Received: 7 April 2022 | Revised: 23 March 2023 | Accepted: 6 May 2023

DOI: 10.33012/navi.601

ORIGINAL ARTICLE

Asynchronous Method of Simultaneous Object Position and Orientation Estimation with Two Transmitters

Jacek Stefanski | **Jaroslaw Sadowski**

Faculty of Electronics, Telecommunications and Informatics, Gdansk University of Technology, Gdansk, Poland

Correspondence
Jacek Stefanski
Email: jstef@eti.pg.edu.pl

Abstract

This paper proposes an object location method for all types of applications, including the Internet of Things. The proposed method enables estimations of the position and orientation of an object on a plane or in space, especially during motion, by means of location signals transmitted simultaneously from two transmitters placed on the object at a known distance from each other. A mathematical analysis of the proposed method and Newton's algorithm for solving the system of nonlinear positional equations is presented. Next, an analysis of a position-dilution-of-precision parameter for the proposed method and a Cramer–Rao lower bound, limiting the accuracy of the method, is presented. Finally, the results of complex simulation studies on the efficiency of the proposed method are described.

Keywords

adoa, asynchronous method, radio navigation, wireless sensor networks

1 | INTRODUCTION

The majority of position estimation systems rely on some type of time synchronization between system elements, e.g., synchronous emission of signals from base stations (BSs) or synchronous detection of signals received from mobile nodes. Few centers in the world deal or have dealt with asynchronous radiolocation methods. At the time of writing of this paper, only a dozen items have been reported in the available literature that, according to the authors, constitute representative comparative material related to asynchronous radiolocation methods applicable to radio communication networks.[1] A brief description of each solution is presented below, but it is worth mentioning that, at this point, these solutions usually concern two-dimensional (2D) cases. In the described solutions, it is also worth noting the different understandings of the concept of asynchronous systems by different authors.

The analyzed asynchronous solutions can be divided into two main groups. The first group includes those based on the transmission of location data in only one

[1]The literature review was carried out with a consideration of the various methods for estimating object positions, the original applications of which were not necessarily related to typically radio solutions. However, as a result of the conducted analyses, it was found that the methods described in this section, intended for systems such as acoustic systems, have great implementation potential in radiolocation systems.

direction, i.e., from the mobile terminal to the network of reference stations with known locations. The second group of solutions includes systems in which, for the purpose of localization, data transmission takes place in two directions. Each of these groups has advantages and disadvantages depending on the application. For example, for the purpose of implementing systems that monitor only the position of moving objects, unquestionable advantages are shown by the solutions from the first group, based on transmission from the object to reference stations. In these solutions, the flow of information in the network is minimized. In contrast, systems based on bidirectional transmission are widely used in commercial solutions that, apart from transmitting signals related to the location process, transmit other useful data, and their sources are located on both sides of the radio communication link.

The first group includes the solutions described, for example, by Li et al. (2004), Vaghefi and Buehrer (2013), and Wang and Leus (2012). In the location system proposed by Li et al. (2004), a finite number of autonomous sensors with known locations are deployed. Generators used to carry out measurements in sensors are not synchronized with each other. The object moving in this area transmits signals at a repetition frequency known on the receiving side, on the basis of which a network of independent sensors estimates the object's location. This system allows many objects to be tracked at the same time, provided that the sensor network is able to distinguish signals from different objects. This method is based on the assumption that the object is in motion, while the estimation of its location is based on the observation of two successive transmitted signals. The disadvantage of this solution is that it determines the object position when the first reference signal is transmitted, therefore introducing some delay between when the measurements are acquired and the measurement results are returned. Thus, the resultant error of the current object position estimation is influenced by the repetition frequency at which location signals are transmitted, which can be compensated to some extent on the receiving side by predicting the direction of the object's movement and knowing its speed of movement. This method can be used to successfully track the path traveled by a moving object.

The radiolocation systems described by Wang and Leus (2012) and Vaghefi and Buehrer (2013) consist of many reference nodes (anchors) with known coordinates and one node with an unknown location. In these systems, it is also assumed that the reference nodes with receivers work synchronously, while the asynchrony is caused by an unknown moment of location signal transmission by a node with an unknown location. The time of arrival (ToA) or time difference of arrival (TDoA) method was used to estimate the location of this node, and the semidefinite programming (SDP) estimator, developed at the beginning of the 1990s and described, e.g., by Vandenberghe and Boyd (1996), was used in the solution proposed by Vaghefi and Buehrer (2013) to estimate the moments of location signal transmission. In contrast to the optimal maximum likelihood (ML) estimator, the suboptimal SDP estimator has no convergence problems, but gives slightly worse results than the ML estimator. Wang et al. (2011) showed that the use of the modified least square (LS) estimator, which does not distinguish the reference node in the process of determining the object position, gives better results than the classic LS approach.

The second group of asynchronous radiolocation systems is also widely represented in the literature. For example, the radiolocation system described by Xiong et al. (2015) consists of two types of nodes: reference (anchor) nodes with known positions, which are connected to the network infrastructure from which they receive energy, and mobile (tag) modes with unknown positions, the coordinates of which are estimated. All reference nodes receive location signals broadcasted by other reference and mobile nodes. Information about the measured ToA is sent to the so-called location server, which estimates the location of mobile nodes based on the received data. In addition, it is assumed that the mobile nodes only periodically transmit location signals but, in order to save energy, do not take part in computationally advanced digital signal processing and do not perform ToA measurements. The process of estimating the location of mobile nodes in this system is carried out in two stages. First, the localization server synchronizes cyclic measurements made by pairs of reference nodes using the classical LS estimator and the reception of signals transmitted by other reference nodes with known coordinates. In the second step, based on the location signals received from the mobile nodes, after the compensation of the clock offsets between the pairs of reference nodes participating in the TDoA measurement, the positions of the mobile nodes are estimated. Because all computationally complex procedures are transferred to the location server, the proposed radiolocation system enables the construction of an energy-saving wireless sensor network. When designing the infrastructure of such a network, one should carefully plan the distribution of reference nodes to enable as many measurements as possible between the reference nodes and mobile nodes under the propagation conditions.

The asynchronous radiolocation system proposed by Wang et al. (2011) is based on the two-way ranging (TWR) algorithm, which is described in detail in the specification of the IEEE 802.15.4a standard (currently replaced by IEEE 802.15.4-2020 (IEEE Std, 2020)) and subjected to measurement verification in the pilot version of the ad hoc sensor network by Sathyan et al. (2011). In the literature, e.g., in the paper presented by Youssef et al. (2006), one can also find a variant of this algorithm, which is based on the symmetric double-sided TWR distance measurement. This measurement leads to a significant reduction in the impact of the error frequency of quartz resonators on the quality of distance estimation between devices, at the cost of twice as many transmitted measurement packages. The original solution of the radiolocation system based on the TWR algorithm assumes that each reference node (anchor) with known coordinates performs a bidirectional distance measurement with the located object (sensor). This solution also assumes that during the exchange of measurement packets between a given anchor–sensor pair, the remaining reference nodes are idle. Thus, for the 2D case, at least three reference stations, i.e., three TWR measurements, are required to unambiguously determine the sensor position, which involves the transmission of at least six measurement packets (and some control packets for data gathering for position estimation). To use the broadcast transmission more effectively in the network described above, Wang et al. (2011) proposed that the reference nodes that are not currently participating in TWR measurements receive asymmetric trip ranking measurement packets. With this modification, the required amount of measurement data can be collected more rapidly (reducing the number of transmitted measurement packets in the network) for a correct estimation of the sensor location, while maintaining the same accuracy. The analysis of this solution shows that a significant reduction in the number of network measurement packets necessary for estimating the sensor location is possible when the number of reference stations is greater than five.

The works published by Choi et al. (2013) and Kim (2009) concern protocol solutions in asynchronous radiolocation networks using the TWR algorithm. There are two types of nodes in the described networks: the so-called readers and tags. Owing to the use of specialized protocol solutions aimed at network structures, in which control functions are supervised by tags, the number of transmitted packets was reduced by 20%–45% compared with the classic solution.

Nawaz et al. (2017) presented a prototype asynchronous network for radiolocation needs in an indoor localization system environment. The constructed network consisted of three unsynchronized receivers and a so-called calibration transmitter, the main task of which was to transmit signals that allowed for the compensation of errors of clocks located in the receivers. This system enabled the position estimation of an object moving on a plane that cyclically emitted location signals.

