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Bifurcation of equilibrium forms of a gas column rotating with constant speed around its axis of symmetry

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A B S T R A C T

We will be concerned with the problem of deformation of the lateral surface of a column that rotates with constant speed around its axis of symmetry. The column is filled by a gas and our goal is to investigate the deformation of the lateral surface depending on the pressure of the gas.

1. Introduction

In recent years, the study of mathematical models that arise in the context of concrete applications has attracted a lot of attention and yielded new methods and tools for investigating different classes of nonlinear ODEs and PDEs. Of particular interest are models that arise in continuum (nonlinear) mechanics of elastic bodies and are subject to different external influences and loads. Let us mention here the well-known von Karman model for the study of buckling of plates and shells. These equations have attracted a lot of interest in the literature (see [1–6]).

There is a broad variety of problems arising in elastic mechanics that can be handled by bifurcation theory. In the present paper we will discuss one of them. Namely, we will study the problem of deformation of the lateral surface of a rotating gas column. We assume that the undeformed column is a cylinder of height H and radius R, where $2R$ is much greater than H and the lateral surface of the column is made of a flexible reinforced material. This means that the shape of the lateral surface of the column is determined by its transverse section. The column rotates with constant speed around its axis of symmetry. The interior of the column is filled by a gas under pressure which is trapped inside the column. Namely, the column is placed between two parallel planes that obstruct the outflow of gas (see Fig. 1).

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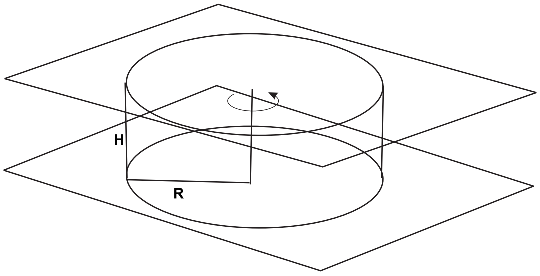

Fig. 1. A model of a rotating gas column.

Equilibrium forms of the gas column are described in polar coordinates by 2π-periodic even C^{m+2}-smooth positive functions $r(\theta)$, $\theta \in [0, 2\pi]$, $m \in \mathbb{N} \cup \{0\}$, satisfying the equation:

$$\frac{r(\theta)}{\sqrt{r^2(\theta) + r'^2(\theta)}} - \frac{r''(\theta)r^2(\theta) - r'^2(\theta)r(\theta)}{(r^2(\theta) + r'^2(\theta))^{3/2}} - \omega^2 m r(\theta) + p = 0,$$

where ω is the rotation speed of the gas column, m is the mass of the gas column, and p is the pressure of gas inside the column. $r(\theta)$ describes the transverse section of the cylinder. The derivation of this equation goes back to a lecture by our colleague prof. Andrei Borisovich, who passed away in 2008, given at our university in 2005. We rewrite the mathematical model as an operator equation and, by using a bifurcation argument, we prove that there exist smooth solutions of the problem which are not radially symmetric. In the last two decades many authors have studied different free boundary problems via bifurcation methods (see for instance [2,7–10]). A lot of important examples, both from a mathematical and an applicable point of view, have appeared in [11]. They come for example from biology, medicine, mechanics or chemistry.

The paper is organized as follows. In Section 2 we derive the equation of equilibrium forms of the gas column following A. Borisovich, and we state the main theorem. Section 3 contains the proof of this result.

We hope that our manuscript will be useful not only for mathematicians working in differential equations but also appeal to physicists and engineers interested in nonlinear mechanics. The developed techniques go beyond this particular model and are applicable to a broad variety of nonlinear models in mechanics that have similar properties.

2. Mathematical model

Let $C_e^m(2\pi)$, $m \in \mathbb{N} \cup \{0\}$, be the Banach space of 2π-periodic even C^m-smooth functions $r(\theta)$ with the standard norm

$$\|r\|_m = \sum_{k=0}^{m} \max_{\theta \in [0,2\pi]} |r^{(k)}(\theta)|, \tag{1}$$

where $r^{(k)}(\theta)$ denotes the kth derivative of $r(\theta)$ for $k = 1, 2, \ldots, m$ and $r^{(0)}(\theta) = r(\theta)$.

