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Analysis of observability and detectability for CSTR model of biochemical processes under uncertain system dynamics and various sets of measured outputs

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An analysis of observability and detectability for continuous stirred tank reactor model of selected biochemical processes has been addressed in this paper. In particular, properties of observability or detectability of the considered system model have been proved under uncertain system dynamics in view of various sets of system measured outputs. It is related to considering system dynamics depending on initial conditions and the impact of inputs taking into account a given measured output. The method of indistinguishable state trajectories (indistinguishable dynamics) and tools based on the Lyapunov second method were used to investigate the observability and detectability properties. The analysis was performed for eight cases of different sets of measured outputs with association to the realistic features of measuring devices. The obtained research results are essential for system state estimation that involves the synthesis of state observers. The proposed approach may be successfully applied to the complex biochemical non-linear uncertain systems modelled as continuous stirred tank reactors.

Key words: biochemical system, indistinguishable dynamics, non-linear systems, observability, process modeling

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1. Introduction

Currently, advanced algorithms are being developed in the area of control theory to perform various tasks, e.g., monitoring, estimation, diagnostics, and control which are needed for effective handling of majority of modern industrial plants and processes. The proper operation of these algorithms is linked to the problem of accessing information about the process variables, e.g., the current state and control outputs of a given process. This information is provided by the measuring devices installed in the plant (system) where the process occurs. In operational practice, only a limited number of process variables are measurable. This is due to the lack of physical measurement capacity, the lack of allocation of the necessary number of sensors, or the quality of measurement information provided is not good enough [19, 33]. Thus, the missing information about process variables needs to be recovered by employing their estimates for example. Typically, the estimation is based on mathematical model of the considered process (system) and measurements of available process variables. In other words, in this approach, the estimation process is premised on utilising direct measurements of other available variables and combining them with the mathematical model of the process given as a set of differential and algebraic equations [1, 2, 17]. In turn, the tool used in the estimation (reconstruction) process is the state observer of various types [1, 2, 16, 21].

It is well-known that the possibility of designing the state observer is strictly related to the observability and detectability properties of the system model. Formally, for the known input-output relationship of the system, observability addresses the ability to fully and explicitly reconstruct (distinguish) its state trajectories in a finite time horizon, whereas detectability (asymptotic observability) allows only an asymptotic convergence to its state trajectories [11, 13, 23]. Specifically, observability invokes the situation, when utilisation of the algorithm exploiting direct output measurements is associated with shaping the dynamical response of the estimates of state variables, i.e., by the selection and tuning values of observer parameters (gains). Whereas, in the case of detectability, this particular choice cannot possibly be done, therefore the state reconstruction is only dependent on the dynamic properties of the process indeed. Thus, the proper investigation of this field asserts crucial information about the potential possibility of using a particular structure of the observer. Moreover, it should be noted that both properties are more difficult to investigate in non-linear systems [4, 5, 11, 13, 15, 20, 23, 25].

The system under consideration is part of a water resource recovery facility (WRRF) and it is composed of two tanks [26]. The first one is a bioreactor where biochemical processes appear, while the second is a settler where treated sewage is separated from the biomass. Two main approaches to modelling such a system are known in the literature. These include activated sludge models (ASMs) and balance models [1, 7, 12, 32]. In this paper, the second approach is used, and the entire system is modelled as a continuous stirred tank reactor (CSTR) with the microbial growth reaction and their mortality with aggregated substrate and biomass concentrations [1, 7]. This is characterised by taking into account the continuous flow of the biomass which is needed for maintaining the appropriate conditions needed to perform biochemical reactions. Therefore, in comparison to, e.g., [4, 25, 28–30], the considered system includes the essential dynamic relationship between the two mentioned tanks. Hence, the performed research provides the analysis of an interesting case study, where distinctive cross interactions have occurred in the state dynamics. These interactions do not appear in a single bioreactor, thus the analysis of observability and detectability must be premised on the consideration of more complicated state relations. It is worth adding that WRRFs are among the systems in which the estimation of process variables is commonly used, e.g., [1,3,6,7,9,23,25,27]. Thus, the topic addressed in the paper is timely and relevant to the development of control and monitoring systems for WRRFs.

Hence, the main aim of this work is to investigate the observability and detectability of a continuous stirred tank reactor model of selected biochemical processes in the presence of uncertainty in the system dynamics. The method of indistinguishable state trajectories (indistinguishable dynamics) is used to prove these properties [8, 11, 13, 18]. Moreover, the approach based on the Lyapunov second method is used to provide sufficient conditions for observability [15, 17]. The analysis presented takes into account various configurations of system measured outputs. Therefore, the various combinations of biomass, substrate, and dissolved oxygen concentrations measurements as physically and technologically available in WRRF are considered [1,7,12]. Fundamentally, this research analysis is necessary to determine whether it is possible to develop a particular state observer depending on the available measured outputs of the system. To summarize, the main contributions of this paper are as follows:

- analysis of observability and detectability of a CSTR model of selected biochemical processes under uncertain system dynamics and various sets of system measured outputs has been devised,
- method of indistinguishable dynamics in combination with an approach based on the Lyapunov second method has been used in the research,
- analysis has been presented for eight cases covering a wide range of selection of system measured outputs, which are physically and technologically available in WRRF.

The paper is organized as follows. In Section 2 the background and problem statement are presented. The considered CSTR model is described in Section 3. Section 4 includes detailed observability and detectability analysis of the considered CSTR model. The paper is concluded in Section 5 and completed with Appendix A.

2. Background

This section delivers the formulation of the considered observability and detectability problem in terms of the method of indistinguishable dynamics and fundamental assumptions associated with this investigation.

2.1. General form of affine dynamic system and its properties

The CSTR model of selected biochemical processes is a highly non-linear dynamic system dependent on multiple inputs. Hence, it can be modelled as the following multi-input multi-output (MIMO) affine system $\boldsymbol{\Sigma}$ [15, 17]:

$$\boldsymbol{\Sigma}: \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) + \sum_{i_\mathrm{p}=1}^{p} \boldsymbol{b}_{i_\mathrm{p}}(\boldsymbol{x}(t)) u_{i_\mathrm{p}}(t) + \sum_{i_\mathrm{m}=1}^{m} \boldsymbol{g}_{i_\mathrm{m}}(\boldsymbol{x}(t)) v_{i_\mathrm{m}}(t), \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0\,, \\ \boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t)), \end{cases} \tag{1}$$

where: $(\dot{\cdot})$ stands for the derivative with respect to t; $t \in \mathbb{T} \subset \mathbb{R}_+ \cup \{0\} \subset \mathbb{R}$ is the time instant, and $\mathbb{R}_+$ denotes the positive part of $\mathbb{R}$; $\forall t \in \mathbb{T}\colon\ \boldsymbol{x}(t) \in \mathbb{X}_\mathrm{n} \subset \mathbb{R}^\mathrm{n}$ is the n-dimensional vector of state variables, $\mathbb{X}_\mathrm{n}$ is a $C^\infty(\cdot)$ manifold of dimension n, and $\boldsymbol{x}_0$ signifies the vector of initial conditions; $\forall t \in \mathbb{T}\colon\ \boldsymbol{u}(t) \in \mathbb{U}_\mathrm{p} \subset \mathbb{R}^\mathrm{p}$ is the p-dimensional vector of control inputs; $\forall t \in \mathbb{T}\colon\ \boldsymbol{v}(t) \in \mathbb{V}_\mathrm{m} \subset \mathbb{R}^\mathrm{m}$ is the m-dimensional vector of unknown (uncertain) inputs; $\forall t \in \mathbb{T}\colon\ \boldsymbol{y}(t) \in \mathbb{Y}_\mathrm{q} \subset \mathbb{R}^\mathrm{q}$ is the q-dimensional vector of measured outputs; $\forall t \in \mathbb{T}\colon\ \boldsymbol{b}_{i_\mathrm{p}}, \boldsymbol{g}_{i_\mathrm{m}}, \boldsymbol{f}\colon\ \mathbb{X}_\mathrm{n} \to \mathrm{T}\mathbb{X}_\mathrm{n}$, $\forall t \in \mathbb{T}\colon\ \boldsymbol{h}\colon\ \mathbb{X}_\mathrm{n} \to \mathbb{Y}_\mathrm{q}$ are smooth maps representing control and unknown inputs, internal system dynamics, and measured outputs, respectively; $\mathrm{T}(\cdot)$ stands for the tangent bundle of vector field.

Assumption 1. *System $\boldsymbol{\Sigma}$ is claimed as complete, i.e., the trajectories of the system state $\boldsymbol{x}(t)$ are defined for every $t \in \mathbb{T}$, every initial condition $\boldsymbol{x}_0$, and for all exogenous inputs which belong to their particular sets.*

Assumption 2. *A general solution of the system $\boldsymbol{\Sigma}$ is simply denoted by $\boldsymbol{x}(t)$ whereas the flow indication, i.e., $\boldsymbol{x}(t) \equiv \boldsymbol{x}(\boldsymbol{x}_0, \boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot), t)$ is used only if necessary. Analogously, the output signal is simply denoted by $\boldsymbol{y}(t)$ instead of $\boldsymbol{y}(\boldsymbol{x}(\boldsymbol{x}_0, \boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot), t))$ or $\boldsymbol{h}(\boldsymbol{x}(\boldsymbol{x}_0, \boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot), t))$.*

Assumption 3. *In the operational conditions all signals and parameters in the system* $\boldsymbol{\Sigma}$ *are non-negative and uniformly bounded, i.e.,* $\forall t \in \mathbb{T}$ *holds:*

$$\mathbf{0}_{(\mathrm{n}\times 1)} \preceq \underline{\boldsymbol{x}} \preceq \boldsymbol{x}(t) \preceq \overline{\boldsymbol{x}},\ \mathbf{0}_{(\mathrm{p}\times 1)} \preceq \underline{\boldsymbol{u}} \preceq \boldsymbol{u}(t) \preceq \overline{\boldsymbol{u}},\ \mathbf{0}_{(\mathrm{m}\times 1)} \preceq \underline{\boldsymbol{v}} \preceq \boldsymbol{v}(t) \preceq \overline{\boldsymbol{v}},$$

where: $\mathbf{0}_{((\cdot)\times 1)}$ *is the zero vector of appropriate dimension;* $\underline{(\cdot)}$ *and* $\overline{(\cdot)}$ *signify the real and positive lower and upper bounds on the particular variable;* $\preceq$ *denotes element-wise operation between particular elements of a given vector or matrix.*

