis [13]. Fig. 1b illustrates the terms in the right-hand part of Eq. (25). The last term in the right-hand part of Eq. (25), proportional to S_3, is zero, but S_1 and S_2 are positive. Evaluation of the factors standing by S_1 and S_2 requires knowledge about the thermodynamic state of the Bingham liquid. Unfortunately, the thermodynamic data which would make possible the evaluation of the compressibility β and the coefficients D_1, D_2, and E_1 are absent in the literature. The only kind of a fluid where the equations of state are known analytically along with all relative coefficients is an ideal gas [4, 5]. The thermodynamics of a Bingham plastic certainly differs from those of an ideal gas, but it seems

useful to recall the properties of an ideal gas in evaluating at least the sign of terms depending on S_1, S_2 or S_3. Thermodynamics of an ideal gas gives $D_1 = -\gamma$, $D_2 = 0$, and $E_1 = 1/(\gamma-1)$, where γ is the ratio of specific heats in an ideal gas. Hence, coefficients depending on S_2 and S_3 are positive in an ideal gas. They are expected to be positive in the case of a Bingham plastic, providing a positive variation in temperature due to loss in acoustic energy. The term associated with S_1 relates to the standard shear viscosity [9], but the origin of the term associated with S_2 is specific for a Bingham plastic's yield stress. This last term depends on the sign of the shear rate.

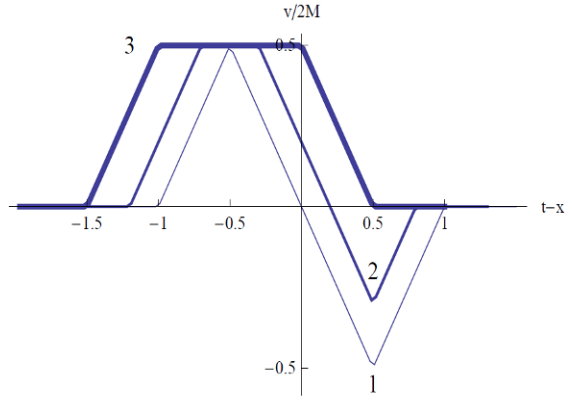

Fig. 1a

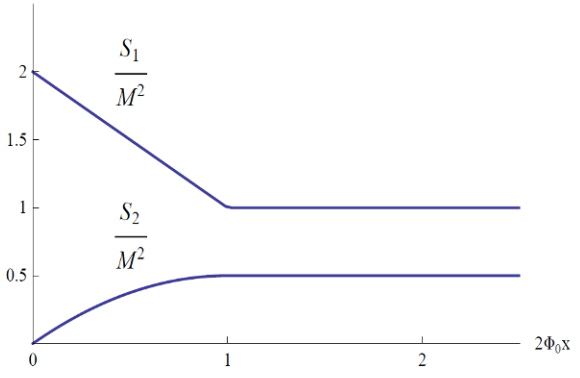

Fig. 1b

Figure 1. (a) Velocity in an acoustic pulse $\frac{1}{2M}v_a(t-x,x)$ at $x = 0$ (1, Eq. (26)), $x = 0.2/\Phi_0$ (2), $x = 0.5/\Phi_0$ (3). (b) Parts of the increase in temperature attributable to the entropy mode after the pulse passes in accordance to Eq. (25). S_3 equals zero.

Eqs. (24) and (25) coincide with the well-known equation governing acoustic heating in the limiting case of a Newtonian ideal gas and periodic sound as an origin of the heating [4, 5]. The yield stress in a Newtonian ideal gas is equals zero, and its internal energy is the well-established function of pressure and density,

$$P_0 = 0, \; e = \frac{p}{(\gamma-1)\rho}. \tag{27}$$

Traditionally, only this case is considered in the theory of nonlinear acoustics. The dimensionless perturbation of velocity in the periodic sound propagating in the positive direction of the Ox axis takes the form

$$v_a = M\exp(-\mu x/2)\sin(t-x), \tag{28}$$

and the excess density attributable to the thermal mode averaged over the sound period decreases with time in accordance to the following equation:

$$\frac{\partial}{\partial t}\langle\rho_e\rangle = -\frac{\mu}{2}(\gamma-1)M^2\exp(-\mu x), \tag{29}$$

where angle brackets denote averaging over the sound period. It may be easily concluded from Eq. (22) that accounting for the thermal conductivity would lead to the same equations as the latter two, but with the overall attenuation $\mu+\delta$ replacing μ. That agrees with the conclusions of studies [4, 5].

6. Conclusions

The main result of this study is the equation governing acoustic heating, Eq. (22). It is a result of consequent decomposition of weakly nonlinear equations governing sound and the entropy mode. The method developed by the author results in instantaneous dynamic equations: it does not need temporal averaging of the conservative equations with respect to sound period. That differs the method from the traditional decomposition of equations for acoustic and non-acoustic motions [4, 5]. Acoustic heating grows with increase of the acoustic Mach number M and the parameters responsible for attenuation and yield stress, μ and Φ_0. It increases also with increasing thermal conductivity δ. In view of the mathematical difficulties, the influence of the thermal conductivity on acoustic heating is not considered in the example.

A very important peculiarity of the dynamics of a Bingham plastic is dependence of the equations governing both sound and the entropy modes on the sign of the shear rate. That requires individual evaluation of the dynamics

of every waveform as well as the heating generated by it. Bipolar waveforms become monopolar beginning at some distance from a transducer. The width of a pulse enlarges with increase of distance from a transducer. In general, the third term forming the acoustic force of heating, proportional to S_3, differs from zero. The pairwise ratios of the three terms in the right-hand side of Eq. (25) depend on a number of quantities describing the thermodynamic state of a Bingham liquid, such as E_1, D_1, and D_2 (Eqs. (3, 10)).

References

[1] E.C. Bingham, U.S. Bureau of Standards Bulletin 13, 309 (1916)

[2] S. Benito et al., Eur. Phys. J. E 25, 225 (2008)

[3] E.D. Kravtsova, E.M. Gil'derbrandt, V.K. Frizorger, Russ. J. Non-Ferr. Met.+ 50, 114 (2009) (in Russian)

[4] O.V. Rudenko, S.I. Soluyan, Theoretical foundations of nonlinear acoustics (Plenum, New York, 1977)

[5] S. Makarov, M. Ochmann, Acustica 82, 579 (1996)

[6] C.L.Hartman et al., J. Acoust. Soc. Am. 91, 513 (1992)

[7] O.V. Rudenko, Phys.-Usp.+ 50, 359 (2007)

[8] A. Perelomova, Acta Acust. United Ac. 89, 754 (2003)

[9] A. Perelomova, Phys. Lett. A 357, 42 (2006)

[10] A. Perelomova, Acta Acust. 94, 382 (2008)

[11] B.-T. Chu, L.S.G. Kovasznay, J. Fluid. Mech. 3, 494 (1958)

[12] B. Riemann, The collected works of Bernard Riemann (Dover, New York, 1953)

[13] V. Gusev, J. Acoust. Soc. Am. 107, 3047 (2000)