APPLICATIONES MATHEMATICAE
29,1 (2002), pp. 33–41

Karol Dziedziul (Gdańsk)

APPLICATION OF MAZUR–ORLICZ'S THEOREM IN AMISE CALCULATION

Abstract. An approximation error and an asymptotic formula are given for shift invariant operators of polynomial order ϱ. Density estimators based on shift invariant operators are introduced and AMISE is calculated.

1. Asymptotic formulas. We assume that $F, G : \mathbb{R}^d \to \mathbb{R}$ are functions such that there are constants $C > 0$ and $0 < q < 1$ such that for all $x \in \mathbb{R}^d$,

$$|F(x)| < Cq^{|x|} \quad \& \quad |G(x)| < Cq^{|x|}, \tag{1}$$

where $|x|^2 = x \cdot x$ and $x \cdot x$ is the scalar product in $\mathbb{R}^d$. Consider the operator given by

$$Qf(x) = \int_{\mathbb{R}^d} K(x,y) f(y) dy, \tag{2}$$

where

$$K(x,y) = \sum_{\alpha \in \mathbb{Z}^d} F(y-\alpha) G(x-\alpha). \tag{3}$$

For $h > 0$, define

$$Q_h = \sigma_h \circ Q \circ \sigma_{1/h}, \tag{4}$$

where

$$\sigma_h f(x) = f(x/h).$$

We call the operators with kernel of type (3) *shift invariant*. Examples of such operators are:

• spline operators: the Ciesielski–Durrmeyer operator (see [C]), a quasi-projection (see [Dz1]), an orthogonal projection (see [BD2], [BHR]),

2000 *Mathematics Subject Classification*: 41A35, 41A60, 62G07, 62G20.

Key words and phrases: shift invariant operators, asymptotic formula, density estimators, central limit, AMISE, asymptotic mean integral square error.

• an orthogonal projection based on multiresolution approximation [M], operators based on shift invariant spaces (see [JZ] and [BDR]; in particular shift invariant spaces constructed by a function which satisfies the Strang–Fix conditions, see [SF]).

Let W_p^r be a Sobolev space (for details see [M]). Let C_0^r be the space of r-differentiable functions with compact support. Set

$$|f|_{r,p} = \sum_{|\beta|=r} \|D^\beta f\|_p, \qquad \|f\|_p = \Big(\int_{\mathbb{R}^d} |f|^p\Big)^{1/p},$$

$$D^\beta f = \frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1} \ldots \partial x_d^{\beta_d}}, \quad \beta = (\beta_1, \ldots, \beta_d), \quad |\beta| = \beta_1 + \ldots + \beta_d.$$

Assume the operator Q *reproduces* all polynomials of degree less than r, i.e. $Q(P) = P$ provided $\deg P < r$. We then say that Q has *polynomial order* r. The following theorem is a generalization of [BHR, Proposition 4, p. 63].

THEOREM 1.1. *Let $1 \le p < \infty$. Assume that Q has polynomial order r. Then there is a constant $C(p) > 0$ such that for all $f \in W_p^r(\mathbb{R}^d)$,*

$$\|Q_h f - f\|_p \le C(p) h^r |f|_{r,p}, \tag{5}$$

Proof. Since the operators Q_h are bounded from L^p to L^p it is sufficient to prove (5) for $f \in C_0^r$. Let $f \in C_0^r$. Let P_x be the Taylor polynomial of f of degree $r-1$ at x. Note that $f(x) = P_x(x)$ and $Q_h f(x) - f(x) = Q_h(f - P_x)(x)$. Now Lemma 1.1 below yields (5) for $1 \le p < \infty$. ■

An easy computation shows the assertion for $p = \infty$ (see proof of [Dz4, Theorem 9.7]).