Assumption 4. *The control inputs to the system* $\boldsymbol{\Sigma}$ *are defined as* $\boldsymbol{u}(t) \in \mathbb{U}_\mathrm{p} \subset \mathbb{R}^\mathrm{p}$. *Vector* $\boldsymbol{u}(t)$ *is the vector of time-dependent functions which are at least* $\boldsymbol{u}(t) \in C^1\left(\mathbb{T}, \mathbb{U}_\mathrm{p}\right)$ *uniformly bounded Lipschitz continuous functions. Let us denote* $\|\cdot\|_\infty$ *as the supremum norm [31], then* $\forall t \in \mathbb{T}$: $\|u_{i_\mathrm{p}}(t)\|_\infty = \sup\left\{|\, u_{i_\mathrm{p}}\,| : \ t \in \mathbb{T}\right\} \leqslant \overline{u}_{i_\mathrm{p}} < \infty$. *Moreover, due to the physical properties of the considered system, it is assumed that all control inputs are the permanent excited positive signals, thus [17]:*

$$\underline{\alpha}_\mathrm{u} \boldsymbol{I}_{\mathrm{p}\times\mathrm{p}} \leqslant \int\limits_t^{T+t} \boldsymbol{u}(\tau)\boldsymbol{u}^\mathrm{T}(\tau)\,\mathrm{d}\tau \leqslant \overline{\alpha}_\mathrm{u} \boldsymbol{I}_{\mathrm{p}\times\mathrm{p}},$$

where: $\underline{\alpha}_\mathrm{u} \in \mathbb{R}_+$, $\overline{\alpha}_\mathrm{u} \in \mathbb{R}_+$ *are the lower and upper bounds of the permanently excited inputs;* $T \in \mathbb{T}$ *is the selected time period;* $\boldsymbol{I}_{\mathrm{p}\times\mathrm{p}}$ *is the identity matrix.*

Assumption 5. *In the context of observer synthesis, the vector of the measured outputs is understood as the following vector of state variables* $\boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t)) = \boldsymbol{C}\boldsymbol{x}(t)$, *where* $\boldsymbol{C} \in \mathbb{R}^{\mathrm{q}\times\mathrm{n}}$ *is the output matrix.*

2.2. Tools for analyzing observability and detectability

In this paper, two main notions are used, i.e., observability and detectability. The classic approach to investigating these properties is based on differential geometry tools [10,13,15]. However, this approach is not always easily applicable. This issue arises due to complicated calculations and problems with checking the injectivity of the observability map [30]. To deal with these problems, the method of indistinguishable state trajectories (indistinguishable dynamics) may be used [2, 8, 10, 13, 15].

Formally, considering the solution $\boldsymbol{x}\left(\boldsymbol{x}_0, \boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot), t\right)$ of the system $\boldsymbol{\Sigma}$ evolving over time $t \in \mathbb{T}$, the following concepts are formulated [2, 13, 22, 23, 25]:

Definition 1. *If there exist two distinct initial conditions* $\boldsymbol{x}_0, \check{\boldsymbol{x}}_0 \in \mathbb{X}_\mathrm{n}$ *which under the same control input vector* $\boldsymbol{u}(\cdot)$ *and two distinct unknown input vectors* $\boldsymbol{v}(\cdot)$ *and* $\check{\boldsymbol{v}}(\cdot)$ *cause the same output behavior, i.e.,* $\boldsymbol{y}\left(\boldsymbol{x}\left(\boldsymbol{x}_0, \boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot), t\right)\right) = \boldsymbol{y}\left(\boldsymbol{x}\left(\check{\boldsymbol{x}}_0, \boldsymbol{u}(\cdot), \check{\boldsymbol{v}}(\cdot), t\right)\right)$ $\forall t \in \mathbb{T}$, *then* $\check{\boldsymbol{x}}_0$ *is strongly u–indistinguishable from* $\boldsymbol{x}_0$.

$\boldsymbol{I}^{\mathrm{UI}}_{(\boldsymbol{x},\boldsymbol{u},\boldsymbol{v})}$ *denotes the set of all strongly u–indistinguishable states from* $\boldsymbol{x}_0$ *and ‘strongly’ means that the property of indistinguishability is associated with the impact of unknown inputs.*

Definition 2. *The system* $\boldsymbol{\Sigma}$ *is strongly u–observable if for any initial condition* $\boldsymbol{x}_0 \in \mathbb{X}_{\mathrm{n}}$*, any control input vector* $\boldsymbol{u}(\cdot)$*, and two distinct unknown input vectors* $\boldsymbol{v}(\cdot)$ *and* $\check{\boldsymbol{v}}(\cdot)$*, the following relation holds:* $\boldsymbol{I}^{\mathrm{UI}}_{(\boldsymbol{x},\boldsymbol{u},\boldsymbol{v})} = \{\boldsymbol{x}_0\}$*. It means that for each particular initial condition,* $\boldsymbol{x}_0$ *is only indistinguishable from ‘himself’, i.e., all of the distinct state trajectories are distinguishable from each other.*

Definition 3. *The system* $\boldsymbol{\Sigma}$ *is strongly u–detectable if for any initial condition* $\boldsymbol{x}_0 \in \mathbb{X}_{\mathrm{n}}$*, all initial conditions meeting the precondition of* $\check{\boldsymbol{x}}_0 \in \boldsymbol{I}^{\mathrm{UI}}_{(\boldsymbol{x},\boldsymbol{u},\boldsymbol{v})}$*, any control input vector* $\boldsymbol{u}(\cdot)$*, and two distinct unknown input vectors* $\boldsymbol{v}(\cdot)$ *and* $\check{\boldsymbol{v}}(\cdot)$ *which cause state indistinguishability, all indistinguishable states (integral curves of its trajectories) mutually converge to each other, i.e.,*
$\lim_{t\to\infty} \|\boldsymbol{x}(\boldsymbol{x}_0, \boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot), t) - \boldsymbol{x}(\check{\boldsymbol{x}}_0, \boldsymbol{u}(\cdot), \check{\boldsymbol{v}}(\cdot), t)\|_2 = 0$*, where* $\|(\cdot)\|_2$ *denotes the Euclidean norm. It means that indistinguishable trajectories asymptotically converge to the ‘common curve’ imposed by particular relation between control inputs, unknown inputs, and measured outputs.*

Remark 1. *The prefix ‘u–(·)’ is associated with the statement that the exogenous signals do not affect the observability property of the system. In the literature, this phenomenon is called uniform observability [2, 10].*

The method of indistinguishable dynamics is premised on the straight interpretation of state trajectory indistinguishability, where particular notions are based on the possibility of unique or asymptotic reconstruction of both state and unknown input trajectories. Several features of this approach are worth highlighting [4, 14, 23–25]:

1. The Lyapunov function-based approach may be applied for the sake of analyzing properties of the system $\boldsymbol{\Sigma}$ such as observability and detectability.
2. Both observability and especially detectability are possible to prove in the presence of unknown inputs.
3. It is possible to depict particular interactions between system states and exogenous inputs, which affect observability and detectability properties.
4. The zeros dynamics known from differential geometry-based control approach, e.g., [15] can be used to investigate observable and unobservable parts of the system.
5. The results may be directly interpreted as global or local.

2.3. Problem statement

In general, the investigation of observability and detectability properties under uncertainty and various sets of system measured outputs for a certain sub-class of the general class of affine non-linear dynamic systems is the aim of the paper. This sub-class is represented by the bioreactor with the settler in WRRF. The considered system model is based on CSTR with the microbial growth reaction and microbial mortality with the aggregated substrate and biomass concentrations. In turn, the reaction kinetics function is the source of uncertainty in the model of system dynamics. Thus, the results of analysis carried out provide the necessary knowledge for the synthesis of a state observer for this type of system.

3. CSTR model of selected biochemical processes

The mechanistic model which describes the selected biochemical processes is derived by invoking the mass balance laws. Hence, the considered CSTR model taking into account the microbial growth reaction and microbial mortality with the aggregated substrate and biomass concentrations yields [7, 9, 12, 29]:

$$\mathbf{\Sigma}_{\mathrm{CSTR}}: \begin{cases} \dot{X}(t) = \mu(t)X(t) - m_{\mathrm{x}}X(t) - (1+r)X(t)D(t) + rX_{\mathrm{r}}(t)D(t), \\ \dot{S}(t) = -Y_{\mathrm{s}}^{-1}\mu(t)X(t) - m_{\mathrm{s}}X(t) + S_{\mathrm{in}}(t)D(t) \\ \qquad\quad - (1+r)S(t)D(t), \\ \dot{DO}(t) = -Y_{\mathrm{o}}^{-1}\mu(t)X(t) - m_{\mathrm{o}}X(t) + DO_{\mathrm{in}}(t)D(t) \\ \qquad\quad - (1+r)DO(t)D(t) + k_{\mathrm{L}}a(t)(DO_{\mathrm{s}} - DO(t)), \\ \dot{X}_{\mathrm{r}}(t) = v(1+r)X(t)D(t) - v(w+r)X_r(t)D(t), \\ X(t_0) = X_0\,, \\ S(t_0) = S_0\,, \\ DO(t_0) = DO_0\,, \\ X_{\mathrm{r}}(t_0) = X_{\mathrm{r}_0}\,, \end{cases} \tag{2}$$

where: $\mu(t)$ $[-]$ denotes the reaction kinetics function (growth rate function); $m_{\mathrm{x}} \in \mathbb{R}_{+}$ $[\mathrm{h}^{-1}]$ is the biomass mortality rate; $m_{\mathrm{s}} \in \mathbb{R}_{+}$ $[\mathrm{h}^{-1}]$, $m_{\mathrm{o}} \in \mathbb{R}_{+}$ $[\mathrm{h}^{-1}]$ are the maintenance coefficients of the substrate and dissolved oxygen concentrations, respectively; $Y_{\mathrm{o}} \in \mathbb{R}_{+}$ $[-]$, $Y_{\mathrm{s}} \in \mathbb{R}_{+}$ $[-]$ denote the yield coefficients of the dissolved oxygen and substrate concentrations, respectively; $S_{\mathrm{in}}(t) \in \mathbb{R}_{+}$ [mg/L], $DO_{\mathrm{in}}(t) \in \mathbb{R}_{+}$ [mg/L] signify the concentrations of the substrate in inflow to the CSTR and dissolved oxygen, respectively; $DO_{\mathrm{s}} \in \mathbb{R}_{+}$ [mg/L] stands for the dissolved oxygen saturation constant; $k_{\mathrm{L}}a(t)$ denotes the gas-liquid transfer function; $D(t) = Q_{\mathrm{in}}(t)/V_{\mathrm{a}}$ $[\mathrm{h}^{-1}]$ is the dilution rate, where $Q_{\mathrm{in}}(t) \in \mathbb{R}_{+}$ $[\mathrm{m}^3/\mathrm{h}]$,

$V_a \in \mathbb{R}_+$ [m^3] are the inflow rate and bioreactor volume; $r = Q_r(t)/Q_{in}(t) \in \mathbb{R}_+$ [−] is the constant proportion between the inflow and recirculated flow, where $Q_r(t) \in \mathbb{R}_+$ [m^3/h] is the recirculated flow rate; $w = Q_w(t)/Q_{in}(t) \in \mathbb{R}_+$ [−] stands for the constant proportion between the sewage flow and inflow, where $Q_w(t) \in \mathbb{R}_+$ [m^3/h] is the sewage flow rate; $v = V_a/V_s \in \mathbb{R}_+$ [−] denotes the constant proportion between the bioreactor's and settler's volumes, where $V_s \in \mathbb{R}_+$ [m^3] is the settler volume; $X(t)$ [mg/L], $S(t)$ [mg/L], $DO(t)$ [mg/L], $X_r(t)$ [mg/L] are the aggregated biomass, aggregated substrate, dissolved oxygen, and aggregated recirculated biomass concentrations, respectively.