In the solution developed by Zachariah et al. (2014), the radiolocation network contains two types of nodes. The first group of nodes is equipped with transceiver devices, while the second group possesses only receivers. The transceiver nodes transmit reference signals derived from their own clock, which are received by all other nodes. The received reference signals are then retransmitted with the shortest possible delay. The role of the receiving nodes is to estimate their position and the position of the transceiver nodes based on the received signals; it is assumed that only some of the transceiver nodes know their coordinates. Therefore, this approach is a modification of the TWR method in an asynchronous network in which the signal delay before retransmission for distance measurements is estimated by other reference nodes. The proposed scenario for locating nodes in the network was subjected to simulation tests that considered typical phenomena occurring in the impulse radio ultra-wideband (UWB) interface. The approximate maximum a posteriori algorithm was used to estimate the location of nodes with unknown coordinates.

At the end of this literature review on asynchronous localization methods, it is worth mentioning the Ph.D. dissertation by He (2016), which is entirely devoted to this subject. The author of the dissertation focused on the analysis of his proposed algorithms in a network consisting of many nodes with known positions (anchors) and nodes whose coordinates are sought. To verify the effectiveness of the proposed asynchronous radiolocation methods, a demonstrator was built, which consisted of nodes with known locations equipped with transmitters or receivers, while the located node had a transceiver unit. The constructed demonstrator used the UWB radio interface to transmit localization pulses. The conducted measurements showed that the proposed algorithms for estimating node locations always lead to a global solution and are characterized by relatively high accuracy.

This paper presents an innovative radiolocation method based on simultaneous transmission of signals from two transmitters placed at a known distance from each other on a localized object (e.g., ends of plane wings, bow and stern of a ship). The signals are received by several reference receivers working asynchronously. In the proposed system, the coordinates of both transmitters are estimated at the same time, which simultaneously gives the position and orientation of the localized object. An exemplary scenario in which two-transmitter-based asynchronous positioning may be used is the navigation and supervision of ships in port canals or planes on the ground, e.g., on airport runways or aprons. In both cases, both the vehicle position and orientation are important. The majority of radio-based positioning and navigation solutions can estimate only the direction of object movement, which may be different than vehicle orientation due to crosswinds in aviation or both wind and sea currents in marine navigation. For transmitters mounted on plane wings or the ship bow/stern, the dimensions of vehicles may be comparable to 0.01—0.1 times the area of the possible distribution of ground BSs; thus, this relation of transmitter separation to ground station distribution will be used in this paper.

This paper is organized as follows: Section 2 describes the proposed method. In Section 3, a mathematical analysis of the method is presented, and Newton's algorithm (NA), dedicated to solving the system of nonlinear positional equations for the method, is described. The next two sections present the position dilution of precision (PDoP) and the Cramér–Rao lower bound (CRLB) for the proposed method. Section 6 presents the results of simulation studies of the effectiveness of the proposed location method. Finally, the last section presents the conclusions of this work.

2 | DESCRIPTION OF THE PROPOSED METHOD

Consider a 2D case in which two synchronized transmitters are placed on a localized object (mobile station [MS]). The transmitters are located at a known distance d from each other and simultaneously transmit localization signals. These signals are received by the BSs (reference stations). By using unique identifiers in each transmitter, the individual BSs can detect both signals individually and determine the difference in distance between both transmitters and receiving antennas. This principle of system operation is described in the patent application by Sadowski and Stefanski (2021).

The proposed method is passive, based on the reception of signals transmitted by a localized MS. The general structure of a radio sensor network in which this method can be implemented is shown in Figure 1. It should be noted that the reverse system structure is also possible. The reception of signals from ground BS transmitters, working asynchronously, by two synchronized receivers using two antennas placed on the localized object (vehicle) will give the same set of position estimation equations. Therefore, the decision regarding system structure should consider the purpose of position data collection and the location for its use.

Based on Figure 1, a system of nonlinear equations can be written in which the MS coordinates (x_1, y_1) and (x_2, y_2) are unknown:

$$\begin{cases} d_{n2} - d_{n1} = \Delta t_n \cdot v = \Delta d_n \quad n = 1,\ldots,N \\ d^2 = \left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 \end{cases} \tag{1}$$

where Δt_n is the time difference between the reception of signals from both transmitters on the MS at the BS, v represents the propagation speed of radio signals

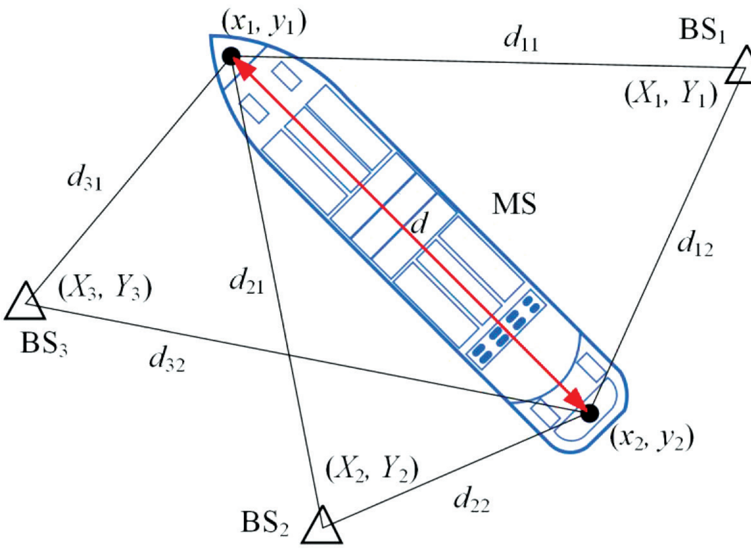


FIGURE 1 Example of a sensor network structure in which the proposed method can be implemented

in a propagation medium (approximately assumed to equal the speed of light in a vacuum, $c \approx 3 \cdot 10^8$ m/s), and N is the number of BSs (in our case, $N = 3$, which is minimal number of reference receivers for a 2D case). Moreover, d_{n1} and d_{n2} can be described by the following relations:

$$\begin{cases} d_{n1} = \sqrt{\left(X_n - x_1\right)^2 + \left(Y_n - y_1\right)^2} \\ d_{n2} = \sqrt{\left(X_n - x_2\right)^2 + \left(Y_n - y_2\right)^2} \end{cases} \tag{2}$$

where (X_n, Y_n) are the BS coordinates. It should be emphasized that the coordinates of both transmitters (x_1, y_1) and (x_2, y_2) are estimated simultaneously, but these coordinates are not independent, as the distance between transmitters is fixed. Therefore, two sets of coordinates for two points on a single object simultaneously give the position of the object, which may be understood as the position of the first transmitter, second transmitter, or any geometric relation between the coordinates of the transmitters and the tracked object (e.g., position at the midpoint between both transmitters), and the orientation of the object in relation to the set of reference receivers. An advantage of the proposed system is that it is possible to estimate the orientation of both stationary and mobile objects, while systems based on single devices attached to localized objects can estimate the direction of movement but are not able to measure the orientation of motionless objects and may introduce orientation errors, e.g., in the case of a difference between plane heading and course in the presence of a crosswind. Equation (1) and Equation (2) show that for the 2D case, three BSs are sufficient for MS position estimation, while for the three-dimensional (3D) case, the minimum number of BSs is five. For obvious reasons, increasing the number of BSs involved in MS position determination for the 2D or 3D case increases the accuracy in calculating the coordinates of the localized object. At this point, it should be emphasized that the analyzed radiolocation system is a special case of an angular system, called an angle difference of arrival (ADoA) system, because the results of distance difference measurements taken by the receiver are directly related to the cosine of the angle between a line passing through both transmitters (object orientation axis) and the direction to the receiver. However, the ambiguity of the cosine function defines two possible angles (α and α' in Figure 2): the correct angle (corresponding to the actual BS coordinates, X_n and Y_n) and a mirrored angle (corresponding to invalid coordinates, X_n' and Y_n'). The well-known algorithms of position estimation for ADoA systems cannot be directly applied without first removing this angle ambiguity. Therefore, this method of position estimation will not be further considered, as the algorithm proposed in Section 3 gives unambiguous position estimation.

The possibility of estimating the coordinates of both transmitters using Equation (1) and the iterative algorithm that will be presented in detail in Section 3 was verified by simulations of the 2D case with different numbers of BSs distributed uniformly on a circle of radius R. For the selected number of BSs, the MS position and orientation were randomly selected, and the number of incorrectly determined positions was recorded in the solution search process. During the simulations, it was assumed that transmitters on the MS are separated by $0.1R$; however, as no measurement errors were introduced in this convergence test, the distance between transmitters had no impact on the probability of convergence of the results to the correct MS coordinates, with some impact only on the number of iterations needed to obtain a given accuracy of position estimation. To simplify the scaling of results, the distance between transmitters will be expressed as a fraction of the radius R, which determines the size of area surrounded by the BSs. For the same reason, the errors of distance difference measurements Δd_n will be modeled in Section 6 with a normal distribution random variable with variation expressed as a fraction of R.