In statistics we need an asymptotic formula for the error in shift invariant operators. Such a formula was proved in [BD3], [BD4] for an interpolation operator and an orthogonal projection. Those proofs are based on a generalization of Mazur–Orlicz's theorem (see [BD3]). This theorem goes back to L. Fejér. Recall that a function g defined on $\mathbb{R}^d$ is called $\mathbb{Z}^d$-*periodic* if for all $x \in \mathbb{R}^d$,

$$g(x) = g(x + \alpha) \quad \text{for all } \alpha \in \mathbb{Z}^d. \tag{6}$$

THEOREM 1.2 (Mazur–Orlicz [MO]). *If for $j = 1, \ldots, m$, g_j are measurable, bounded, $\mathbb{Z}^d$-periodic functions and f_j are measurable functions with*

$$\int_{\mathbb{R}^d} |f_j(x)|^p \, dx < \infty$$

for some $1 \le p < \infty$, then

$$\int_{\mathbb{R}^d} \Big| \sum_{j=1}^m f_j(x) g_j(x/h) \Big|^p dx \to \int_{[0,1)^d} \int_{\mathbb{R}^d} \Big| \sum_{j=1}^m f_j(t) g_j(x) \Big|^p dt\, dx \quad \textit{as } h \to 0. \tag{7}$$

Earlier results concerning the asymptotic formula can be found in [C], [Dz1] for spline operators, and in [DU], [BD2] for an orthogonal projection in L^2. See also [DLP]. Let $[]^\beta(x) = x^\beta = x_1^{\beta_1}\dots x_d^{\beta_d}$. We present a new and simpler proof of the asymptotic formula for the error in shift invariant operators.

THEOREM 1.3. *Assume that Q has maximal polynomial order ϱ. Let $1 \le p < \infty$ and $f \in W_p^\varrho(\mathbb{R}^d)$. Then*

$$(8)\qquad \lim_{h\to 0^+}\left\|\frac{Q_hf-f}{h^\varrho}\right\|_p^p = \int_{\mathbb{R}^d}\bigg(\int_{[0,1]^d}\bigg|\sum_{|\beta|=\varrho}\frac{1}{\beta!}D^\beta f(t)(Q([]^\beta)(x)-x^\beta)\bigg|^p\,dx\bigg)\,dt.$$

Proof. It is sufficient to prove (8) for the dense subset $C_0^{\varrho+1}$ of $W_p^\varrho(\mathbb{R}^d)$ since

$$\left\|\frac{Q_hf-f}{h^\varrho}\right\|_p \le C|f|_{\varrho,p}.$$

Fix $f \in C_0^{\varrho+1}$. Let P_x be the Taylor polynomial of degree ϱ of f at x. By the triangle inequality (we take $F(x) = Q_h(P_x)(x) \neq P_x(x)$)

$$\left\|\frac{Q_hf-f}{h^\varrho}\right\|_p \le \left\|\frac{Q_h(f-P.)}{h^\varrho}\right\|_p + \left\|\frac{Q_hP.-P.}{h^\varrho}\right\|_p$$

and

$$\left\|\frac{Q_hP.-P.}{h^\varrho}\right\|_p \le \left\|\frac{Q_hf-f}{h^\varrho}\right\|_p + \left\|\frac{Q_h(f-P.)}{h^\varrho}\right\|_p.$$

If we prove that there is C such that for all $f \in C_0^{\varrho+1}$,

$$(9)\qquad \|Q_h(f-P.)\|_p \le Ch^{\varrho+1}|f|_{\varrho+1,p},$$

then the proof of (8) is completed by showing that

$$(10)\qquad \lim_{h\to 0^+}\left\|\frac{Q_hP.-P.}{h^\varrho}\right\|_p^p = \int_{\mathbb{R}^d}\bigg(\int_{[0,1]^d}\bigg|\sum_{|\beta|=\varrho}\frac{1}{\beta!}D^\beta f(t)(Q([]^\beta)(x)-x^\beta)\bigg|^p\,dx\bigg)\,dt.$$