The entire system modelled by (2) is shown in Fig. 1.

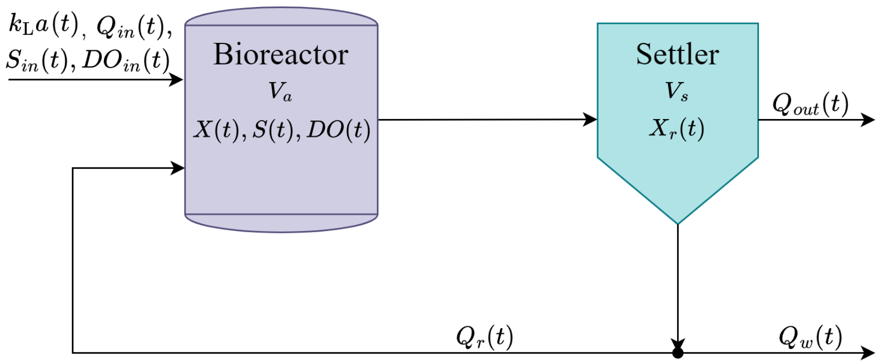

Figure 1: Diagram of the modelled system

For further consideration, according to Assumption 3, the set of all possible system states is defined as:

$$\begin{aligned} \mathbf{\Omega} = \{ & (X(t), S(t), DO(t), X_r(t)) \in \mathbb{R}^4_+ \cup \{\mathbf{0}_{4\times 1}\} : \forall t \in \mathbb{T} \\ & 0 \leqslant \underline{X}(t) \leqslant X(t) \leqslant \overline{X}, \quad 0 \leqslant \underline{S}(t) \leqslant S(t) \leqslant \overline{S}, \\ & 0 \leqslant \underline{DO}(t) \leqslant DO(t) \leqslant \overline{DO}, \quad 0 \leqslant \underline{X}_r(t) \leqslant X_r(t) \leqslant \overline{X}_r\}, \end{aligned} \tag{3}$$

It is worth adding that $\mathbf{\Omega} \subset \mathbb{X}_n \subset \mathbb{R}^4$ is an invariant set known with the condition of the general theory of biochemical processes dynamics [1, 7, 26, 27]. Moreover, according to [24] the system (2) is Lyapunov stable and its state trajectories are bounded $\forall t \in \mathbb{T}$.

CSTR model $\mathbf{\Sigma}_{CSTR}$ includes the following biochemical phenomena [1, 1, 7, 9, 12, 26]:

1) the microbial growth in the bioreactor is premised on the reaction kinetics function $\mu(t)$,

2) the inflow to the given tank and its outflow is imposed by the dilution rate $D(t)$,

3) the gas-liquid transfer of dissolved oxygen delivered to the bioreactor is given by the positive term including the oxygen mass transfer coefficient $k_{\mathrm{L}}a(t)$,
4) the microbial mortality is given as follows. For the biomass death $X(t) \to X_{\mathrm{d}}(t)$, while for the substrate and dissolved oxygen maintenance $DO(t) + S(t) + X(t) \to X(t)$, where X_{d} [mg/L] is the dead biomass concentration,
5) the inflow concentrations $S_{\mathrm{in}}(t)$ and $DO_{\mathrm{in}}(t)$ are delivered to the bioreactor from the outside of the process, e.g., other bioreactors.

It is easy to state that CSTR model $\mathbf{\Sigma}_{\mathrm{CSTR}}$ neatly fits the general form of dynamic system (1). Hence, the state variables and particular inputs of the considered CSTR model are defined as follows:

$$\begin{aligned} \boldsymbol{x}(t) &= \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{bmatrix}^{\mathrm{T}} \\ &\triangleq \begin{bmatrix} X(t) & S(t) & DO(t) & X_{\mathrm{r}}(t) \end{bmatrix}^{\mathrm{T}}, \quad v(t) \triangleq \mu(t), \\ \boldsymbol{u}(t) &= \begin{bmatrix} u_1(t) & u_2(t) & u_3(t) & u_4(t) \end{bmatrix}^{\mathrm{T}} \\ &\triangleq \begin{bmatrix} D(t) & k_{\mathrm{L}}a(t) & S_{\mathrm{in}}(t) \cdot D(t) & DO_{\mathrm{in}}(t) \cdot D(t) \end{bmatrix}^{\mathrm{T}}. \end{aligned} \tag{4}$$

Thus, and according to Assumption 3, the dilution rate, gas-liquid transfer function, concentrations of the substrate and dissolved oxygen in inflow to the CSTR signify the bounded inputs to plant, i.e., $0 \leqslant D(t) \leqslant \overline{D} \in \mathbb{R}_+$, $0 \leqslant k_{\mathrm{L}}a(t) \leqslant \overline{k_{\mathrm{L}}a} \in \mathbb{R}_+$, $0 \leqslant S_{\mathrm{in}}(t) \leqslant \overline{S}_{\mathrm{in}} \in \mathbb{R}_+$, and $0 \leqslant DO_{\mathrm{in}}(t) \leqslant \overline{DO}_{\mathrm{in}} \in \mathbb{R}_+$. However, since in model (2) appears multiplication between the dilution rate and inflow concentrations, the new (artificial) inputs are proposed in (4) as the parts of the model for analysis purposes.

Various types of models of reaction kinetics function can be found in the literature [1,3,7,30]. They differ due to the number of constituting compounds or because of distinct structures representing the biochemical phenomenon. However, due to the lack of sufficient knowledge about the values of parameters or the structure complexity, it is often hard to propose particular kinetics functions. Therefore, for the observability and detectability analysis purposes, it is assumed that $\mu(t)$ is the uniformly bounded unknown input $v(t)$, i.e., $0 \leqslant \mu(t) \leqslant \overline{\mu} \in \mathbb{R}_+$. Otherwise, i.e. $v(t) \equiv 0\ \forall t \in \mathbb{T}$, the reaction kinetics function is treated as the state-dependent part of the internal system dynamics. Thus, it is given as a product of two distinct Monod components [7,9,24,27,29,30]:

$$\mu(t) = \mu_{\max} \frac{S(t)}{K_{\mathrm{s}} + S(t)} \frac{DO(t)}{K_{\mathrm{o}} + DO(t)}, \tag{5}$$

where $\forall t \in \mathbb{T}$: $\mu_{\max} \in \mathbb{R}_+$ [h^{-1}], $K_\mathrm{s} \in \mathbb{R}_+$ [mg/L], $K_\mathrm{o} \in \mathbb{R}_+$ [mg/L] denote the coefficients of maximum specific growth rate, substrate saturation, and dissolved oxygen saturation, respectively.

Remark 2. *The issue of analyzing the observability and detectability properties of biochemical systems is strictly associated with not only the selection of measured outputs but also with appropriate considering the form of reaction kinetics function. As it has been shown in, e.g., [4, 6, 24, 29, 30], the Monod component entails indistinguishable state trajectories of the system for the given initial conditions only in the (biomass) wash-out state, where $X(t)$ is zero. In detail, the functional form of Monod component compared to, e.g., the Haldane component does not affect the verdict of the observability and detectability analysis due to its monotonicity.*

Remark 3. *According to the physical requirements of the wastewater treatment process the wash-out state is highly undesired. However, even knowing that, in order to carry out a comprehensive analysis of observability and detectability, this state has been also studied.*

4. Analysis of observability and detectability

As stated above the method of indistinguishable state trajectories (indistinguishable dynamics) is used to analyze the properties of observability and detectability. To apply this tool, the following considerations are done [4, 5, 14, 23, 25, 28–30]. To begin with, the extended system, which embodies the 'original – $\boldsymbol{x}(t)$' system and 'copied – $\boldsymbol{z}(t)$' system, is derived as the extended model, which has analogous form to the general form of affine system (1). Hence, the extended system has analogous form to the general form of the system Σ_{CSTR} invoked in (2) and its dynamics is the eight-dimensional manifold of the original state space and 'error' space $[\boldsymbol{x}^\mathrm{T}(t),\ \boldsymbol{\varepsilon}^\mathrm{T}(t)]^\mathrm{T} \in \boldsymbol{\Omega} \times \boldsymbol{\Xi} = \boldsymbol{\Psi} \subset \mathbb{R}^4_+ \times \mathbb{R}^4$. The 'error' variables are defined as follows:

$$\boldsymbol{\varepsilon}(t) \triangleq \boldsymbol{x}(t) - \boldsymbol{z}(t) \ \rightarrow\ \boldsymbol{z}(t) = \boldsymbol{x}(t) - \boldsymbol{\varepsilon}(t). \tag{6}$$

Moreover, the difference between uncertain parts of the dynamics of the 'original' and 'copied' systems is determined as follows:

$$\varepsilon_\mu(t) \triangleq \mu(t) - \mu_\mathrm{z}(t) \ \rightarrow\ \mu_\mathrm{z}(t) = \mu(t) - \varepsilon_\mu(t), \tag{7}$$

where $\varepsilon_\mu(t) \in \mathbb{R}$ are the 'error' of reaction kinetics function, and $\mu_\mathrm{z}(t) \in \mathbb{R}_+ \cup \{0\}$ denotes the reaction kinetics function for 'copied' system.