Tests were performed for one million cases for each BS configuration. Initially, the starting point for the first iteration of position estimation was fixed in the center of the area surrounded by BSs, with a constant object orientation defined by transmitter coordinates (0; 0) and (0.1R; 0). This fixed starting point should not be considered as a limitation of the proposed method, but only as an example selected for simulation. The MS position can be arbitrary, either inside or outside the circle of the BSs, and the starting point for iterative position estimation can also be chosen arbitrarily. The obtained results are shown in Table 1 in the column marked as "One initial position with fixed orientation."

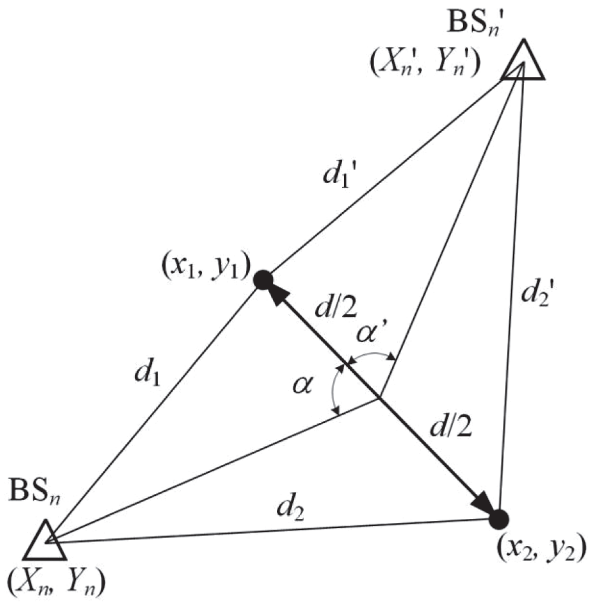


FIGURE 2 The ADoA case in the system under consideration

TABLE 1
Percentage of Incorrect Solutions as a Function of BS Number

Number of base stations	Percentage of incorrect solutions	
	One initial position with fixed orientation	One initial position with variable orientation
3	34.9	28.2
4	---	17.4
5	13.5	1.69
6	12.1	4.57
7	10.0	0.41
8	8.6	0.83
9	7.7	0.064
10	7.2	0.39
11	6.0	0.038
12	5.8	0.22

As expected, as the BS number increases, the percentage of cases with incorrect results (algorithm divergence or convergence to wrong coordinates) decreases. A very interesting case was observed for four BSs. For the assumptions made in this case, the BSs are located at the vertices of a square, which leads to the impossibility of inverting the Jacobian matrix when estimating the object position in practically all cases considered, as long as the starting point for the first iteration is defined as mentioned above (both transmitters and two BSs in one line). However, iterative algorithms are known to be sensitive to the choice of initial conditions; in our case, this sensitivity is primarily connected to the low probability of convergence when the true object orientation differs greatly from the initial assumptions. Taking this into account, convergence tests were repeated with a fixed initial position in the center of the area but with variable orientations of the transmitter pair. For the case of N BSs, the iterative algorithm from Section 3 was run N times with N different orientations of the starting point calculated in such a way that the initial positions of both transmitters were not in line with any BS. From the set of N results from N repetitions of position estimation, the best match was selected as the match with the lowest difference between the measured and estimated distance differences (Equation (1)). This approach not only enabled convergence in the scenario with four BSs but also significantly reduced the probability of incorrect results, as presented in the column marked as "One initial position with variable orientation" in Table 1. By analyzing the data in Table 1, one may observe that the results obtained for an odd number of BSs are generally more promising than those for an even number of ground receivers. This finding may indicate that the performance of the iterative position estimation algorithm is influenced by not only the starting point, but also the symmetry of BS deployment. Both of these aspects may be investigated in future works, but the scope of this paper is to illustrate the system structure and prove that position estimation using two transmitters is possible. The algorithm proposed in Section 3 is simply an example of one method, but not the only method, for solving Equation (1).

A comprehensive analysis of possible methods for solving the system in Equation (1), as well as the results obtained, is presented in the following sections.

3 | CALCULATING THE OJBECT POSITION

The system in Equation (1) can be used to estimate the MS position, i.e., the coordinates of both transmitting antennas (x_1, y_1) and (x_2, y_2), which define the position and orientation of a mobile object at the same time, assuming that the following are known:

- BS coordinates,
- distance differences Δd_n between the two transmitters and BSs, and
- distance d between transmitters.

In the literature on this subject, it is difficult to find direct algorithms that lead to the solution of systems of nonlinear equations described by Equation (1). For the purposes of this study, the generalized NA approach used by Foy (1976) in his algorithm, has been adapted to solve the problem raised in this paper. At this point, it should be clearly emphasized that only one algorithm was used to solve the system in Equation (1), because the main goal of this paper is to present a new localization method, rather than to optimize methods for solving this system of equations.

Assuming the simplest 2D case with $N = 3$, the problem of MS position estimation in the presented method requires the solution of a system of nonlinear equations in the following form:

$$\mathrm{f}(x_1, y_1, x_2, y_2) = \begin{cases} d_{12} - d_{11} = \Delta d_1 \\ d_{22} - d_{21} = \Delta d_2 \\ d_{32} - d_{31} = \Delta d_3 \\ d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \end{cases} \tag{3}$$

where the unknowns are the coordinates of the transmitters (x_1, y_1) and (x_2, y_2) in the MS. The linearized version of the above system of equations can be written as follows:

$$\mathbf{J}(x_1, y_1, x_2, y_2) \cdot \mathbf{h} = -\mathbf{f}(x_1, y_1, x_2, y_2) \tag{4}$$

where $\mathbf{J}(x_1, y_1, x_2, y_2)$ is the Jacobian matrix described by Equation (5) and vector $\mathbf{h} = [h_{x_1}, h_{y_1}, h_{x_2}, h_{y_2}]^T$ represents the correction of the transmitter coordinates in subsequent iterations:

$$\mathbf{J}(x_1, y_1, x_2, y_2) = \begin{bmatrix} \frac{x_1 - X_1}{d_{11}} & \frac{y_1 - Y_1}{d_{11}} & \frac{X_1 - x_2}{d_{12}} & \frac{Y_1 - y_2}{d_{12}} \\ \frac{x_1 - X_2}{d_{21}} & \frac{y_1 - Y_2}{d_{21}} & \frac{X_2 - x_2}{d_{22}} & \frac{Y_2 - y_2}{d_{22}} \\ \frac{x_1 - X_3}{d_{31}} & \frac{y_1 - Y_3}{d_{31}} & \frac{X_3 - x_2}{d_{32}} & \frac{Y_3 - y_2}{d_{32}} \\ \frac{x_2 - x_1}{d} & \frac{y_2 - y_1}{d} & \frac{x_1 - x_2}{d} & \frac{y_1 - y_2}{d} \end{bmatrix} \tag{5}$$

The algorithm used to estimate the MS position has been implemented in MATLAB.

4 | PDoP FOR THE PROPOSED METHOD

The accuracy of the position estimation depends on the geometrical distribution of the reference stations (BSs) in relation to the localized object. The accuracy is often characterized by the PDoP parameter, which indicates the influence of measurement errors on the position estimation, as clearly presented by Tsui (2000). A higher PDoP value indicates that a lower accuracy of the position estimation in the radiolocation system is obtained despite maintaining a constant level of accuracy in radio signal parameter measurements. The PDoP coefficient can be determined from the following relation, which can be found, e.g., in Bard and Ham (1999) and Shin and Sung (2002):

$$P_D = \sqrt{tr[(\mathbf{J}^T \mathbf{J})^{-1}]} \tag{6}$$

where $\mathbf{J}$ is the Jacobian matrix described by Equation (5) and $tr[\bullet]$ is the trace of the matrix. To analyze the distribution of PDoP coefficient values for the developed method, the model proposed in the previous section was adopted, i.e., BSs were distributed uniformly on a circle of radius R. For the randomly generated coordinates of the first transmitter, an azimuth was randomly selected on which the second transmitter was placed at a given distance. Example results from the numerical calculations are shown in Figure 3, where five BSs are assumed and the distance between transmitters is $0.01R$ (Figure 3(a)), $0.03R$ (Figure 3(b)), or $0.1R$ (Figure 3(c)). We decided not to average the PDoP calculation results for different azimuths to the second transmitter to show the possible dispersion of position estimation accuracy caused by the variable orientation of the mobile object. Therefore, the dominant color of the dots in Figure 3 corresponds to the average PDoP, while the color variation in a selected area shows the dispersion of the position estimation accuracy in this area, which is due to the dependence of PDoP on the MS orientation. A lower PDoP value corresponds to a more advantageous distribution of the reference stations because it translates into a smaller MS position error.