The technical proof of (9) is postponed to Lemma 1.1. Let

$$P_x = T_x + R_x,$$

where T_x is homogeneous of degree ϱ and $\deg R_x < \varrho$. Since $Q(R_x) = R_x$ we have

$$(11)\qquad \frac{Q_h(P_x)(t)-P_x(t)}{h^\varrho} = \frac{Q_h(T_x)(t)-T_x(t)}{h^\varrho} = Q(T_x)(t/h) - T_x(t/h) = \sum_{|\beta|=\varrho}\frac{1}{\beta!}D^\beta f(x)(Q([]^\beta)(t/h)-(t/h)^\beta).$$

Consequently, from (11) we get

$$\left\|\frac{Q_h P_\cdot - P_\cdot}{h^\varrho}\right\|_p^p = \int_{\mathbb{R}^d} \Big| \sum_{|\beta|=\varrho} \frac{1}{\beta!} D^\beta f(x)(Q([]^\beta)(x/h) - (x/h)^\beta)\Big|^p \, dx.$$

An easy calculation shows (cf. [Dz3, Lemma 3.3]) that the functions

$$Q([]^\beta)(x) - x^\beta = (-1)^{|\beta|} \sum_{\alpha\in\mathbb{Z}^d} \int_{\mathbb{R}^d} (x-y)^\beta F(y-\alpha)\, dy\, G(x-\alpha)$$

are $\mathbb{Z}^d$-periodic. Now the Mazur–Orlicz Theorem (7) implies (10).

Lemma 1.1. *Let $1 \le p < \infty$. Let P_x be the Taylor polynomial of degree $k-1$ of a function f. There is C such that for all $f \in C_0^k$,*

$$\|Q_h(f - P_\cdot)\|_p \le Ch^k |f|_{k,p}. \tag{12}$$

Proof. By Taylor's formula,

$$\|Q_h(f-P_\cdot)\|_p^p = \int_{\mathbb{R}^d} \Big| \sum_{\alpha\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \int_0^1 \sum_{|\beta|=k} \frac{1}{\beta!} D^\beta f(x + s(hy-x))(1-s)^{k-1}\, ds \\ \times (hy-x)^\beta F(y-\alpha)\, dy\, G(x/h-\alpha)\Big|^p \, dx.$$

To prove (12), using assumption (1), it is sufficient to estimate

$$J_\beta = \int_{\mathbb{R}^d} \Big| \sum_{\alpha\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \int_0^1 |D^\beta f(x+s(hy-x))|\, ds \\ \times |hy-x|^k q^{|y-\alpha|}\, dy\, q^{|x/h-\alpha|}\Big|^p \, dx.$$

We apply Jensen's inequality three times:

$$\Big(\int_0^1 g(s)\, ds\Big)^p \le \int_0^1 |g(x)|^p dx,$$

$$\Big(\sum_{\alpha\in\mathbb{Z}^d} |a_\alpha| q^{|x-\alpha|}\Big)^p \le C_1 \sum_{\alpha\in\mathbb{Z}^d} |a_\alpha|^p q^{|x-\alpha|},$$

where C_1 is independent of x, i.e. $C_1 = \max_x (\sum_{\alpha\in\mathbb{Z}^d} q^{|x-\alpha|})^{p-1}$,

$$\Big(\int_{\mathbb{R}^d} |g(y)| q^{|y-\alpha|}\, dy\Big)^p \le C_2 \int_{\mathbb{R}^d} |g(y)|^p q^{|y-\alpha|}\, dy,$$

where C_2 is independent of α. Consequently,

$$J_\beta \le C \sum_{\alpha\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 |D^\beta f(x+s(hy-x))|^p\, ds \\ \times |hy-x|^{pk} q^{|y-\alpha|}\, dy\, q^{|x/h-\alpha|}\, dx.$$

Letting $x/h-\alpha=u$ yields

$$J_\beta \le Ch^d \sum_{\alpha\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 |D^\beta f(hu+h\alpha+s(hy-hu-h\alpha))|^p\,ds \times |hy-hu-h\alpha|^{pk} q^{|y-\alpha|}\,dy\,q^{|u|}\,du$$

and by obvious changes of variables

$$\begin{aligned} J_\beta &\le Ch^d \sum_{\alpha\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 |D^\beta f(hu+h\alpha+sh(z-u))|^p\,ds \\ &\quad\times h^{pk}|z-u|^{pk} q^{|z|}\,dz\,q^{|u|}\,du \\ &= Ch^{d+pk} \sum_{\alpha\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 |D^\beta f(hu+h\alpha+shv)|^p\,ds|v|^{pk} q^{|u+v|}\,dv\,q^{|u|}\,du. \end{aligned}$$