The total energy of the gas column, denoted by E_t, is defined to be

$$E_t = E_1 - E_2 + E_3,$$

where

- E_1 stands for the potential energy,
- E_2 is the kinetic energy,
- E_3 denotes the energy of gas.

The energy E_1 is given by

$$E_1(r) = \alpha \int_0^{2\pi} \sqrt{r^2(\theta) + r'^2(\theta)} d\theta, \tag{2}$$

where $\alpha > 0$ is an elasticity coefficient of the flexible material, $r \in C_e^{m+2}(2\pi)$ describes the boundary of the cross section of the gas column and therefore $r(\theta) > 0$ for each $\theta \in [0, 2\pi]$. There is no loss of generality in assuming that $\alpha = 1$. The energy E_2 is defined by

$$E_2(r) = \int_0^{2\pi} \frac{\omega^2 m}{2} r^2(\theta) d\theta, \tag{3}$$

where $\omega > 0$ is the rotation speed of the gas column and $m > 0$ is the mass of the gas column. Finally, the energy E_3 is defined as

$$E_3(r, p) = \int_0^{2\pi} pr(\theta) d\theta, \tag{4}$$

where $p > 0$ is the pressure of gas. Hence the formula for the total energy has the form

$$E_t(r, p) = \int_0^{2\pi} \left(\sqrt{r^2(\theta) + r'^2(\theta)} - \frac{\omega^2 m}{2} r^2(\theta) + pr(\theta) \right) d\theta. \tag{5}$$

Dividing (5) by 2π, we obtain a functional E given by

$$E(r, p) = \frac{1}{2\pi} \int_0^{2\pi} \left(\sqrt{r^2(\theta) + r'^2(\theta)} - \frac{\omega^2 m}{2} r^2(\theta) + pr(\theta) \right) d\theta. \tag{6}$$

In what follows we refer to E as the energy functional. It is easy to check that

$$\begin{aligned} E'_r(r, p)h &= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r(\theta)}{\sqrt{r^2(\theta) + r'^2(\theta)}} - \frac{r''(\theta) r^2(\theta) - r'^2(\theta) r(\theta)}{(r^2(\theta) + r'^2(\theta))^{3/2}} \right) h(\theta) d\theta \\ &\quad - \frac{1}{2\pi} \int_0^{2\pi} \left(\omega^2 m r(\theta) - p \right) h(\theta) d\theta \end{aligned}$$

for each $p \in \mathbb{R}_+$, $r, h \in C_e^{m+2}(2\pi)$ and $r(\theta) > 0$ for $\theta \in [0, 2\pi]$. Consequently, critical points of the functional E (i.e. equilibrium forms of the gas column) are 2π-periodic even C^{m+2}-smooth positive solutions of the second order differential equation

$$\frac{r(\theta)}{\sqrt{r^2(\theta) + r'^2(\theta)}} - \frac{r''(\theta) r^2(\theta) - r'^2(\theta) r(\theta)}{(r^2(\theta) + r'^2(\theta))^{3/2}} - \omega^2 m r(\theta) + p = 0. \tag{7}$$

Substituting $r(\theta) \equiv \text{const}$ into (7), we get an algebraic equation with a solution given by

$$R_p = \frac{1 + p}{m\omega^2}, \tag{8}$$

which means that the cross section of the gas column is a circle of radius R_p. To sum up, for all $m, \omega \in \mathbb{R}_+$ there exists a family of radially symmetric solutions of Eq. (7) given by

$$\Gamma = \{(R_p, p) \colon p \in \mathbb{R}_+\}, \tag{9}$$

where R_p is defined by (8).

3. Bifurcation problem

We now want to find all values of parameter p for which the radially symmetric solution R_p loses its stability. For this purpose we will study bifurcation from the set of radial solutions with respect to p.