In the area surrounded by reference stations, the PDoP coefficient for the considered method (for five BSs) changes from a few to several dozen, and as expected, the PDoP in all areas strongly depends on the distance between MS transmitters: a lower distance (smaller vehicle dimension) increases PDoP and decreases the possible accuracy of position estimation. This trend can be considered the most substantial disadvantage of our system, because very high PDoP values impose high demands on the precision of the signal propagation time difference measurements, much higher than in synchronous positioning systems based on the ToA or TDoA principle in similar BS deployment scenarios.

A reduction in accuracy outside the area defined by the reference stations, as clearly shown in all maps in Figure 3, is expected for many positioning systems. However, for the case of two-transmitter asynchronous positioning, especially high degradation of position estimation accuracy is observed near the circle at which all BSs are deployed. This effect is typical in systems based on the ADoA positioning method, in which regions of ambiguity are defined by circles with any two BSs taking part in position estimation and the MS position. Therefore, an even deployment of all BSs on a circle is advantageous when the MS is close to center of the circle; however, it is almost impossible to estimate the position of an MS close to the circle in this scenario. To avoid such problems, an uneven distribution of BSs should be used; in particular, no three BSs should be placed on the same circle or on the same line.

A distribution of BSs around the area of system operation may be a good positioning model for aircraft at an airport or ships on inland canals or bays deeply cut into the land. However, when all BSs must be installed on one side of the supervised area, e.g., in the case of marine navigation near a straight stretch of seashore, a PDoP increase will likely be observed, similar to that of typical ToA/TDoA systems in such conditions. However, as the PDoP value in our system depends not only on the MS position but also on the transmitter orientation, an optimal selection of BS locations should account for the trajectory of MS movement in the supervised area. A comprehensive analysis of optimal BS deployment for the proposed system may be pursued in future work.

5 | CRAMER–RAO LOWER BOUND

The CRLB parameter describes the potential effectiveness of the radiolocation system. This parameter defines the minimum mean square error of an unbiased estimator, as presented by McDonough and Whalen (1995), which, in our case, is a solution of a system of equations describing the applied localization method. For an unbiased case, let us assume that $\boldsymbol{\theta}$ is an unknown deterministic vector (in our case, a vector of estimated distance differences containing estimated

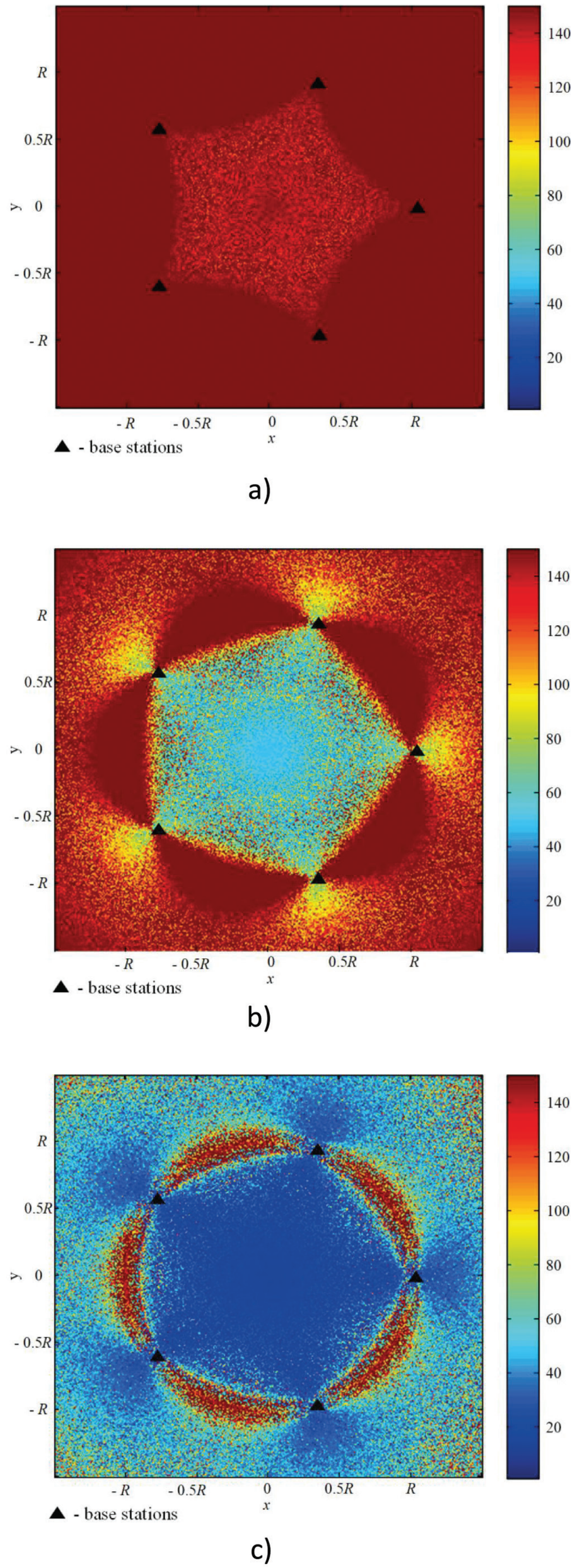


FIGURE 3 Distribution of PDoP values for five BSs: a) MS transmitter spacing of 0.01*R*, b) MS transmitter spacing of 0.03*R*, c) MS transmitter spacing of 0.1*R*

coordinates x_1, y_1, x_2, y_2), which is estimated using the $\boldsymbol{\rho}$ observation vector (in our case, a vector whose elements represent distance differences Δd_n) from the probability distribution of the density $p(\boldsymbol{\rho};\boldsymbol{\theta})$. In this scenario, the variance of any unbiased $\hat{\boldsymbol{\theta}}$ estimator of vector $\boldsymbol{\theta}$ is limited from the bottom by the inverse of the Fisher information matrix:

$$\mathrm{var}\left(\hat{\boldsymbol{\theta}}\right) \geq \frac{1}{\mathbf{I}\left(\boldsymbol{\theta}\right)} \tag{7}$$

where:

$$\mathbf{I}\left(\boldsymbol{\theta}\right) = -E\left[\frac{\partial^2 \ln p(\boldsymbol{\rho};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^2}\right] \tag{8}$$

and $E[\bullet]$ is the expected value. More information on this topic can be found in Kay (1993). In radiolocation systems with a Gaussian error distribution of measurements, $p(\boldsymbol{\rho};\boldsymbol{\theta})$ takes the following form:

$$p(\boldsymbol{\rho};\boldsymbol{\theta}) = -\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{1}{2\sigma^2}(\boldsymbol{\rho}-\boldsymbol{\theta})^2\right] \tag{9}$$

where σ is the standard deviation of the measurements.

Equation (1) is the starting point for deriving the CRLB for the proposed method. By accounting for Equation (9) and the considerations described by Chang and Sahai (2006), the probability density function $p(\boldsymbol{\rho};\boldsymbol{\theta})$ in 2D space can be described by the following equation:

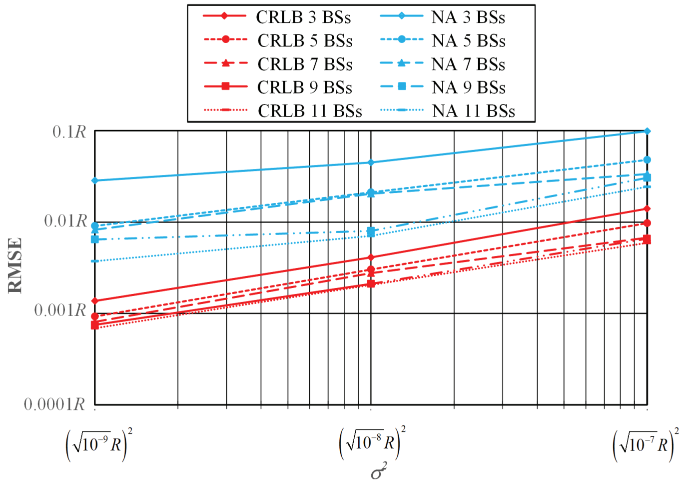
$$\begin{aligned} p(\boldsymbol{\rho};\boldsymbol{\theta}) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\Bigg\{-\frac{1}{2\sigma^2}\sum_{n=1}^{N}\Bigg[\Delta d_n - \Bigg(\sqrt{(X_n-x_2)^2+(Y_n-y_2)^2} \\ -\sqrt{(X_n-x_1)^2+(Y_n-y_1)^2}\Bigg)\Bigg]^2\Bigg\} \end{aligned} \tag{10}$$