Let us split the integrals:

$$\begin{aligned} J_\beta &\le Ch^{d+pk} \sum_{n=1}^{\infty} \sum_{\alpha\in\mathbb{Z}^d} \int_{n-1<|u|<n} \int_{\mathbb{R}^d} \int_0^1 |D^\beta f(hu+h\alpha+shv)|^p\,ds \\ &\quad\times |v|^{pk} q^{|u+v|}\,dv\,q^{n-1}\,du \\ &\le Ch^{d+pk} \sum_{n=1}^{\infty} q^{n-1} \sum_{\alpha\in\mathbb{Z}^d} \int_{n-1<|u|<n} \sum_{j=1}^{\infty} q^{j-1} \\ &\quad\times \int_{j-1<|u+v|<j} \int_0^1 |D^\beta f(hu+h\alpha+shv)|^p\,ds\,|v|^{pk}\,dv\,du. \end{aligned}$$

Note that if $|v+u|<j$ and $|u|<n$ then

$$|v| < |v+u| + |u| < j+n.$$

Thus

$$\begin{aligned} J_\beta &\le Ch^{d+pk} \sum_{n=1}^{\infty} q^{n-1} \sum_{\alpha\in\mathbb{Z}^d} \int_{|u|<n} \sum_{j=1}^{\infty} q^{j-1} \\ &\quad\times \int_{|v|<j+n} \int_0^1 |D^\beta f(hu+h\alpha+shv)|^p\,ds\,|v|^{pk}\,dv\,du. \end{aligned}$$

Changing the order of the integrations we get

$$\begin{aligned} J_\beta &\le Ch^{d+pk} \sum_{n=1}^{\infty} q^{n-1} \sum_{j=1}^{\infty} q^{j-1} \\ &\quad\times \int_{|v|<j+n} \int_0^1 \sum_{\alpha\in\mathbb{Z}^d} \int_{|u|<n} |D^\beta f(hu+h\alpha+shv)|^p\,du\,ds\,|v|^{pk}\,dv. \end{aligned}$$

Note that if $|v| < j+n$ then

$$h^d \sum_{\alpha\in\mathbb{Z}^d} \int_{|u|<n} |D^\beta f(hu+h\alpha+shv)|^p\,du \le \sum_{\alpha\in\mathbb{Z}^d} \int_{|x-h\alpha|<h(2n+j)} |D^\beta f(x)|^p\,dx \le (4n+2j)^d \int_{\mathbb{R}^d} |D^\beta f(x)|^p\,dx$$

and moreover

$$\int_{|v|<j+n} |v|^{pk}\,dv = C(j+n)^{pk+d}.$$

Consequently,

$$J_\beta \le C\sum_{n=1}^{\infty} q^{n-1}h^{pk}\sum_{j=1}^{\infty} q^{j-1}(4n+2j)^d(j+n)^{pk+d}\int_{\mathbb{R}^d}|D^\beta f(x)|^p\,dx \le Ch^{pk}\int_{\mathbb{R}^d}|D^\beta f|^p.$$

This finishes the proof of the lemma. ∎

Let $X_1,\ldots,X_n$ be a random sample from a distribution with density $f\in W_2^\varrho$. We define a density estimator based on the kernel K by

(13) $$f_{h,n}(x) = \frac{1}{n}\sum_{j=1}^{n} K_h(x,X_j),$$

where

$$K_h(x,y) = (1/h)^d K(x/h,y/h).$$

Note that

(14) $$Ef_{h,n} = Q_h f.$$

As usual we consider the estimation error given by

(15) $$\mathrm{MISE}(f,h) = E\Big[\int_{\mathbb{R}^d}[f_{h,n}-f]^2\Big].$$

It is known that

(16) $$\mathrm{MISE}(f,h) = E\Big[\int_{\mathbb{R}^d}[f_{h,n}-Q_hf]^2\Big] + \int_{\mathbb{R}^d}[Q_hf-f]^2.$$