As it has been pointed out in Section 3, the two situations are considered. The first, when the reaction kinetics function $\mu(t)$ is assumed as an unknown input, and the second when $\mu(t)$ is claimed as a component of model dynamics (2). Thus, for the investigation of observability and detectability purposes, the following $\mu_z(t)$ yields:

$$\mu_z(t) = \mu_{\max} \frac{x_2(t) - \varepsilon_2(t)}{K_s + x_2(t) - \varepsilon_2(t)} \frac{x_3(t) - \varepsilon_3(t)}{K_o + x_3(t) - \varepsilon_3(t)} . \tag{8}$$

The analysis based on the introduced methodology is given as follows [23, 25, 28–30]. It is assumed that the initial conditions of both systems are not equal. Thus, different integral curve trajectories of the system state, i.e., $\boldsymbol{x}(t) \neq \boldsymbol{z}(t)$ $\forall t \in \mathbb{T}$ are given. Moreover, measured outputs $\boldsymbol{y}(t)$ and control inputs $\boldsymbol{u}(t)$ associated with both systems are the same for $t \in \mathbb{T}$. Hence, observability occurs, when the only possible state solutions of 'original' and 'copied' state dynamics are always equal, which is the same as the state trajectory indistinguishability. In turn, the detectability property takes place, when the indistinguishable state solutions mutually tend to one another asymptotically in finite time, which is associated with asymptotic stability of 'error' dynamics. The 'error' dynamics is given as follows:

$$\Sigma_E : \begin{cases} \dot{\varepsilon}_1(t) = \varepsilon_\mu(t)x_1(t) + \varepsilon_1(t)\left(\mu(t) - \varepsilon_\mu(t)\right) - m_x\varepsilon_1(t) \\ \qquad\quad - (1+r)\varepsilon_1(t)u_1(t) + r\varepsilon_4(t)u_1(t), \\ \dot{\varepsilon}_2(t) = -Y_s^{-1}\left[\varepsilon_\mu(t)x_1(t) + \varepsilon_1(t)\left(\mu(t) - \varepsilon_\mu(t)\right)\right] - m_s\varepsilon_1(t) \\ \qquad\quad - (1+r)\varepsilon_2(t)u_1(t), \\ \dot{\varepsilon}_3(t) = -Y_o^{-1}\left[\varepsilon_\mu(t)x_1(t) + \varepsilon_1(t)\left(\mu(t) - \varepsilon_\mu(t)\right)\right] - m_o\varepsilon_1(t) \\ \qquad\quad - (1+r)\varepsilon_3(t)u_1(t) - \varepsilon_3(t)u_2(t), \\ \dot{\varepsilon}_4(t) = v(1+r)\varepsilon_1(t)u_1(t) - v(w+r)\varepsilon_4(t)u_1(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0 . \end{cases} \tag{9}$$

To increase the clarity of the above description, Figs. 2 and 3 show the trajectories of error $\varepsilon_1(t)$ for two different situations.

Figure 2 depicts how detectable trajectories converge to the zero point of equilibrium when all trajectories of error $\varepsilon_1(t)$ are globally asymptotically stable for any initial conditions and for identity relationship between $y(t)$ and $\boldsymbol{u}(t)$ impacts their time evolution. In turn, the situation when $x_1(t)$ is non-detectable is presented in Fig. 3, where error trajectories $\varepsilon_1(t)$ do not converge to the zero equilibrium point. Obviously, for any initial conditions of $x_1(t)$, observable (distinguishable) trajectories of $\varepsilon_1(t)$ must always be zero.

It is worth emphasizing, that for error dynamics (9) two types of error of reaction kinetics function $\varepsilon_\mu(t)$ defined in (6) can be used. Then $\mu(t)$ is unknown and $\varepsilon_\mu(t)$ is treated directly as the difference between distinct unknown inputs.

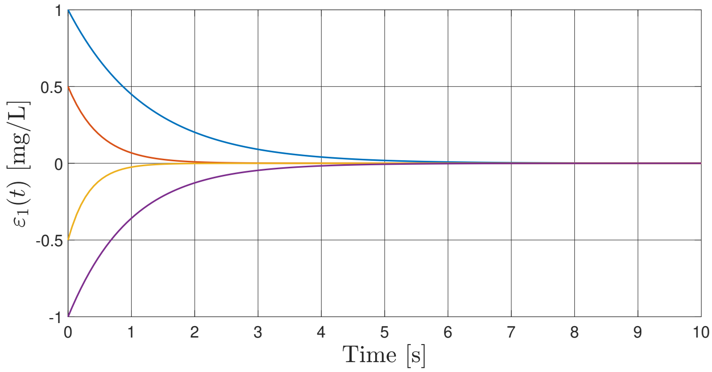

Figure 2: The error trajectories $\varepsilon_1(t)$ illustrating the time behavior of detectable state variable $x_1(t)$ for different initial conditions $x_1(t_0)$

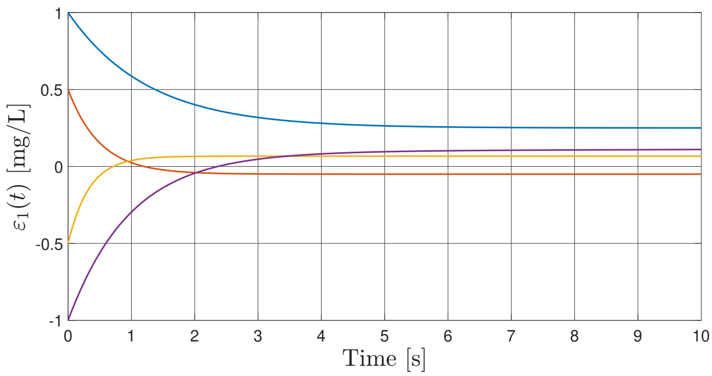

Figure 3: The error trajectories $\varepsilon_1(t)$ illustrating the time behavior of non-detectable state variable $x_1(t)$ for different initial conditions $x_1(t_0)$

In the second case, when $\mu(t)$ is considered as the part of the dynamics, the proposed $\mu_z(t)$ is combined with (7), and finally substituted into model (9). Therefore, systems Σ_{CSTR} and Σ_E constitute indistinguishable dynamics and by incorporating the relevant notions from [5, 14, 25] the following definitions are introduced:

- System Σ_{CSTR} is globally u–observable if and only if for every input-output system's behavior and every $\boldsymbol{\varepsilon}(t_0)$ there is only a solution given by $\boldsymbol{\varepsilon}(t) = \overline{\boldsymbol{\varepsilon}} = 0\ \forall t \in \mathbb{T}$, where $\overline{\boldsymbol{\varepsilon}}$ is an equilibrium point of error dynamics (so-called 'zero point').
- System Σ_{CSTR} is locally u–observable if and only if the property of global u–observability is given only in the certain neighborhood $\boldsymbol{\mathcal{V}} \subset \boldsymbol{\Xi}$ of zero point.

- System Σ_{CSTR} is globally u–detectable if and only if for every input-output system's behavior and every initial condition $\boldsymbol{\varepsilon}(t_0) \neq \overline{\boldsymbol{\varepsilon}}$, zero point of Σ_{E} is a globally attractive point for all trajectories defined in $\mathbb{T}$.
- System Σ_{CSTR} is locally u–detectable if and only if the property of for global u–detectability is given only in the certain neighborhood $\boldsymbol{\mathcal{V}} \subset \boldsymbol{\Xi}$ of zero point.

Remark 4. *Error dynamics (9) does not contain $u_3(t)$ and $u_4(t)$ defined in (4). It is because of straight subtraction between 'original' and 'copied' system dynamics.*

The analysis presented below in this section focuses on the fundamental situation where the reaction kinetics function is unknown. However, to show the impact of this uncertainty, the second part of the analysis shows studies for the situation when this function is known.

4.1. Part 1 – unknown reaction kinetics function

A detailed analysis of four most interesting cases and a synthetic (tabular) description of all possibilities limited to a reasonable number of two measured outputs is presented in this section.

4.1.1. Measured output – biomass concentration $X(t)$

The measured output is $y(t) = h(\boldsymbol{x}(t)) = x_1(t)$. It implies that $\forall t \in \mathbb{T}\ \varepsilon_1(t) = 0$ and $\dot{\varepsilon}_1(t) = 0$, thus model (9) is converted into the following differential-algebraic equation (DAE) system:

$$\begin{cases} 0 = \varepsilon_\mu(t)x_1(t) + r\varepsilon_4(t)u_1(t), \\ \dot{\varepsilon}_2(t) = -Y_{\text{s}}^{-1}\varepsilon_\mu(t)x_1(t) - (1+r)\varepsilon_2(t)u_1(t), \\ \dot{\varepsilon}_3(t) = -Y_{\text{o}}^{-1}\varepsilon_\mu(t)x_1(t) - (1+r)\varepsilon_3(t)u_1(t) - \varepsilon_3(t)u_2(t), \\ \dot{\varepsilon}_4(t) = -v(w+r)\varepsilon_4(t)u_1(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0\,. \end{cases} \tag{10}$$

By substituting the first algebraic equation into the second and third differential equations, system (10) yields:

$$\begin{cases} 0 = \varepsilon_\mu(t)x_1(t) + r\varepsilon_4(t)u_1(t), \\ \dot{\varepsilon}_2(t) = Y_{\text{s}}^{-1}r\varepsilon_4(t)u_1(t) - (1+r)\varepsilon_2(t)u_1(t), \\ \dot{\varepsilon}_3(t) = Y_{\text{o}}^{-1}r\varepsilon_4(t)u_1(t) - (1+r)\varepsilon_3(t)u_1(t) - \varepsilon_3(t)u_2(t), \\ \dot{\varepsilon}_4(t) = -v(w+r)\varepsilon_4(t)u_1(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0\,. \end{cases} \tag{11}$$

Now, there are two variants to be considered. The first takes place when state variable $x_1(t)$ is positive, whereas the second is linked with a wash-out state, where $x_1(t)$ becomes zero.

In the first option, when $x_1(t) > 0$ and input $u_1(t)$ is always positive, the last dynamic equation from (10) is globally asymptotically stable. Therefore, $\varepsilon_4(t) \to 0$, which means that $x_4(t)$ is globally strongly u–detectable. Hence, dynamics (11) is reduced to:

$$\begin{cases} \quad 0 = \varepsilon_\mu(t)x_1(t) + r\gamma_1(t)u_1(t), \\ \dot{\varepsilon}_2(t) = Y_\mathrm{s}^{-1} r\gamma(t)u_1(t) - (1+r)\varepsilon_2(t)u_1(t), \\ \dot{\varepsilon}_3(t) = Y_\mathrm{o}^{-1} r\gamma_1(t)u_1(t) - (1+r)\varepsilon_3(t)u_1(t) - \varepsilon_3(t)u_2(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0\,, \end{cases} \tag{12}$$

where $\varepsilon_4(t) \equiv \gamma_1(t) = \varepsilon_4(t_0)\exp\left[-\nu(w+r)\int_{t_0}^{t} u_1(\tau)\,\mathrm{d}\tau\right] \to 0$ when $t \to \infty$ is the solution of the third differential equation from (11).

Therefore, taking into account that both inputs $u_1(t)$ and $u_2(t)$ are always positive, the remaining dynamic equations in (12) asymptotically tend to zero, which means that state variables $x_2(t)$ and $x_3(t)$ are globally strongly u–detectable. Moreover, since kinetics-related error $\varepsilon_\mu(t)$ is associated with positive signals $x_1(t)$ and $u_1(t)$ and globally asymptotically stable $\gamma_1(t)$, the reaction kinetics function is globally u–detectable. The fact that the right-hand sides of differential equations from (12) are always negative can be explained in an analogous way to [29], where the classical partial integration approach has been applied.