The Fisher information matrix takes the following form:

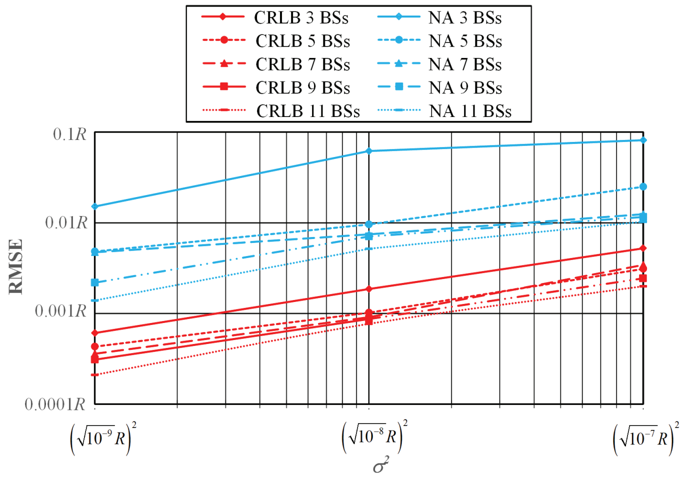
$$\mathbf{I}\left(\boldsymbol{\theta}\right) = \begin{bmatrix} -E\left[\frac{\partial^2 \ln p_1(\boldsymbol{\rho};\boldsymbol{\theta})}{\partial x_1^2}\right] & \cdots & -E\left[\frac{\partial^2 \ln p_1(\boldsymbol{\rho};\boldsymbol{\theta})}{\partial x_1 \partial y_2}\right] \\ \vdots & \ddots & \vdots \\ -E\left[\frac{\partial^2 \ln p_2(\boldsymbol{\rho};\boldsymbol{\theta})}{\partial y_2 \partial x_1}\right] & \cdots & -E\left[\frac{\partial^2 \ln p_2(\boldsymbol{\rho};\boldsymbol{\theta})}{\partial y_2^2}\right] \end{bmatrix} \tag{11}$$

where the individual elements of the matrix are presented in Appendix A.

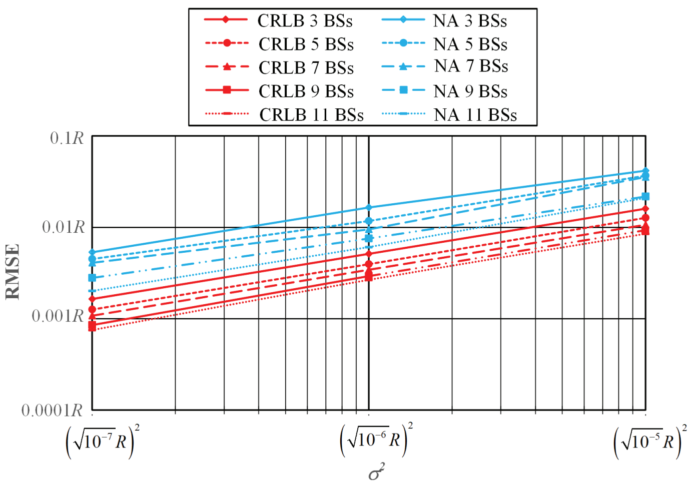
The above relationships were applied to perform numerical calculations and simulation tests. The results are presented in Figure 4 for three different values of distance between the transmitters: 0.01R (Figure 4(a)), 0.03R (Figure 4(b)), and 0.1R (Figure 4(c)). Numerical calculations and simulation studies were conducted to determine the average root mean squared error (RMSE) in the function of variance of errors in distance difference measurements between the transmitters and BS. The RMSE error of the localized object was determined on the basis of CRLB analysis and was also calculated from NA. The distances between transmitters,



a)



b)



c)

FIGURE 4 RMSE of MS position estimation in the function of the variance of distance difference measurement errors (σ^2) for NA and CRLB for different numbers of BSs: a) MS transmitter spacing of 0.01R, b) MS transmitter spacing of 0.03R, c) MS transmitter spacing of 0.1R

position estimation errors, and variance of distance measurement errors were related to the dimension R in the adopted model of the simulation environment. The individual axes of the graphs in Figure 4 have been purposefully scaled to the size (R) of the research area. This approach enables easy scaling of the obtained results to the size of various areas, where the accuracy of the distance difference measurements is a given percentage of the radius. For example, for a circle-shaped area with $R = 1{,}000$ m, the variance $(0.001R)^2$ represents 1 m^2, i.e., the standard deviation of errors in the measurement of distance differences represents 0.1% of the radius. When analyzing Figure 4, it should be noted that the error ranges of the distance difference measurements are not the same in all charts, because a higher measurement accuracy is required for a smaller transmitter spacing to obtain a comparable quality of position estimation. In practical implementations, the distance difference measurement errors should be kept at least two orders of magnitude smaller than the distance d between transmitters to achieve reasonable position estimation accuracy; smaller measurement errors translate to better system accuracy. Therefore, the error variance ranges in Figure 4 have been selected in such a way as to show the boundary conditions of the practical usability of the proposed system.

The results presented in Figure 4 are as expected. Depending on the assumed variance of the errors of the distance difference measurements, the CRLB varies between $0.001R$ and $0.02R$. Consequently, for the example considered above, the accuracy of the radiolocation method is limited to a range of 1–20 m in a circular area with a radius of 1 km. Comparing the results for different transmitter spacings, it can be concluded that scaling the distance between the transmitters and the measurement error of the distance difference in the same way gives a comparable accuracy of position estimation. Therefore, in the next section, the results of position estimation accuracy are limited to only one transmitter spacing value, and all values are expressed as a fraction of R, which simplifies the scaling of results for different sizes of supervised areas, transmitter spacings, and measurement errors.

Compared with the CRLB, the accuracy obtained for NA is worse by several factors, depending on the configuration variants of the sensor network (BS number). Therefore, it may be possible to find a better algorithm for position calculation that provides more accurate results than NA, but in this paper, we focus only on the introduction of a new method for simultaneously estimating the position and orientation of a mobile object. Algorithm optimization may be the next step in future work.

6 | SIMULATION RESULTS

The quality of position estimation in the developed radiolocation method was evaluated based on the results of simulation tests. The following model was adopted in the simulation analyses: BSs are placed evenly on a circle of radius R; inside the circle, the coordinates of the first transmitter on the object are randomly generated. Next, a second transmitter is placed on the randomly generated azimuth at a given distance d from the position of the first transmitter. From these coordinates, distance difference values Δd_n are calculated. Using NA, the positions of both transmitters on the object are estimated from these distance differences. In the process of solving the system in Equation (1), an initial vector with the coordinates of the two transmitters equal to (0, 0) and (d, 0), respectively, is taken for each case under consideration. Fixed starting points for the first NA iteration are frequently used in publications, as this approach corresponds to the situation in which the approximate position or orientation of the tracked object is not known (e.g., from previous measurements); however, the selection of initial coordinates and orientation of the MS influences the speed of convergence of NA. Therefore, one may use measurement data for an initial evaluation of the approximate orientation of the object by using other methods, such as a genetic or grid search algorithm, or by using data from previous observations of the MS position in the case of position tracking to increase the convergence probability and reduce the number of iterations. It was also assumed that the distance between the transmitters placed on the MS is equal to $d = 0.1R$. From the previous analysis of the CRLB (Section 5), it is known that scaling the distance between transmitters while scaling the distance difference measurement errors by the same factor results in a comparable quality of position estimation.

The simulation tool was developed in the universal mathematical computational environment MATLAB. During the simulation tests, the measurement error of the distance differences Δd_n between two transmitters and BSs was considered to follow a normal distribution (δ_d). To model this error, the *randn* function was used as follows:

$$\delta_d = \sigma \cdot randn \tag{12}$$

where σ is the standard deviation of the distance difference measurements. During the simulation study, each case was repeated 10,000 times, following the general concept of simulation described by Stefanski and Sadowski (2020).