Definition 3.1. We call $(R_p, p) \in \Gamma$ a bifurcation point of Eq. (7) with respect to the set Γ if there exists a branch of non-radially symmetric solutions $(r(t), p(t))$ of (7), depending on $|t| < \varepsilon$, with $r(0) = R_p$ and $p(0) = p$.

Our theorem reads as follows.

Theorem 3.1. *Let $p_k = \frac{k^2}{4} - 1$, $k \geq 4$, $k \in 2\mathbb{N}$. Then there exists a set of non-radially symmetric solutions $(r(t), p(t))$ of Eq. (7), depending on $|t| < \varepsilon$, with*

$$r(t)(\theta) = R_{p(t)} + t \cdot \sqrt{2} \cos\left(\frac{k\theta}{2}\right) + o(|t|)$$

and $r(0) = R_{p_k}$, $p(0) = p_k$. Hence $(R_{p_k}, p_k) \in \Gamma$ is a bifurcation point of (7).

Set

$$X = C_e^{m+2}(2\pi) \quad \text{and} \quad Y = C_e^m(2\pi) \qquad (m \in \mathbb{N} \cup \{0\}).$$

Fix $p_0 \in \mathbb{R}_+$. Consider the point $(R_{p_0}, p_0) \in \Gamma$. Let $X_\delta(0)$ and $(\mathbb{R}_+)_\delta(p_0)$ denote the balls in X and $\mathbb{R}_+$, respectively, of radius δ around 0 and p_0. For $\varrho \in X_\delta(0)$ and $p \in (\mathbb{R}_+)_\delta(p_0)$ we define

$$r(\theta) = R_p + \varrho(\theta), \tag{10}$$

where $\theta \in [0, 2\pi]$ and R_p is given by (8). Substituting (10) into (6), we get the energy functional $\hat{E}\colon X_\delta(0) \times (\mathbb{R}_+)_\delta(p_0) \to \mathbb{R}$ given by

$$\hat{E}(\varrho, p) = \frac{1}{2\pi} \int_0^{2\pi} \left(\sqrt{(R_p + \varrho)^2 + \varrho'^2} - \frac{m\omega^2}{2} (R_p + \varrho)^2 + p(R_p + \varrho) \right) d\theta. \tag{11}$$

It is a simple matter to check that

$$\begin{aligned} \hat{E}'_\varrho(\varrho, p)h = {} & \frac{1}{2\pi} \int_0^{2\pi} \frac{(R_p + \varrho)}{\sqrt{(R_p + \varrho)^2 + \varrho'^2}} h d\theta \\ & - \frac{1}{2\pi} \int_0^{2\pi} \frac{\varrho''(R_p + \varrho)^2 - \varrho'^2(R_p + \varrho)}{((R_p + \varrho)^2 + \varrho'^2)^{\frac{3}{2}}} h d\theta \\ & - \frac{1}{2\pi} \int_0^{2\pi} \left(m\omega^2 (R_p + \varrho) - p \right) h d\theta. \end{aligned} \tag{12}$$

Let $\hat{F}\colon X_\delta(0) \times (\mathbb{R}_+)_\delta(p_0) \to Y$ be given by

$$\hat{F}(\varrho, p) = \frac{(R_p + \varrho)}{\sqrt{(R_p + \varrho)^2 + \varrho'^2}} - \frac{\varrho''(R_p + \varrho)^2 - \varrho'^2(R_p + \varrho)}{((R_p + \varrho)^2 + \varrho'^2)^{\frac{3}{2}}} - m\omega^2 (R_p + \varrho) + p.$$

Let us denote by $\langle \cdot, \cdot \rangle$ the standard inner product in $L^2(2\pi)$, i.e.

$$\langle g, h \rangle = \frac{1}{2\pi} \int_0^{2\pi} g(\theta) h(\theta) d\theta, \quad g, h \in L^2(2\pi).$$

Proposition 3.2. *The map $\hat{F}$ is a variational gradient of the functional $\hat{E}$ with respect to the inner product in $L^2(2\pi)$, i.e.*

$$\hat{E}'_\varrho(\varrho, p)h = \left\langle \hat{F}(\varrho, p), h \right\rangle$$

for all $\varrho \in X_\delta(0)$, $h \in X$ and $p \in \mathbb{R}_+$.