In turn, in the second option, where $x_1(t) = 0$, the algebraic equation from (11) is always equal to zero. It causes that expression $r\varepsilon_4(t)u_1(t) = 0 \;\forall t \in \mathbb{T}$, which means that due to the positiveness of $u_1(t)$, $\varepsilon_4(t)$ must be always equal to zero. Hence, the fourth state variable is globally strongly u–observable. In the end, the whole dynamics (11) is reduced to:

$$\begin{cases} \dot{\varepsilon}_2(t) = -(1+r)\varepsilon_2(t)u_1(t), \\ \dot{\varepsilon}_3(t) = -(1+r)\varepsilon_3(t)u_1(t) - \varepsilon_3(t)u_2(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0\,. \end{cases} \tag{13}$$

Taking into account that both inputs $u_1(t)$ and $u_2(t)$ are always positive, the remaining dynamic equations in (13) globally asymptotically tend to zero, which means that they are globally strongly u–detectable. However, in comparison to the first variant, it is not possible to make any statement about the observability and detectability of $\mu(t)$, hence this variable is treated as non-detectable.

4.1.2. Measured output – substrate concentration $S(t)$

The measured output is $y(t) = h(\boldsymbol{x}(t)) = x_2(t)$. It implies that $\forall t \in \mathbb{T}\ \varepsilon_2(t) = 0$ and $\dot{\varepsilon}_2(t) = 0$, thus model (9) is converted into the following DAE system:

$$\begin{cases} Y_{\mathrm{s}} m_{\mathrm{s}} \varepsilon_1(t) = -\left[\varepsilon_\mu(t) x_1(t) + \mu(t)\varepsilon_1(t) - \varepsilon_\mu(t)\varepsilon_1(t)\right], \\ \qquad \dot{\varepsilon}_1(t) = -\left[Y_{\mathrm{s}} m_{\mathrm{s}} + m_{\mathrm{x}} + (1+r)u_1(t)\right]\varepsilon_1(t) + r\varepsilon_4(t)u_1(t), \\ \qquad \dot{\varepsilon}_3(t) = \left[Y_{\mathrm{s}} Y_{\mathrm{o}}^{-1} m_{\mathrm{s}} - m_{\mathrm{o}}\right]\varepsilon_1(t) - \left[(1+r)u_1(t) + u_2(t)\right]\varepsilon_3(t), \\ \qquad \dot{\varepsilon}_4(t) = v(1+r)\varepsilon_1(t)u_1(t) - v(w+r)\varepsilon_4(t)u_1(t), \\ \qquad \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0 . \end{cases} \tag{14}$$

Analogously to 4.1.1, the considerations are divided into two variants, i.e., when $x_1(t)$ is always positive or when it becomes zero.

In the first variant, when $x_1(t) > 0$, if dynamics (14) is asymptotically stable, then it is supposed that $x_1(t)$, $x_3(t)$, and $x_4(t)$ are globally strongly u–detectable and $\mu(t)$ is globally u–detectable. To show that, it must be proved, that the first and fourth differential equations from (14) have global attractor in $(\bar{\varepsilon}_1, \bar{\varepsilon}_4) = (0, 0)$ equilibrium point. This particular sub-dynamics of $\varepsilon_1(t)$ and $\varepsilon_4(t)$ can be decoupled from the third differential equation due to the non-occurrence of $\varepsilon_3(t)$ in mentioned equations. This issue can be resolved in the following way. By invoking the method of indistinguishable dynamics [4, 22, 23, 25], the Lyapunov analysis can be employed [17].

Remark 5. *It is worth emphasizing that the current literature where the method of indistinguishable dynamics has been applied (e.g. [4,23]) does not show examples of using the second Lyapunov method. However, some tools like the equilibrium point linearization, the first Lyapunov method, or Poincaré-Bendixson theorem have been used.*

Thus, assuming the following Lyapunov function $\mathcal{V}\left(\varepsilon_1(t), \varepsilon_4(t)\right) :\ \boldsymbol{\Xi} \to \mathbb{R}_+$:

$$\mathcal{V}\left(\varepsilon_1(t), \varepsilon_4(t)\right) = 0.5\left(\varepsilon_1^2(t) + v^{-1}\varepsilon_4^2(t)\right),$$

the analysis of detectability is performed (the proof that there is only one zero equilibrium point related to dynamics (14) is shown in Appendix A). Thus, the time derivative of $\mathcal{V}(\varepsilon_1(t), \varepsilon_4(t))$ is as follows:

$$\dot{\mathcal{V}}\left(\varepsilon_1(t), \varepsilon_4(t)\right) = -\left(Y_{\mathrm{s}} m_{\mathrm{s}} + m_{\mathrm{x}} + (1+r)u_1(t)\right)\varepsilon_1^2(t) \\ + (2r+1)u_1(t)\varepsilon_1(t)\varepsilon_4(t) - (w+r)u_1(t)\varepsilon_4^2(t) < 0. \tag{15}$$

Next, by taking some transformation using Schwartz inequality [3, 17], the part of the inequality (15) converts to:

$$\varepsilon_1(t)\varepsilon_4(t) \leqslant |\varepsilon_1(t)\varepsilon_4(t)| \leqslant |\varepsilon_1(t)||\varepsilon_4(t)| \leqslant 0.5\left(\varepsilon_1^2(t) + \varepsilon_4^2(t)\right).$$

Thus, the right-hand side of inequality (15) can be assessed by:

$$\begin{aligned}\dot{\mathcal{V}}\left(\varepsilon_1(t), \varepsilon_4(t)\right) \leqslant &-\left(Y_{\mathrm{s}} m_{\mathrm{s}} + m_{\mathrm{x}} + (1+r)u_1(t)\right)\varepsilon_1^2(t) + 0.5(2r+1)u_1(t)\varepsilon_1^2(t) \\ &+ 0.5(2r+1)u_1(t)\varepsilon_4^2(t) - (w+r)u_1(t)\varepsilon_4^2(t) < 0. \end{aligned} \tag{16}$$

Therefore, by reordering the components of (16), the following inequalities are given:

$$\begin{cases} 0 > -Y_{\mathrm{s}} m_{\mathrm{s}} - m_{\mathrm{x}} - \left[(1+r) - 0.5(2r+1)\right] u_1(t) \\ 0 > -\left((w+r) - 0.5(2r+1)\right) u_1(t) \end{cases}$$

$$\rightarrow \quad \begin{cases} 0 > -\left(0.5u_1(t) + Y_{\mathrm{s}} m_{\mathrm{s}} + m_{\mathrm{x}}\right) \\ w > 0.5 \end{cases}$$

Hence, it is shown $\forall t \in \mathbb{T}$ that $(\overline{\varepsilon}_1, \overline{\varepsilon}_4) = (0, 0)$ is globally asymptotically stable equilibrium point when $w > 0.5$, which means that $x_1(t)$ and $x_4(t)$ are globally strongly u–detectable. Knowing that, dynamics (14) can be presented as:

$$\begin{cases} Y_{\mathrm{s}} m_{\mathrm{s}} \varepsilon_1(t) = -\Big[\varepsilon_\mu(t)x_1(t) + \mu(t)\varepsilon_1(t) - \varepsilon_\mu(t)\varepsilon_1(t)\Big], \\ \qquad \dot{\varepsilon}_3(t) = \left[Y_{\mathrm{s}} Y_{\mathrm{o}}^{-1} m_{\mathrm{s}} - m_{\mathrm{o}}\right]\gamma_2(t) - \left[(1+r)u_1(t) + u_2(t)\right]\varepsilon_3(t), \\ \qquad \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0, \end{cases} \tag{17}$$

where $\gamma_2(t) \equiv \varepsilon_1(t)$ is the solution of the first differential equation, which vanishes asymptotically to zero for any initial condition. Therefore, considering the positiveness of both inputs and the property of $\gamma_2(t)$, the third state variable is globally strongly u–detectable. If $\varepsilon_1(t) \rightarrow 0$ and $\varepsilon_3(t) \rightarrow 0$ then $\mu(t)$ is globally strongly u–detectable.

In the second option, when the $x_1(t) = 0$, the investigation results are as follows. When $x_1(t)$ is eliminated from (14), the new indistinguishable dynamics is given as:

$$\begin{cases} \qquad 0 = -\Big[Y_{\mathrm{s}} m_{\mathrm{s}} + \mu(t) - \varepsilon_\mu(t)\Big]\varepsilon_1(t), \\ \dot{\varepsilon}_1(t) = -\left[Y_{\mathrm{s}} m_{\mathrm{s}} + m_{\mathrm{x}} + (1+r)u_1(t)\right]\varepsilon_1(t) + r\varepsilon_4(t)u_1(t), \\ \dot{\varepsilon}_3(t) = \left[Y_{\mathrm{s}} Y_{\mathrm{o}}^{-1} m_{\mathrm{s}} - m_{\mathrm{o}}\right]\varepsilon_1(t) - \left[(1+r)u_1(t) + u_2(t)\right]\varepsilon_3(t), \\ \dot{\varepsilon}_4(t) = v(1+r)\varepsilon_1(t)u_1(t) - v(w+r)\varepsilon_4(t)u_1(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0. \end{cases} \tag{18}$$

The investigation of global strong u–detectability of $x_1(t)$, $x_3(t)$, and $x_4(t)$ can be performed in the same way as in the situation when the biomass concentration ($x_1(t)$) is positive. It is because the differential equations contained in dynamics (18) are the same as in dynamics (14). Thus, by applying the presented methodology, i.e. (15)–(17), it can be shown that $x_1(t)$, $x_3(t)$, and $x_4(t)$ are globally strongly u–detectable, due to asymptotic convergence of particular differential equations to zero point. However, due to the form of the algebraic equation from (18), nothing can be said on the u–detectability of $\mu(t)$. In other words, if $\varepsilon_1(t)$ tends to zero asymptotically, then the rest of invoked expression cannot indicate anything about the asymptotic convergence of $\varepsilon_\mu(t)$, which coincide with $Y_s m_s + \mu(t)$ expression, which is not vanishing in time $\mathbb{T}$.