Based on the results obtained from the simulation studies, the cumulative distribution functions (CDFs) of the absolute error δ, described in Equation (13) were plotted:

$$\delta = \sqrt{\left(\hat{x} - x\right)^2 + \left(\hat{y} - y\right)^2} \tag{13}$$

where $(\hat{x}, \hat{y})$ are the MS coordinate estimates and (x, y) are the actual MS coordinates. Here, the following assumptions were made:

$$\hat{x} = \frac{\hat{x}_1 + \hat{x}_2}{2}, \quad \hat{y} = \frac{\hat{y}_1 + \hat{y}_2}{2} \tag{14}$$

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2} \tag{15}$$

Because NA is an iterative algorithm, it is sensitive to the initial solution vector. When the vector of initial solutions is closer to the correct solution, the algorithm finds this solution more rapidly. In our considerations, the initial vector was always the same, corresponding to a fixed position at the origin and a constant orientation. During the simulation studies, a convergence criterion was adopted in the form of a norm described by the relation $||(\mathbf{J}^{T} \cdot \mathbf{J})^{-1} \cdot \mathbf{J}^{T} \cdot \mathbf{f}||$. The maximum number of iterations was 1,000. If the norm did not reach the set threshold after 1,000 iterations, the simulation program was interrupted and another random MS position was selected. However, in many cases, changing the orientation (x_2, y_2) while maintaining a constant position (x_1, y_1) for the starting point is sufficient for convergence. For a sensor network topology with 3–11 BSs, the accuracy of MS position estimation was analyzed using NA and three values of standard deviation σ for the distance difference measurement errors. In order to generalize the conclusions, the obtained results are expressed based on the radius (R) of the studied area. The

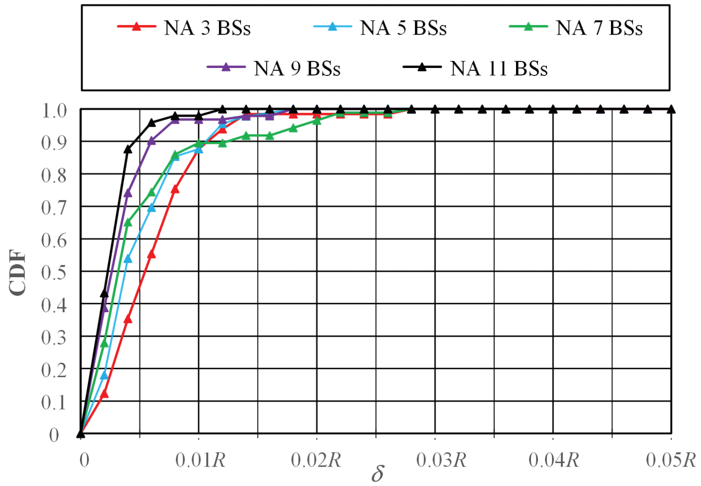


FIGURE 5 CDFs of the absolute position error δ for the proposed method using NA
It was assumed that $\sigma = \sqrt{10^{-7}}R$ for the measurement error of the distance differences.

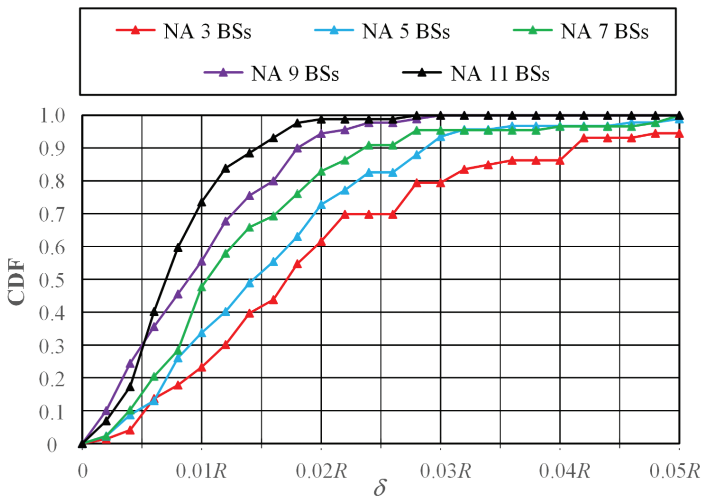


FIGURE 6 CDFs of the absolute position error δ for the proposed method using NA
It was assumed that $\sigma = \sqrt{10^{-6}}R = 0.001R$ for the measurement error of the distance differences.

results of the simulation tests are presented in Figures 5–7. The obtained results are as expected. For example, for $\sigma = \sqrt{10^{-7}}R$ with 7 BSs, the absolute error δ did not exceed $0.01R$ in 90% of the cases; for a radius of $R = 1{,}000$ m, this result translates into an MS position estimation error of no worse than 10 m. For the same sensor network configuration with $\sigma = \sqrt{10^{-6}}R = 0.001R$, the δ error in 90% of the cases is no worse than 25 m. As might be expected, the worst results were obtained for the largest standard deviation of the distance difference measurements (Figure 7). Recalling the above example, in which $R = 1{,}000$ m for 11 BSs, the absolute position error in 90% of the cases is no worse than 50 m.

The undoubted advantage of the developed method is the possibility of simultaneously estimating the object's position and orientation. Applying the simulation model described above with the described assumptions, tests were carried out, which resulted in a family of curves. The CDF of the orientation error δ_{orient} was defined according to the following relationship:

$$\delta_{orient} = \left| \arctan\left(\frac{y_2 - y_1}{x_2 - x_1}\right) - \arctan\left(\frac{\hat{y}_2 - \hat{y}_1}{\hat{x}_2 - \hat{x}_1}\right) \right| \tag{16}$$

The results of the simulation tests are presented in Figures 8–10. From the analysis of the orientation error results for all considered cases, it should be noted that most of the results are limited to an error of up to 10 degrees. For this value of orientation error, the CDF takes values between 85% and 97%. Only for the case with 3 BSs does the CDF value slightly exceed 70% for the smallest accepted standard deviation for the error in distance difference measurements. From these results, it can be concluded that the developed method for locating objects on the basis of two simultaneously transmitted signals has great application potential. The achievable accuracy of position and orientation estimation may be estimated from the results of our simulations by properly scaling the measurement errors and the area of system operation.

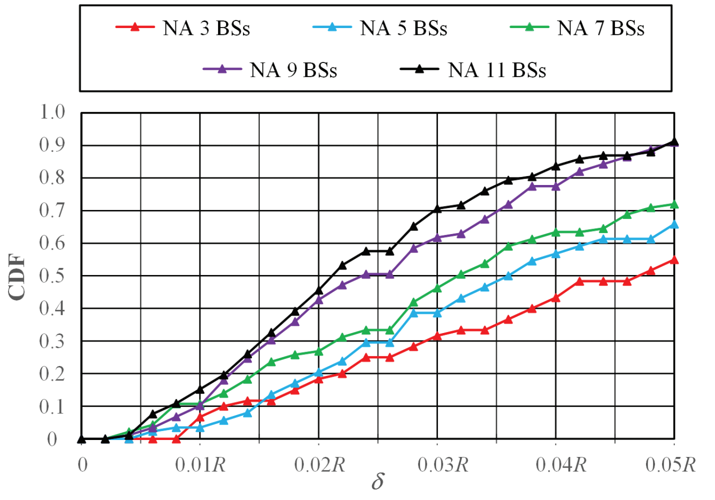


FIGURE 7 CDFs of the absolute position error δ for the proposed method using NA It was assumed that $\sigma = \sqrt{10^{-5}}R$ for the measurement error of the distance differences.

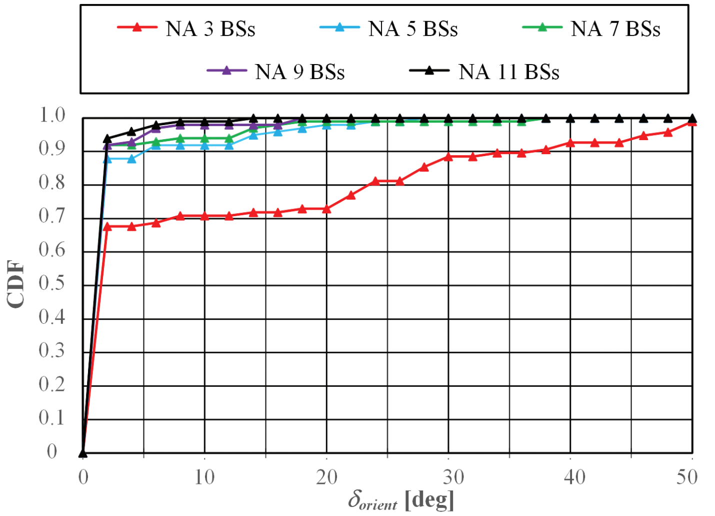


FIGURE 8 CDFs of the orientation error δ_{orient} for the proposed method using NA It was assumed that $\sigma = \sqrt{10^{-7}}R$ for the measurement error of the distance differences.