The asymptotic formula for the second factor in (16) is given in (8). We prove that

THEOREM 1.4. *Assume that Q has maximal polynomial order $\varrho>0$. If $nh^d\to\infty$, $h\to 0$ then*

(17) $$\lim_{nh^d\to\infty} nh^dE\Big[\int_{\mathbb{R}^d}[f_{h,n}-Q_hf]^2\Big] = \int_{\mathbb{R}^d}\Big[\int_{[0,1]^d}K^2(x,y)\,dy\Big]\,dx,$$

where

$$\int_{\mathbb{R}^d}\Big[\int_{[0,1]^d} K^2(x,y)\,dy\Big]\,dx = \sum_{\alpha\in\mathbb{Z}^d} \eta(\alpha)\xi(\alpha) \tag{18}$$

and

$$\eta = G * \breve{G},\qquad \xi = F * \breve{F},\qquad \breve{G}(x) = G(-x),\qquad \breve{F}(x) = F(-x).$$

Proof. Note that

$$\begin{aligned} E\Big[\int_{\mathbb{R}^d}[f_{h,n} - Ef_{h,n}]^2\Big] &= \frac{1}{n^2}\sum_{j=1}^{n}\int_{\mathbb{R}^d} E[K_h(x,X_j) - EK_h(x,X_j)]^2\,dx \\ &= \frac{1}{n^2}\sum_{j=1}^{n}\int_{\mathbb{R}^d}(E[K_h^2(x,X_j)] - [EK_h(x,X_j)]^2)\,dx. \end{aligned}$$

If $h \to 0$ then by (5),

$$\int_{\mathbb{R}^d}[EK_h(x,X_j)]^2\,dx = \int_{\mathbb{R}^d}(Q_h f)^2 \to \int_{\mathbb{R}^d} f^2.$$

On the other hand

$$\int_{\mathbb{R}^d} EK_h^2(x,X_j)\,dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_h^2(x,y)f(y)\,dy\,dx.$$

From Fubini's theorem

$$\int_{\mathbb{R}^d} EK_h^2(x,X_j)\,dx = \frac{1}{h^d}\int_{\mathbb{R}^d}\Big[\int_{\mathbb{R}^d} K^2(u,y/h)\,du\Big] f(y)\,dy.$$

Note that for all $\alpha\in\mathbb{Z}^d$,

$$\int_{\mathbb{R}^d} K^2(x,y+\alpha)\,dx = \int_{\mathbb{R}^d} K^2(x,y)\,dx. \tag{19}$$

From Mazur–Orlicz's theorem we get

$$\lim_{h\to 0}\int_{\mathbb{R}^d}\Big[\int_{\mathbb{R}^d} K^2(u,y/h)\,du\Big] f(y)\,dy = \int_{[0,1]^d}\Big[\int_{\mathbb{R}^d} K^2(u,y)\,du\Big]\,dy\int_{\mathbb{R}^d} f(y)\,dy.$$

We thus get (17). A simple calculation leads to (18). ∎

REMARKS. 1. From (16)–(8) we get

$$\begin{aligned} \mathrm{MISE}(f,h) \sim \mathrm{AMISE} &:= \frac{1}{nh^d}\int_{\mathbb{R}^d}\Big[\int_{[0,1]^d} K^2(x,y)\,dy\Big]\,dx \\ &\quad + h^{2\varrho}\int_{\mathbb{R}^d}\Big(\int_{[0,1]^d}\Big|\sum_{|\beta|=\varrho}\frac{1}{\beta!}D^\beta f(t)Q([]^\beta)(x) - x^\beta\Big|^2\,dx\Big)\,dt. \end{aligned}$$

So the best choice of $h > 0$ which minimizes (16) is

$$h \sim n^{-1/(2\varrho+d)}.$$

2. Using the methods of [Dz2] one can prove the central limit theorem. This theorem generalizes the results for wavelet estimators [DL1]–[DL2] in $\mathbb{R}^d$ and box spline estimators [Dz2]. These results are motivated by the result for the Rosenblatt–Parzen estimator [H].