4.1.3. Measured output – dissolved oxygen concentration $DO(t)$

The measured output is $y(t) = h(\boldsymbol{x}(t)) = x_3(t)$. It implies that $\forall t \in \mathbb{T}\ \varepsilon_3(t) = 0$ and $\dot{\varepsilon}_3(t) = 0$, thus model (9) is converted into the following DAE system:

$$\begin{cases} \dot{\varepsilon}_1(t) = \varepsilon_\mu(t)x_1(t) + \mu(t)\varepsilon_1(t) - \varepsilon_\mu(t)\varepsilon_1(t) - m_x\varepsilon_1(t) \\ \qquad\quad - (1+r)\varepsilon_1(t)u_1(t) + r\varepsilon_4(t)u_1(t), \\ \dot{\varepsilon}_2(t) = -Y_s^{-1}\left[\varepsilon_\mu(t)x_1(t) + \mu(t)\varepsilon_1(t) - \varepsilon_\mu(t)\varepsilon_1(t)\right] - m_s\varepsilon_1(t) \\ \qquad\quad - (1+r)\varepsilon_2(t)u_1(t), \\ \quad 0 = -Y_o^{-1}\left[\varepsilon_\mu(t)x_1(t) + \mu(t)\varepsilon_1(t) - \varepsilon_\mu(t)\varepsilon_1(t)\right] - m_o\varepsilon_1(t), \\ \dot{\varepsilon}_4(t) = v(1+r)\varepsilon_1(t)u_1(t) - v(w+r)\varepsilon_4(t)u_1(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0\,. \end{cases} \tag{19}$$

By invoking to the third equation in (19), the following transformation is given:

$$\begin{cases} \dot{\varepsilon}_1(t) = -Y_o m_o\varepsilon_1(t) - m_x\varepsilon_1(t) - (1+r)\varepsilon_1(t)u_1(t) + r\varepsilon_4(t)u_1(t), \\ \dot{\varepsilon}_2(t) = Y_s^{-1}Y_o m_o\varepsilon_1(t) - m_s\varepsilon_1(t) - (1+r)\varepsilon_2(t)u_1(t), \\ Y_o m_o\varepsilon_1(t) = -\left[\varepsilon_\mu(t)x_1(t) + \mu(t)\varepsilon_1(t) - \varepsilon_\mu(t)\varepsilon_1(t)\right], \\ \dot{\varepsilon}_4(t) = v(1+r)\varepsilon_1(t)u_1(t) - v(w+r)\varepsilon_4(t)u_1(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0\,. \end{cases} \tag{20}$$

If $x_1(t) > 0$, then it is assumed, analogously to the considerations presented in Section 4.1.2, that $x_1(t)$, $x_2(t)$, $x_4(t)$ are globally strongly u–detectable provided that dynamics (20) is asymptotically stable. To show that this property holds, the same analysis premised on the second Lyapunov method can be performed as it is done in Section 4.1.2. Thus, by substituting dynamics (14) by dynamics

(20), $\varepsilon_1(t)$ and $\varepsilon_4(t)$ tend asymptotically to zero for any initial conditions in time $\mathbb{T}$ if and only if parameter $w > 0.5$. Hence, it is shown that $(\overline{\varepsilon}_1, \overline{\varepsilon}_4) = (0, 0)$ is globally asymptotically stable equilibrium point, which means that $x_1(t)$ and $x_4(t)$ are globally strongly u–detectable. Knowing that, dynamics (20) can be presented as:

$$\begin{cases} Y_o m_o \varepsilon_1(t) = -\left[\varepsilon_\mu(t) x_1(t) + \mu(t)\varepsilon_1(t) - \varepsilon_\mu(t)\varepsilon_1(t)\right], \\ \qquad \dot{\varepsilon}_2(t) = \left[Y_o Y_s^{-1} m_o - m_s\right] \gamma_3(t) - (1+r)u_1(t)\varepsilon_2(t), \\ \qquad \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0, \end{cases} \tag{21}$$

where $\gamma_3(t) \equiv \varepsilon_1(t)$ is the solution of the first differential equation, which vanishes asymptotically. Therefore, considering the positiveness of both control inputs and the property of $\gamma_3(t)$, state variable $x_2(t)$ is globally strongly u–detectable.

In turn, if $x_1(t) = 0$, then the investigation results are as follows. When $x_1(t)$ is cancelled from (20), then the new indistinguishable dynamics is given as:

$$\begin{cases} \qquad 0 = -\left[Y_o m_o + \mu(t) - \varepsilon_\mu(t)\right] \varepsilon_1(t), \\ \dot{\varepsilon}_1(t) = -\left[Y_o m_o + m_x + (1+r)u_1(t)\right] \varepsilon_1(t) + r\varepsilon_4(t)u_1(t), \\ \dot{\varepsilon}_2(t) = \left[Y_o Y_s^{-1} m_o - m_s\right] \varepsilon_1(t) - (1+r)u_1(t)\varepsilon_2(t), \\ \dot{\varepsilon}_4(t) = v(1+r)\varepsilon_1(t)u_1(t) - v(w+r)\varepsilon_4(t)u_1(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0. \end{cases} \tag{22}$$

The investigation of global strong u–detectability of $x_1(t)$, $x_2(t)$, and $x_4(t)$ can be performed in the same way as in the situation when the biomass concentration ($x_1(t)$) is positive. It is because the differential equations contained in dynamics (18) are the same as in dynamics (14). Thus, by applying the presented methodology, i.e. (15)–(17), it can be shown that $x_1(t)$, $x_2(t)$, and $x_4(t)$ are globally strongly u–detectable, due to asymptotic convergence of particular differential equations to zero point. However, due to the form of the algebraic equation from (22), nothing can be said on the u–detectability of $\mu(t)$. In other words, if $\varepsilon_1(t)$ tends to zero asymptotically, then the rest of invoked expression cannot indicate anything about the asymptotic convergence of $\varepsilon_\mu(t)$, which coincide with $Y_o m_o + \mu(t)$ expression, which is not vanishing in time $\mathbb{T}$.

4.1.4. Measured output – recirculated biomass concentration $X_r(t)$

The measured output is $y(t) = h(\boldsymbol{x}(t)) = x_4(t)$. It implies that $\forall t \in \mathbb{T}$ $\varepsilon_4(t) = 0$ and $\dot{\varepsilon}_4(t) = 0$, thus model (9) is converted into the following DAE system:

$$\begin{cases} \dot{\varepsilon}_1(t) = \varepsilon_\mu(t)x_1(t) + \mu(t)\varepsilon_1(t) - \varepsilon_\mu(t)\varepsilon_1(t) - m_\mathrm{x}\varepsilon_1(t) \\ \qquad\quad - (1+r)\varepsilon_1(t)u_1(t), \\ \dot{\varepsilon}_2(t) = -Y_\mathrm{s}^{-1}\left[\varepsilon_\mu(t)x_1(t) + \mu(t)\varepsilon_1(t) - \varepsilon_\mu(t)\varepsilon_1(t)\right] - m_\mathrm{s}\varepsilon_1(t) \\ \qquad\quad - (1+r)\varepsilon_2(t)u_1(t), \\ \dot{\varepsilon}_3(t) = -Y_\mathrm{o}^{-1}\left[\varepsilon_\mu(t)x_1(t) + \mu(t)\varepsilon_1(t) - \varepsilon_\mu(t)\varepsilon_1(t)\right] - m_\mathrm{o}\varepsilon_1(t) \\ \qquad\quad - (1+r)\varepsilon_3(t)u_1(t) - \varepsilon_3(t)u_2(t), \\ \qquad 0 = v(1+r)\varepsilon_1(t)u_1(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0 . \end{cases} \tag{23}$$

By invoking to the last equation in (23), if $\forall t \in \mathbb{T}\ u_1(t) \neq 0$ then $\forall t \in \mathbb{T}\ \varepsilon_1(t) = 0$ and $\dot{\varepsilon}_1(t) = 0$, hence $x_1(t)$ is globally strongly u–observable. It leads to the following transformation of (23):

$$\begin{cases} \qquad 0 = \varepsilon_\mu(t)x_1(t), \\ \dot{\varepsilon}_2(t) = -Y_\mathrm{s}^{-1}\varepsilon_\mu(t)x_1(t) - (1+r)\varepsilon_2(t)u_1(t), \\ \dot{\varepsilon}_3(t) = -Y_\mathrm{o}^{-1}\varepsilon_\mu(t)x_1(t) - (1+r)\varepsilon_3(t)u_1(t) - \varepsilon_3(t)u_2(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0 . \end{cases} \tag{24}$$

Because the left-hand side of the algebraic equation in (24) must be equal to zero $\forall t \in \mathbb{T}$, the considerations about observability and detectability of $x_2(t)$ and $x_3(t)$ are divided into two cases. The first is associated with the fact that $x_1(t)$ is zero, whereas the other one is linked to the fact that the kinetics function error $\varepsilon_\mu(t)$ is equal to zero $\forall t \in \mathbb{T}$, assuming that $x_1(t)$ is always positive.

In the first case, where $x_1(t)$ becomes zero, the behavior of kinetics function error $\varepsilon_\mu(t)$ cannot be directly established. Additionally, due to the boundedness of the state trajectories, the kinetics function does not tend infinitely. Taking into account that the inputs are the permanent excited positive signals, the dynamics of $\varepsilon_2(t)$ and $\varepsilon_3(t)$ from (24) asymptotically tend to zero. Therefore, in the first case $x_2(t)$ and $x_3(t)$ are globally strongly u–detectable, whereas nothing can be said about the observability or detectability of kinetics function $\mu(t)$. In turn, in the second case, when $x_1(t)$ is always positive a different result is obtained. The algebraic equation becomes zero $\forall t \in \mathbb{T}$ when error $\varepsilon_\mu(t)$ reveals this particular property, which makes $\mu(t)$ strongly u–observable. Next, to make the right-hand side of the third equation in (24) equals zero, both $\varepsilon_2(t)$ and $\varepsilon_3(t)$ and of course its derivatives must tend to zero asymptotically. Hence, in the second case, $x_2(t)$ and $x_3(t)$ are globally strongly u–detectable.

4.1.5. Part 1 – summary

Analogous analysis to cases described in Sections 4.1.1–4.1.4 has been carried out for the combinations of measured outputs and its results, together with results of Sections 4.1.1–4.1.4 are summarised in Table 1.

Table 1: Part 1 – results of the observability or detectability analysis

Case		Variable				
No.	wash-out	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_4(t)$	$\mu(t)$
(1)	NO	M	D	D	D	O
	YES	M	D	D	D	ND
(2)	NO	D	M	D	D	D
	YES	D	M	D	D	ND
(3)	NO	D	D	M	D	D
	YES	D	D	M	D	ND
(4)	NO	O	D	D	M	O
	YES	O	D	D	M	ND
(5)	NO	D	M	M	O	D
	YES	D	M	M	O	ND
(6)	NO	M	D	D	M	O
	YES	M	O	D	M	ND
(7)	NO	O	M	D	M	O
	YES	O	M	D	M	ND
(8)	NO	O	D	M	M	O
	YES	O	D	M	M	ND

The symbols used in Table 1 denote ‘O’ – observable, ‘D’ – detectable, ‘M’ – measured, and ‘ND’ – non-detectable.

4.2. Part 2 – known reaction kinetics function

A detailed analysis of three cases not presented in [5] and a synthetic (tabular) description of all possibilities limited to a reasonable number of two measured outputs is presented in this section.