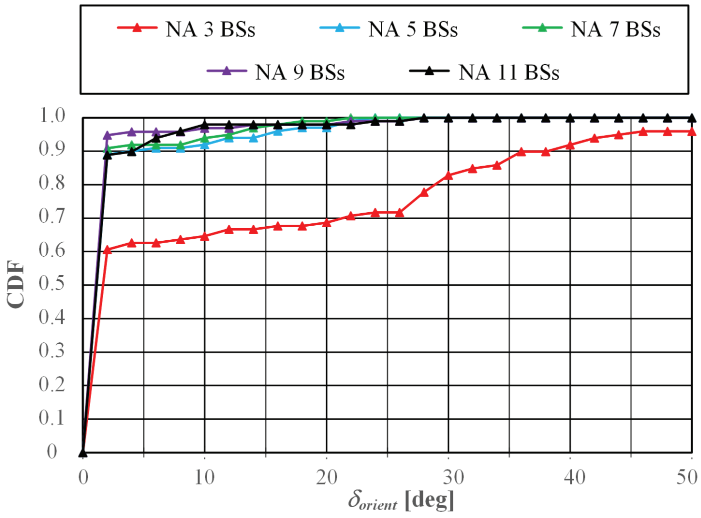


FIGURE 9 CDFs of the orientation error δ_{orient} for the proposed method using NA It was assumed that $\sigma = \sqrt{10^{-6}}R = 0.001R$ for the measurement error of the distance differences.

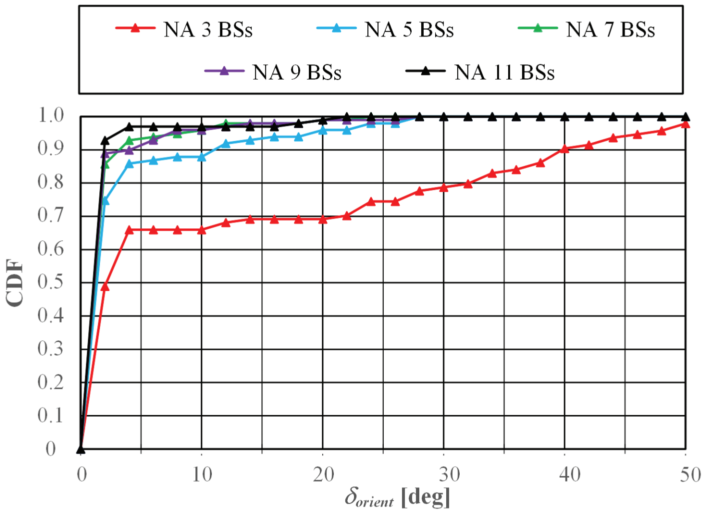


FIGURE 10 CDFs of the orientation error δ_{orient} for the proposed method using NA It was assumed that $\sigma = \sqrt{10^{-5}}R$ for the measurement error of the distance differences.

7 | CONCLUSION

The paper presents an innovative asynchronous radiolocation method for various applications. One novelty of this method is the simultaneous transmission of locating signals by two transmitters at a known distance placed on the traced object. A mathematical analysis of the proposed method was performed, considering the implementation details of NA to solve the system of positional equations.

The PDoP coefficient values were analyzed, assuming that this coefficient is determined in a manner analogous to that of classical solutions for radiolocation systems, i.e., based on a knowledge of the Jacobian matrix. For obvious reasons, some regions in the studied area are particularly prone to high PDoP values because this coefficient depends on the configuration of the sensor network (number and site of BSs). The problem of optimal BS deployment for our system needs further analysis, as the quality of position estimation depends not only on BS deployment but also on the orientation of the pair of transmitters on the localized vehicle.

In Section 5 of this paper, the CRLB, i.e., the limit of accuracy of this method, has been presented, with the assumption of a Gaussian distribution of measurement errors. The CRLB provides a reference point for evaluating the developed algorithms for object position estimation. A set of curves representing the RMSE of the object position estimation as a function of the absolute measurement error σ was presented. The error ranges of the distance difference measurements for the CRLB analysis were selected in such a way as to show the boundary conditions of practical usability of the proposed system. The results obtained using NA for relatively small values of the measurement error variance differ by at most three-fold from the CRLB limit.

The results of complex simulation studies of the proposed asynchronous method for three cases (three different values of the measurement error variance) were presented. Three sets of characteristics of the CDF as a function of absolute error and orientation error were obtained. These results show that the proposed method, described by a system of nonlinear equations, enables one to obtain the actual position and orientation of the localized object with the use of NA as an example.

The proposed method can be widely used, especially for locating objects of large dimensions, such as ships. The application of the proposed solution in aviation may be difficult, primarily limited to the positioning of aircraft on runways and airport aprons; however, mobile autonomous, unmanned, or pilot-assisted platforms, whether ground-based, sea-based, or airborne, could also be an important market sector. The advantage of the proposed solution over currently used navigation and location systems, whether airborne or sea-based, is the acquisition of not only location information, but also information about the orientation of the supervised object. Such solutions are not currently used in practice. For example, the orientation of ships in an automatic identification system is usually derived from an on-board gyrocompass, rather than being autonomously determined by the location system. However, actual interest in the proposed technique for simultaneously determining the position and orientation of an object will largely depend on the work and decisions of the administrative units responsible for overseeing regulations on the safety of implementation of operations (air or sea) or defining the minimum equipment of vehicles/platforms necessary for admission to traffic.

REFERENCES

Bard, J. D., & Ham, F. M. (1999). Time difference of arrival dilution of precision and applications. *IEEE Transactions on Signal Processing*, *47*(2), 521–523. http://doi.org/10.1109/78.740135

Chang, C., & Sahai, A. (2006). Cramer-Rao type bounds for localization. *EURASIP Journal of Applied Signal Processing*, 1–13. http://doi.org/10.1155/ASP/2006/94287

Choi, H., Baek, Y., & Lee, B. (2013). Efficient congestion control utilizing message eavesdropping in asynchronous range-based localization. *ETRI Journal*, *35*(1), 35–40. http://doi.org/10.4218/etrij.13.0112.0302

Foy, W. H. (1976). Position-location solutions by Taylor-series estimation. *IEEE Transactions on Aerospace and Electronic Systems*, *12*(2), 187–194. http://doi.org/10.1109/TAES.1976.308294

He, S. (2016). *Asynchronous time difference of arrival positioning system and implementation.* [Doctoral dissertation, University of Victoria].

IEEE, J. (2020). IEEE standard for low-rate wireless networks. *IEEE Std 802.15.4-2020*. http://doi.org/10.1109/IEEESTD.2020.9144691

Kay, S. M. (1993). *Fundamentals of statistical signal processing: Estimation theory*. Prentice Hall.

Kim, H. (2009). A ranging scheme for asynchronous location positioning systems. *Proc. of the 6th Workshop on Positioning, Navigation and Communication (WPNC 2009),* Hannover, Germany. 89–94. http://doi.org/10.1109/WPNC.2009.4907809

Li, T., Ekpenyong, A., & Huang, Y. A. (2004). A location system using asynchronous distributed sensors. *Proc. of the IEEE INFOCOM 2004 – The Conference on Computer Communications*, Hong Kong, China. 620–628. http://doi.org/10.1109/INFCOM.2004.1354533

McDonough, R. N., & Whalen, A. D. (1995). *Detection of signals in noise* (2nd ed.). Academic Press.

Nawaz, H., Bozkurt, A., and Tekin, I. (2017). A novel power efficient asynchronous time difference of arrival indoor localization system using CC1101 radio transceivers. *Microwave and Optical Technology Letters*, *59*(3), 550–555. http://doi.org/10.1002/mop.30342

Sadowski, J., & Stefanski, J. (2021). *A method of simultaneously determining the position and orientation of an object, especially one in motion*. (Patent Appl. No. P.439191) Poland. (In Polish).

Sathyan, T., Humphrey, D., & Hedley, M. (2011). WASP: A system and algorithms for accurate radio localization using low-cost hardware. *IEEE Transactions on Systems, Man, and Cybernetics, Part C: Applications and Reviews*, *41*(2), 211–222. http://doi.org/10.1109/TSMCC.2010.2051027

Shin, D., & Sung, T. (2002). Comparisons of error characteristics between TOA and TDOA positioning. *IEEE Transactions on Aerospace and Electronic Systems*, *38*(1), 307–311. http://doi.org/10.1109/7.993253

Stefanski, J., & Sadowski, J. (2020). Sensor position estimation method for IoT using mobile reference node. *IEEE Access*, *8*, 79287–79298. http://doi.org/10.1109/ACCESS.2020.2990385

Tsui, J. B. Y. (2000). *Fundamentals of global positioning system receivers. A software approach.* Wiley & Sons.