We are interested in finding critical points of the energy functional $\hat{E}\colon X_\delta(0) \times (\mathbb{R}_+)_\delta(p_0) \to \mathbb{R}$ subject to constraints:

$$\frac{1}{2\pi}\int_0^{2\pi} \varrho(\theta)\cos(\theta)d\theta = 0 \tag{13}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \varrho(\theta)d\theta = 0. \tag{14}$$

For this purpose, we apply the method of Lagrange multipliers. Let us remark that the condition (13) together with the evenest of $\varrho(\theta)$ excludes a displacement of the axis of symmetry of the cylinder. The condition (14) gives a loss of radial symmetry, i.e. it excludes $\varrho(\theta) = const$. Finally, these conditions exclude $\varrho(\theta) = \cos(\theta)$.

Define

$$\varepsilon(\varrho, \lambda_1, \lambda_2, p) = \hat{E}(\varrho, p) + \frac{\lambda_1}{2\pi}\int_0^{2\pi} \varrho(\theta)\cos(\theta)d\theta + \frac{\lambda_2}{2\pi}\int_0^{2\pi} \varrho(\theta)d\theta \ ,$$

where $\varrho \in X_\delta(0)$, $p \in (\mathbb{R}_+)_\delta(p_0)$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ are Lagrange multipliers.

Set $x = (\varrho, \lambda_1, \lambda_2)$. We have

$$\varepsilon'_x(x,p)(h, h_1, h_2) = \varepsilon'_\varrho(x,p)h + \varepsilon'_{\lambda_1}(x,p)h_1 + \varepsilon'_{\lambda_2}(x,p)h_2 \tag{15}$$

for all $h \in X_\delta(0)$ and $h_1, h_2 \in \mathbb{R}$. An easy computation shows that

$$\begin{aligned} \varepsilon'_\varrho(\varrho, \lambda_1, \lambda_2, p)h = & \frac{1}{2\pi}\int_0^{2\pi} \frac{(R_p + \varrho)}{\sqrt{(R_p+\varrho)^2 + \varrho'^2}}\, hd\theta \\ & - \frac{1}{2\pi}\int_0^{2\pi} \frac{\varrho''(R_p+\varrho)^2 - \varrho'^2(R_p+\varrho)}{((R_p+\varrho)^2 + \varrho'^2)^{\frac{3}{2}}}\, hd\theta \\ & - \frac{1}{2\pi}\int_0^{2\pi} \left(m\omega^2(R_p+\varrho) - p\right) hd\theta \\ & + \frac{\lambda_1}{2\pi}\int_0^{2\pi} h\cos(\theta)d\theta + \frac{\lambda_2}{2\pi}\int_0^{2\pi} hd\theta, \end{aligned} \tag{16}$$

$$\varepsilon'_{\lambda_1}(\varrho, \lambda_1, \lambda_2, p)h_1 = \frac{h_1}{2\pi}\int_0^{2\pi} \varrho(\theta)\cos(\theta)d\theta$$

and

$$\varepsilon'_{\lambda_2}(\varrho, \lambda_1, \lambda_2, p)h_2 = \frac{h_2}{2\pi}\int_0^{2\pi} \varrho(\theta)d\theta.$$

Let $\mathcal{F}\colon X_\delta(0) \times \mathbb{R}^2 \times (\mathbb{R}_+)_\delta(p_0) \to Y \times \mathbb{R}^2$ be given by

$$\mathcal{F}(x,p) = \left(\tilde{F}(\varrho, \lambda_1, \lambda_2, p), \frac{1}{2\pi}\int_0^{2\pi} \varrho(\theta)\cos(\theta)d\theta, \frac{1}{2\pi}\int_0^{2\pi} \varrho(\theta)d\theta\right), \tag{17}$$

where

$$\tilde{F}(\varrho, \lambda_1, \lambda_2, p) = \hat{F}(\varrho, p) + \lambda_1\cos(\theta) + \lambda_2$$

and $x = (\varrho, \lambda_1, \lambda_2)$.