4.2.1. Measured output – substrate concentration $S(t)$

The measured output is $y(t) = h(\boldsymbol{x}(t)) = x_2(t)$. It implies that $\forall t \in \mathbb{T}\ \varepsilon_2(t) = 0$ and $\dot{\varepsilon}_2(t) = 0$, thus model (9) is converted into the following DAE system:

$$\begin{cases} \dot{\varepsilon}_1(t) = \mu(t)x_1(t) - \mu_z(t)\left(x_1(t) - \varepsilon_1(t)\right) - m_x\varepsilon_1(t) \\ \qquad - (1+r)\varepsilon_1(t)u_1(t) + r\varepsilon_4(t)u_1(t), \\ \quad 0 = -Y_s^{-1}\left[\mu(t)x_1(t) - \mu_z(t)\left(x_1(t) - \varepsilon_1(t)\right)\right] - m_s\varepsilon_1(t), \\ \dot{\varepsilon}_3(t) = -Y_o^{-1}\left[\mu(t)x_1(t) - \mu_z(t)\left(x_1(t) - \varepsilon_1(t)\right)\right] - m_o\varepsilon_1(t) \\ \qquad - (1+r)\varepsilon_3(t)u_1(t) - \varepsilon_3(t)u_2(t), \\ \dot{\varepsilon}_4(t) = v(1+r)\varepsilon_1(t)u_1(t) - v(w+r)\varepsilon_4(t)u_1(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0\,. \end{cases} \tag{25}$$

By invoking to the second equation in (25), the following transformation is given:

$$\begin{cases} \dot{\varepsilon}_1(t) = -Y_s m_s\varepsilon_1(t) - m_x\varepsilon_1(t) - (1+r)\varepsilon_1(t)u_1(t) + r\varepsilon_4(t)u_1(t), \\ Y_s m_s\varepsilon_1(t) = -\left[\mu(t)x_1(t) - \mu_z(t)\left(x_1(t) - \varepsilon_1(t)\right)\right], \\ \dot{\varepsilon}_3(t) = Y_o^{-1}Y_s m_s\varepsilon_1(t) - m_o\varepsilon_1(t) - (1+r)\varepsilon_3(t)u_1(t) - \varepsilon_3(t)u_2(t), \\ \dot{\varepsilon}_4(t) = v(1+r)\varepsilon_1(t)u_1(t) - v(w+r)\varepsilon_4(t)u_1(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0\,. \end{cases} \tag{26}$$

If $x_1(t) > 0$, then it is assumed, analogously to the considerations presented in Section 4.1.2, that $x_1(t)$, $x_3(t)$, $x_4(t)$ are globally u–detectable provided that dynamics (26) is asymptotically stable. To show that this property holds, the same analysis premised on the second Lyapunov method can be performed as it is done in Section 4.1.2. Thus, by substituting dynamics (14) by dynamics (26), $\varepsilon_1(t)$ and $\varepsilon_4(t)$ tend asymptotically to zero for any initial conditions in time $\mathbb{T}$ if and only if parameter $w > 0.5$. Hence, it is shown that $(\overline{\varepsilon}_1, \overline{\varepsilon}_4) = (0, 0)$ is globally asymptotically stable equilibrium point, which means that $x_1(t)$ and $x_4(t)$ are globally u–detectable. Knowing that, dynamics (26) can be presented as:

$$\begin{cases} Y_s m_s\varepsilon_1(t) = -\left[\mu_z(t)x_1(t) + \mu(t)\varepsilon_1(t) - \mu_z(t)\varepsilon_1(t)\right], \\ \dot{\varepsilon}_3(t) = \left[Y_s Y_o^{-1} m_s - m_o\right]\gamma_4(t) - \left[(1+r)u_1(t) + u_2(t)\right]\varepsilon_3(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0\,, \end{cases} \tag{27}$$

where $\gamma_4(t) \equiv \varepsilon_1(t)$ is the solution of the first differential equation, which vanishes asymptotically. Therefore, considering the positiveness of both control inputs and the property of $\gamma_4(t)$, the third state variable is globally u–detectable. It is worth emphasizing that the behavior of $\mu_z(t)$ function, which is dependent on $\varepsilon_3(t)$ does not cause any indistinguishable trajectories, due to the fact that if $\varepsilon_3(t) \to 0$ then $\mu_z(t) \to 0$ if $t \to \infty$.

In turn, when $x_1(t) = 0$, then the investigation results are as follows. When $x_1(t)$ is eliminated from (26), then the new indistinguishable dynamics is given as:

$$\begin{cases} \quad 0 = -\left[Y_s m_s + \mu(t)\right] \varepsilon_1(t), \\ \dot{\varepsilon}_1(t) = -\left[Y_s m_s + m_x + (1+r)u_1(t)\right] \varepsilon_1(t) + r\varepsilon_4(t)u_1(t), \\ \dot{\varepsilon}_3(t) = \left[Y_s Y_o^{-1} m_s - m_o\right] \varepsilon_1(t) - \left[(1+r)u_1(t) + u_2(t)\right] \varepsilon_3(t), \\ \dot{\varepsilon}_4(t) = v(1+r)\varepsilon_1(t)u_1(t) - v(w+r)\varepsilon_4(t)u_1(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0 . \end{cases} \tag{28}$$

Taking into account that the first algebraic equation from (28) disappears when $\varepsilon_1(t)$ and $\dot{\varepsilon}_1(t)$ are equal to zero, $x_1(t)$ is claimed as globally u–observable. Therefore, dynamics (28) is conversed to:

$$\begin{cases} \quad 0 = r\varepsilon_4(t)u_1(t), \\ \dot{\varepsilon}_3(t) = -\left[(1+r)u_1(t) + u_2(t)\right] \varepsilon_3(t), \\ \dot{\varepsilon}_4(t) = -v(w+r)\varepsilon_4(t)u_1(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0 . \end{cases} \tag{29}$$

Knowing that, the algebraic equation from (29) becomes zero when $\forall t \in \mathbb{T}$ $u_1(t) \neq 0$. Hence, state variable $x_4(t)$ is u–observable. Also, considering that $u_1(t)$ and $u_2(t)$ are persistent excited signals, $\varepsilon_3(t)$ related differential equation is globally asymptotically stable in equilibrium zero point, which means that $x_3(t)$ is globally u–detectable.

4.2.2. Measured output – dissolved oxygen concentration $DO(t)$

The measured output is $y(t) = h(\boldsymbol{x}(t)) = x_3(t)$. It implies that $\forall t \in \mathbb{T}$ $\varepsilon_3(t) = 0$ and $\dot{\varepsilon}_3(t) = 0$, thus model (9) is converted into the following DAE system:

$$\begin{cases} \dot{\varepsilon}_1(t) = \mu(t)x_1(t) - \mu_z(t)\left(x_1(t) - \varepsilon_1(t)\right) - m_x\varepsilon_1(t) \\ \qquad\quad - (1+r)\varepsilon_1(t)u_1(t) + r\varepsilon_4(t)u_1(t), \\ \dot{\varepsilon}_2(t) = -Y_s^{-1}\left[\mu(t)x_1(t) - \mu_z(t)\left(x_1(t) - \varepsilon_1(t)\right)\right] - m_s\varepsilon_1(t) \\ \qquad\quad - (1+r)\varepsilon_2(t)u_1(t), \\ \quad 0 = -Y_o^{-1}\left[\mu(t)x_1(t) - \mu_z(t)\left(x_1(t) - \varepsilon_1(t)\right)\right] - m_o\varepsilon_1(t), \\ \dot{\varepsilon}_4(t) = v(1+r)\varepsilon_1(t)u_1(t) - v(w+r)\varepsilon_4(t)u_1(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0 . \end{cases} \tag{30}$$

By invoking to the second equation in (30), the following transformation is given:

$$\begin{cases} \dot{\varepsilon}_1(t) = -Y_o m_o \varepsilon_1(t) - m_x \varepsilon_1(t) - (1+r)\varepsilon_1(t)u_1(t) + r\varepsilon_4(t)u_1(t), \\ \dot{\varepsilon}_2(t) = Y_s^{-1} Y_o m_o \varepsilon_1(t) - m_s \varepsilon_1(t) - (1+r)\varepsilon_2(t)u_1(t), \\ Y_o m_o \varepsilon_1(t) = -\left[\mu(t)x_1(t) - \mu_z(t)\left(x_1(t) - \varepsilon_1(t)\right)\right], \\ \dot{\varepsilon}_4(t) = v(1+r)\varepsilon_1(t)u_1(t) - v(w+r)\varepsilon_4(t)u_1(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0\,. \end{cases} \tag{31}$$

If $x_1(t) > 0$, then it is assumed, analogously to the considerations presented in Section 4.1.2, that $x_1(t)$, $x_2(t)$, $x_4(t)$ are globally u–detectable provided that dynamics (31) is asymptotically stable. To show that this property holds, the same analysis premised on the second Lyapunov method can be performed as it is done in Section 4.1.2. Thus, by substituting dynamics (14) by dynamics (31), $\varepsilon_1(t)$ and $\varepsilon_4(t)$ tend asymptotically to zero for any initial conditions in time $\mathbb{T}$ if and only if parameter $w > 0.5$. Hence, it is shown that $(\overline{\varepsilon}_1, \overline{\varepsilon}_4) = (0, 0)$ is globally asymptotically stable equilibrium point, which means that $x_1(t)$ and $x_4(t)$ are globally u–detectable. Knowing that, dynamics (31) can be presented as:

$$\begin{cases} Y_o m_o \varepsilon_1(t) = -\left[\mu_z(t)x_1(t) + \mu(t)\varepsilon_1(t) - \mu_z(t)\varepsilon_1(t)\right], \\ \dot{\varepsilon}_2(t) = \left[Y_o Y_s^{-1} m_o - m_s\right]\gamma_5(t) - (1+r)u_1(t)\varepsilon_2(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0\,, \end{cases} \tag{32}$$

where $\gamma_5(t) \equiv \varepsilon_1(t)$ is the solution of the first differential equation, which vanishes asymptotically. Therefore, considering the positiveness of both control inputs and the property of $\gamma_5(t)$, the second state variable is globally u–detectable. It is worth emphasizing that the behavior of $\mu_z(t)$, which is dependent on $\varepsilon_2(t)$ does not cause any indistinguishable trajectories, due to the fact that if $\varepsilon_2(t) \to 0$ then $\mu_z(t) \to 0$ if $t \to \infty$.

In the second option, when $x_1(t) = 0$, the investigation results are as follows. When $x_1(t)$ is cancelled from (31), the new indistinguishable dynamics is given as:

$$\begin{cases} 0 = -\left[Y_o m_o + \mu(t)\right]\varepsilon_1(t), \\ \dot{\varepsilon}_1(t) = -\left[Y_o m_o + m_x + (1+r)u_1(t)\right]\varepsilon_1(t) + r\varepsilon_4(t)u_1(t), \\ \dot{\varepsilon}_2(t) = \left[Y_o Y_s^{-1} m_o - m_s\right]\varepsilon_1(t) - (1+r)u_1(t)\varepsilon_2(t), \\ \dot{\varepsilon}_4(t) = v(1+r)\varepsilon_1(t)u_1(t) - v(w+r)\varepsilon_4(t)u_1(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0\,. \end{cases} \tag{33}$$

Taking into account that the algebraic equation from (28) vanishes when $\varepsilon_1(t)$ and $\dot{\varepsilon}_1(t)$ are always equal to zero, $x_1(t)$ is claimed as globally u–observable.