Vaghefi, R. M., & Buehrer, R. M. (2013). Asynchronous time-of-arrival-based source localization. *Proc. of the IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP 2013)* Vancouver, Canada. 4086–4090. http://doi.org/10.1109/ICASSP.2013.6638427

Vandenberghe, L., & Boyd, S. (1996). Semidefinite programming. *Society for Industrial and Applied Mathematics Review*, *38*(1), 49–95. http://doi.org/10.1137/1038003

Wang, Y., & Leus, G. (2012). Reference-free time-based localization for an asynchronous target. *EURASIP Journal on Advances in Signal Processing*, 1–21. http://doi.org/10.1186/1687-6180-2012-19

Wang, Y., Leus, G., & Ma, X. (2011). Time-based localization for asynchronous wireless sensor networks. *Proc. of the IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP 2011),* Prague, Czech Republic. 3284–3287. http://doi.org/10.1109/ICASSP.2011.5946723

Xiong, H., Chen, Z., An, W., & Yang, B. (2015). Robust TDOA localization algorithm for asynchronous wireless sensor networks. *International Journal of Distributed Sensor Networks*, *11*(5), 598747. http://doi.org/10.1155/2015/598747

Youssef, M., Youssef, A., Rieger, C., Shankar, U., & Agrawala, A. (2006). PinPoint: An asynchronous time-based location determination system. *Proc. of the 4th International Conference on Mobile Systems, Applications and Services (MobiSys '06),* Uppsala, Sweden. 165–176. http://doi.org/10.1145/1134680.1134698

Zachariah, D., de Angelis, A., Dwivedi, S., & Händel, P. (2014). Schedule-based sequential localization in asynchronous wireless networks. *EURASIP Journal on Advances in Signal Processing*, 1–12. http://doi.org/10.1186/1687-6180-2014-16

How to cite this article: Stefanski, J., & Sadowski, J. (2023). Asynchronous method of simultaneous object position and orientation estimation with two transmitters. *NAVIGATION, 70*(4). https://doi.org/10.33012/navi.601

APPENDIX A

The individual elements of the Fisher matrix in Equation (12) were described by the following relationships (Equation (A.1) to Equation (A.16)):

$$\frac{\partial^2 \ln p_1(\rho;\theta)}{\partial x_1^2} = -\frac{1}{\sigma^2}\sum_{n=1}^{N}\left[\frac{A}{\sqrt{B}} - \frac{A\cdot\left(x_1 - X_n\right)^2}{B^{\frac{3}{2}}} + \frac{\left(x_1 - X_n\right)^2}{B}\right] \tag{A.1}$$

$$\frac{\partial^2 \ln p(\rho;\theta)}{\partial x_1 \partial y_1} = -\frac{1}{\sigma^2}\sum_{n=1}^{N}\left[\frac{\left(x_1 - X_n\right)\cdot\left(Y_n - y_1\right)\cdot A}{B^{\frac{3}{2}}} - \frac{\left(x_1 - X_n\right)\cdot\left(Y_n - y_1\right)}{B}\right] \tag{A.2}$$

$$\frac{\partial^2 \ln p(\rho;\theta)}{\partial x_1 \partial x_2} = -\frac{1}{\sigma^2}\sum_{n=1}^{N}\left[\frac{(x_1 - X_n)\cdot(X_n - x_2)}{\sqrt{B\cdot C}}\right] \tag{A.3}$$

$$\frac{\partial^2 \ln p(\rho;\theta)}{\partial x_1 \partial y_2} = -\frac{1}{\sigma^2}\sum_{n=1}^{N}\left[\frac{(x_1 - X_n)\cdot(Y_n - y_2)}{\sqrt{B\cdot C}}\right] \tag{A.4}$$

$$\frac{\partial^2 \ln p(\rho;\theta)}{\partial y_1 \partial x_1} = -\frac{1}{\sigma^2}\sum_{n=1}^{N}\left[\frac{(y_1 - Y_n)\cdot(X_n - x_1)\cdot A}{B^{\frac{3}{2}}} - \frac{(y_1 - Y_n)\cdot(X_n - x_1)}{B}\right] \tag{A.5}$$

$$\frac{\partial^2 \ln p(\rho;\theta)}{\partial y_1^2} = -\frac{1}{\sigma^2}\sum_{n=1}^{N}\left[\frac{A}{\sqrt{B}} - \frac{A\cdot(y_1 - Y_n)^2}{B^{\frac{3}{2}}} + \frac{(y_1 - Y_n)^2}{B}\right] \tag{A.6}$$

$$\frac{\partial^2 \ln p(\rho;\theta)}{\partial y_1 \partial x_2} = -\frac{1}{\sigma^2}\sum_{n=1}^{N}\left[\frac{(y_1 - Y_n)\cdot(X_n - x_2)}{\sqrt{B\cdot C}}\right] \tag{A.7}$$

$$\frac{\partial^2 \ln p(\rho;\theta)}{\partial y_1 \partial y_2} = -\frac{1}{\sigma^2}\sum_{n=1}^{N}\left[\frac{(y_1 - Y_n)\cdot(Y_n - y_2)}{\sqrt{B\cdot C}}\right] \tag{A.8}$$

$$\frac{\partial^2 \ln p(\rho;\theta)}{\partial x_2 \partial x_1} = -\frac{1}{\sigma^2}\sum_{n=1}^{N}\left[\frac{(X_n - x_2)\cdot(X_n - x_1)}{\sqrt{B\cdot C}}\right] \tag{A.9}$$

$$\frac{\partial^2 \ln p(\rho;\theta)}{\partial x_2 \partial y_1} = -\frac{1}{\sigma^2}\sum_{n=1}^{N}\left[\frac{(X_n - x_2)\cdot(Y_n - y_1)}{\sqrt{B\cdot C}}\right] \tag{A.10}$$

$$\frac{\partial^2 \ln p(\rho;\theta)}{\partial x_2^2} = -\frac{1}{\sigma^2}\sum_{n=1}^{N}\left[-\frac{A}{\sqrt{C}} + \frac{A\cdot(X_n - x_2)^2}{C^{\frac{3}{2}}} + \frac{(X_n - x_2)^2}{C}\right] \tag{A.11}$$

$$\frac{\partial^2 \ln p(\rho;\theta)}{\partial x_2 \partial y_2} = -\frac{1}{\sigma^2}\sum_{n=1}^{N}\left[\frac{(X_n - x_2)\cdot(Y_n - y_2)\cdot A}{C^{\frac{3}{2}}} + \frac{(X_n - x_2)\cdot(Y_n - y_2)}{C}\right] \tag{A.12}$$

$$\frac{\partial^2 \ln p(\rho;\theta)}{\partial y_2 \partial x_1} = -\frac{1}{\sigma^2}\sum_{n=1}^{N}\left[-\frac{(Y_n - y_2)\cdot(X_n - x_1)}{\sqrt{B\cdot C}}\right] \tag{A.13}$$

$$\frac{\partial^2 \ln p(\rho;\theta)}{\partial y_2 \partial y_1} = -\frac{1}{\sigma^2}\sum_{n=1}^{N}\left[-\frac{(Y_n - y_2)\cdot(Y_n - y_1)}{\sqrt{B\cdot C}}\right] \tag{A.14}$$

$$\frac{\partial^2 \ln p(\rho;\theta)}{\partial y_2 \partial x_2} = -\frac{1}{\sigma^2}\sum_{n=1}^{N}\left[\frac{(Y_n - y_2)\cdot(X_n - x_2)\cdot A}{C^{\frac{3}{2}}} + \frac{(Y_n - y_2)\cdot(X_n - x_2)}{C}\right] \tag{A.15}$$

$$\frac{\partial^2 \ln p(\rho;\theta)}{\partial y_2^2} = -\frac{1}{\sigma^2}\sum_{n=1}^{N}\left[-\frac{A}{\sqrt{C}} + \frac{A\cdot(Y_n - y_2)^2}{C^{\frac{3}{2}}} + \frac{(Y_n - y_2)^2}{C}\right] \tag{A.16}$$

where:

$$A = \Delta d_n - \left(\sqrt{(X_n - x_2)^2 + (Y_n - y_2)^2} - \sqrt{(X_n - x_1)^2 + (Y_n - y_1)^2} \right) \tag{A.17}$$

$$B = (X_n - x_1)^2 + (Y_n - y_1)^2 \tag{A.18}$$

$$C = (X_n - x_2)^2 + (Y_n - y_2)^2 \tag{A.19}$$

$$D = d^2 - \left((x_2 - x_1)^2 + (y_2 - y_1)^2 \right) \tag{A.20}$$