References

[BD1] M. Beśka and K. Dziedziul, *The saturation theorem for interpolation and the Bernstein–Schnabl operator*, Math. Comp. 70 (2001), 705–717.

[BD2] —, —, *The saturation theorem for orthogonal projection*, in: *Advances in Multivariate Approximation*, W. Haussmann *et al.* (eds.), Wiley, 1999, 73–83.

[BD3] —, —, *Asymptotic formula for the error in cardinal interpolation*, Numer. Math. 89 (2001), 445–456.

[BD4] —, —, *Asymptotic formula for the error in orthogonal projection*, Math. Nachr. 233 (2002), 47–53.

[BDR] C. de Boor, R. A. DeVore and A. Ron, *Approximation from shift-invariant subspaces of* $L_2(\mathbb{R}^d)$, Trans. Amer. Math. Soc. 341 (1994), 787–806.

[BHR] C. de Boor, K. Höllig and S. Riemenschneider, *Box Splines*, Springer, 1993.

[C] Z. Ciesielski, *Asymptotic nonparametric spline density estimation in several variables*, in: Internat. Ser. Numer. Math. 94, Birkhäuser, Basel, 1990, 25–53.

[DU] I. Daubechies and M. Unser, *On the approximation power of convolution-based least squares versus interpolation*, IEEE Trans. Signal Process. 45 (1997), 1697–1711.

[DL1] P. Doukhan and J. R. León, *Quadratic deviation of projection density estimates*, Rebrape 7 (1993), 37–63.

[DL2] —, —, *Déviation quadratique d'estiimateurs de densité par projections orthogonales*, C. R. Acad. Sci. Paris Sér. I Math. 310 (1990), 425–430.

[DLP] P. Doukhan, J. Léon et F. Portal, *Une mesure de la déviation quadratique d'estimateurs non paramétriques*, Ann. Inst. H. Poincaré Probab. Statist. 22 (1986), 37–66.

[Dz1] K. Dziedziul, *The saturation theorem for quasi-projections*, Studia Sci. Math. Hungar. 35 (1999), 99–111.

[Dz2] —, *Central limit theorem for square error of multivariate nonparametric box spline density estimators*, Appl. Math. (Warsaw) 28 (2001), 437–456.

[Dz3] —, *Asymptotic formulas in cardinal interpolation and orthogonal projection*, in: *Recent Progress in Multivariate Approximation*, W. Haussmann *et al.* (eds.), Internat. Ser. Numer. Math. 137, Birkhäuser, Basel, 2001, 139–157.

[Dz4] —, *Box Splines*, Wyd. Politechniki Gdańskiej, 1997 (in Polish).

[H] P. Hall, *Central limit theorem for integrated square error of multivariate nonparametric density estimators*, J. Multivariate Anal. 14 (1984), 1–16.

[JZ] K. Jetter and D.-X. Zhou, *Order of linear approximation from shift-invariant spaces*, Constr. Approx. 11 (1995), 423–438.

[MO] S. Mazur et W. Orlicz, *Sur quelques propriétés de fonctions périodiques*, Studia Math. 9 (1940), 1–16.

[M] Y. Meyer, *Wavelets and Operators*, Cambridge Univ. Press, Cambridge, 1992.

[SF] G. Strang and G. Fix, *A Fourier analysis of the finite element variational method*, in: Constructive Aspects of Functional Analysis, G. Geymonat (ed.), Ed. Cremonese, Roma, 1973, 793–840.

Faculty of Applied Mathematics
Technical University of Gdańsk
G. Narutowicza 11/2
80-952 Gdańsk, Poland
E-mail: kdz@mifgate.pg.gda.pl

Received on 5.11.2001;
revised version on 22.3.2002 (1600)