Therefore, dynamics (28) is conversed to:

$$\begin{cases} 0 = r\varepsilon_4(t)u_1(t), \\ \dot{\varepsilon}_2(t) = -(1+r)u_1(t)\varepsilon_2(t), \\ \dot{\varepsilon}_4(t) = -v(w+r)\varepsilon_4(t)u_1(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0 \,. \end{cases} \tag{34}$$

Knowing that, the first algebraic equation from (29) becomes zero when $\forall t \in \mathbb{T}\ u_1(t) \neq 0$. Hence, state variable $x_4(t)$ is u–observable. Also, taking into account that $u_1(t)$ is the persistent excited signal, $\varepsilon_2(t)$ related to differential equation is globally asymptotically stable in equilibrium zero point, which means that $x_2(t)$ is globally u–detectable.

4.2.3. Measured output – recirculated biomass concentration X_r

The measured output is $y(t) = h(\boldsymbol{x}(t)) = x_4(t)$. It implies that $\forall t \in \mathbb{T}\ \varepsilon_4(t) = 0$ and $\dot{\varepsilon}_4(t) = 0$, thus model (9) is converted into the following DAE system:

$$\begin{cases} \dot{\varepsilon}_1(t) = \mu(t)x_1(t) - \mu_\mathrm{z}(t)\left(x_1(t) - \varepsilon_1(t)\right) - m_\mathrm{x}\varepsilon_1(t) - (1+r)\varepsilon_1(t)u_1(t), \\ \dot{\varepsilon}_2(t) = -Y_\mathrm{s}^{-1}\left[\mu(t)x_1(t) - \mu_\mathrm{z}(t)\left(x_1(t) - \varepsilon_1(t)\right)\right] - m_\mathrm{s}\varepsilon_1(t) \\ \qquad\quad - (1+r)\varepsilon_2(t)u_1(t), \\ \dot{\varepsilon}_3(t) = -Y_\mathrm{o}^{-1}\left[\mu(t)x_1(t) - \mu_\mathrm{z}(t)\left(x_1(t) - \varepsilon_1(t)\right)\right] - m_\mathrm{o}\varepsilon_1(t) \\ \qquad\quad - (1+r)\varepsilon_3(t)u_1(t) - \varepsilon_3(t)u_2(t), \\ \quad 0 = v(1+r)\varepsilon_1(t)u_1(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0 \,. \end{cases} \tag{35}$$

By invoking to the last equation in (35), if $\forall t \in \mathbb{T}\ u_1(t) \neq 0$ then $\forall t \in \mathbb{T}\ \varepsilon_1(t) = 0$ and $\dot{\varepsilon}_1(t) = 0$, hence $x_1(t)$ is globally u–observable. It leads to the following transformation of (35):

$$\begin{cases} 0 = \mu(t)x_1(t) - \mu_\mathrm{z}(t)x_1(t), \\ \dot{\varepsilon}_2(t) = -Y_\mathrm{s}^{-1}\left(\mu(t) - \mu_\mathrm{z}(t)\right)x_1(t) - (1+r)\varepsilon_2(t)u_1(t), \\ \dot{\varepsilon}_3(t) = -Y_\mathrm{o}^{-1}\left(\mu(t) - \mu_\mathrm{z}(t)\right)x_1(t) - (1+r)\varepsilon_3(t)u_1(t) - \varepsilon_3(t)u_2(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0 \,. \end{cases} \tag{36}$$

Considering the first equation in (36) this system can be rewritten as:

$$\begin{cases} 0 = \mu(t)x_1(t) - \mu_\mathrm{z}(t)x_1(t), \\ \dot{\varepsilon}_2(t) = -(1+r)\varepsilon_2(t)u_1(t), \\ \dot{\varepsilon}_3(t) = -(1+r)\varepsilon_3(t)u_1(t) - \varepsilon_3(t)u_2(t), \\ \boldsymbol{\varepsilon}(t_0) = \boldsymbol{\varepsilon}_0 \,. \end{cases} \tag{37}$$

Since the left-hand side of algebraic equation in (37) must be equal to zero $\forall t \in \mathbb{T}$, the considerations about observability and detectability of $x_2(t)$ and $x_3(t)$ are divided into two situations. The first is associated with the fact that $x_1(t)$ is zero, whereas the other one is linked to the fact that the kinetics functions $\mu(t)$ and error $\mu_z(t)$ are equal to each other $\forall t \in \mathbb{T}$, assuming that $x_1(t)$ is always positive.

In the first situation, where $x_1(t)$ becomes zero, the behavior of kinetics functions is dependent only on inputs and initial conditions of $x_2(t)$ and $x_3(t)$. Taking into account that the inputs are the permanent excited positive signals, the dynamics of $\varepsilon_2(t)$ and $\varepsilon_3(t)$ from (37) asymptotically tend to zero. Moreover, due to fact that $\mu_z(t)$ and $\mu(t)$ are Monod terms, when $\varepsilon_2(t) \to 0$ and $\varepsilon_3(t) \to 0$ then $\mu_z(t) \to \mu(t)$ asymptotically. Additionally, due to the boundedness of the state trajectories, the kinetics functions do not tend infinitely. Therefore, in the first situation $x_2(t)$ and $x_3(t)$ are globally strongly u–detectable. In turn, in the second situation, when $x_1(t)$ is always positive the same result is obtained; however, its understanding is slightly different. To make the right-hand side of the third equation in (37) equals zero, both $\varepsilon_2(t)$ and $\varepsilon_3(t)$ and of course its derivatives must tend to zero. Hence, in the second situation, $x_2(t)$ and $x_3(t)$ are globally strongly u–detectable.

4.2.4. Part 2 – summary

Analogous analysis to cases described in Sections 4.2.1–4.2.3 has been carried out for the combinations of measured outputs and its results, together with results of Sections 4.2.1–4.2.3 are summarized in Table 2.

Table 2: Part 2 – results of the observability or detectability analysis

Case		Variable			
No.	wash-out	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_4(t)$
(1)	NO	D	M	D	D
	YES	O	M	D	O
(2)	NO	D	D	M	D
	YES	O	D	M	O
(3)	NO	O	D	D	M
	YES	O	D	D	M
(4)	NO	M	D	D	D
	YES	M	D	D	D

Continued on the next page

Table 2 [cont.]

Case		Variable			
No.	wash-out	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_4(t)$
(5)	NO	O	M	M	O
	YES	O	M	M	O
(6)	NO	M	D	D	M
	YES	M	D	D	M
(7)	NO	O	M	O	M
	YES	O	M	D	M
(8)	NO	O	O	M	M
	YES	O	D	M	M

5. Conclusions

In this paper, the analysis of the observability and detectability of continuous stirred tank reactor model of selected biochemical processes has been investigated. In particular, the properties of observability or detectability of the considered system model have been discussed under uncertain system dynamics and taking into account various sets of measured outputs. The method of indistinguishable state trajectories (indistinguishable dynamics) and the Lyapunov second method have been used as analysis tools. Eight different cases covering a wide range of possible combinations of the measured outputs have been considered. Delivered comprehensive analysis shows how the given structure of the system mathematical model and selection of particular measured outputs affect the observability and detectability properties. The proposed approaches may be successfully applied to the complex biochemical non-linear uncertain systems modeled as continuous stirred tank reactors.

The obtained research results are essential for system state estimation that involves the synthesis of state observers. Thus, they may be used for the state observer synthesis and diagnostic systems development for biochemical processes.

A. Analysis of equilibrium points of indistinguishable dynamics

In Section 4 a gradual analysis of the observability and detectability for the particular selection of measured outputs is presented. For the sake of completing the research, it is necessary to prove the hypothesis that only one equilibrium point of decoupled sub-dynamics of (14) is given as $(\overline{\varepsilon}_1, \overline{\varepsilon}_4) = (0, 0)$. Essentially, the derived two-dimensional system is linear with respect to its states $\varepsilon_1(t)$ and $\varepsilon_4(t)$

(detailing bi-linear [15,17]). Hence, let us define the set of two algebraic equations derived directly from (14):

$$\begin{cases} 0 = -\left[Y_s m_s + m_x + (1+r)\overline{u}_1\right]\overline{\varepsilon}_1 + r\overline{\varepsilon}_4\overline{u}_1\,, \\ 0 = v(1+r)\overline{u}_1\overline{\varepsilon}_1 - v(w+r)\overline{u}_1\overline{\varepsilon}_4\,, \end{cases} \tag{38}$$

where $\overline{\varepsilon}_1, \overline{\varepsilon}_4, \overline{u}_1 \in \mathbb{R}$ are constants, which represent particular coordinates of the equilibrium point.

The site performed division by v and $\overline{u}_1$ in the second equation can be done due to the positiveness of invoked constants. Thus, the set of equations (38) can be written as follows:

$$\begin{gathered} \mathbf{0}_{2\times 1} = \underbrace{\begin{bmatrix} -\alpha_{\mathcal{A}} & r\overline{u}_1 \\ (1+r) & -(w+r) \end{bmatrix}}_{\mathcal{A}\in\mathbb{R}^{2\times 2}} \begin{bmatrix} \overline{\varepsilon}_1 \\ \overline{\varepsilon}_4 \end{bmatrix}, \\ \alpha_{\mathcal{A}} \triangleq \left[Y_s m_s + m_x + (1+r)\overline{u}_1\right] \in \mathbb{R}_+ \;\; \forall t \in \mathbb{T}. \end{gathered} \tag{39}$$

To prove the proposed hypothesis, it must be shown that $\ker(\mathcal{A}) = \{(\overline{\varepsilon}_1, \overline{\varepsilon}_4) = (0,0)\}$, where $\ker(\cdot)$ denotes kernel of the matrix. This situation appears when the rows of $\mathcal{A}$ are linearly independent $\forall t \in \mathbb{T}$ [17].

Theorem 1. *The rows of matrix $\mathcal{A}$ from (39) are linearly independent, i.e.* $\operatorname{rank}(\mathcal{A}) = 2$, *for any positive values of parameters (m_s, m_x r, w and Y_s) and any value of positive input $u_1(t)$ in time $\mathbb{T}$.*

Proof. Since matrix $\mathcal{A}$ is square the checking its singularity is necessary for the sake of the rank investigation. Thus, the calculation of its determinant yields:

$$\begin{aligned} \det(\mathcal{A}) &= \alpha_{\mathcal{A}}(w+r) - r\overline{u}_1(1+r) \\ &= \alpha_{\mathcal{A}}w + \left[Y_s m_s + m_x + (1+r)\overline{u}_1\right] r - (1+r)r\overline{u}_1 \\ &= \alpha_{\mathcal{A}}w + \left[Y_s m_s + m_x\right] r. \end{aligned} \tag{40}$$

The form of obtained determinant does not include any division or subtraction operations. Hence, taking into account that all parameters from (40) are always positive, there is no possibility to make $\det(\mathcal{A}) = 0$ under any circumstances. Hence, rows of $\mathcal{A}$ are linearly independent, which causes there to be only one zero equilibrium point, which ends the proof. □

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