Lemma 3.3. *For each $p \in (\mathbb{R}_+)_\delta(p_0)$, $\mathcal{F}(\cdot,p)\colon X_\delta(0)\times\mathbb{R}^2 \to Y\times\mathbb{R}^2$ is a variational gradient of $\varepsilon(\cdot,p)\colon X_\delta(0)\times\mathbb{R}^2 \to \mathbb{R}$ with respect to the scalar product in the Hilbert space $L^2(2\pi)\times\mathbb{R}^2$, i.e. for all $x=(\varrho,\lambda_1,\lambda_2)\in X_\delta(0)\times\mathbb{R}^2$, $h\in X$ and $h_1,h_2\in\mathbb{R}$,*

$$\varepsilon'_x(x,p)(h,h_1,h_2) = \langle \mathcal{F}(x,p),(h,h_1,h_2)\rangle .$$

Let us now consider the equation

$$\mathcal{F}(x,p)=0. \tag{18}$$

Eq. (18) possesses a trivial family of solutions

$$\hat{\Gamma} = \{(0,p)\in X\times\mathbb{R}^2\times\mathbb{R}_+ : p\in(\mathbb{R}_+)_\delta(p_0)\}.$$

Let us recall that $(0,p_0)\in\hat{\Gamma}$ is a bifurcation point of (18) if there exists a branch of nontrivial solutions $(x(t),p(t))$ of (18), parametrized by $|t|<\epsilon$, such that $x(0)=0$, $p(0)=p_0$.

Let us remark that we have reduced the problem of bifurcation from the set of radial solutions of (7) at (R_{p_0},p_0) to the problem of bifurcation from trivial solutions of (18) at $(0,p_0)$. The task is now to prove the theorem below.

Theorem 3.4. *Let $p_k = \frac{k^2}{4}-1$ for $k\geq 4$, $k\in 2\mathbb{N}$. Then there is a smooth curve of nontrivial solutions $(x(t),p(t))$ of (18), parametrized by $|t|<\epsilon$, with*

$$x(t)(\theta) = (\varrho(t)(\theta),\lambda_1(t),\lambda_2(t))$$

and $x(0)=(0,0,0)$, $p(0)=p_k$, where

$$\varrho(t)(\theta) = t\cdot\sqrt{2}\cos\left(\frac{k\theta}{2}\right)+o(|t|)$$

and $\varrho(0)=0$. Hence $(0,p_k)\in\hat{\Gamma}$ is a bifurcation point of Eq. (18).

3.1. Proof of Theorem 3.4

Our proof is based on the Crandall–Rabinowitz theorem on simple bifurcation points (see [12]). More precisely, we will apply a gradient (variational) version of the Crandall–Rabinowitz theorem due to A.Yu. Borisovich (see [1,3,13]). For the convenience of the reader we state this theorem. It will cause no confusion if we use the same letters in the abstract result as in our issue.

Theorem 3.5. *Assume that H is a Hilbert space with a scalar product $(\cdot,\cdot)_H$. Let X and Y be Banach spaces continuously embedded in H. Suppose that a C^r-operator $F\colon X_\delta(0)\times\mathbb{R}_\delta(\tau_0)\to Y$ and a C^{r+1}-functional $E\colon X_\delta(0)\times\mathbb{R}_\delta(\tau_0)\to\mathbb{R}$, where $r\geq 2$, satisfy the following conditions:*

1. $F(0,\tau)=0$ *for* $\tau\in\mathbb{R}_\delta(\tau_0)$,
2. $\dim\ker F'_x(0,\tau_0)=1$, $F'_x(0,\tau_0)e=0$, $(e,e)_H=1$,
3. $\operatorname{codim}\ \operatorname{im} F'_x(0,\tau_0)=1$,
4. $E'_x(x,\tau)h=(F(x,\tau),h)_H$ *for* $(x,\tau)\in X_\delta(0)\times\mathbb{R}_\delta(\tau_0)$ *and* $h\in X$,
5. $E'''_{xx\tau}(0,\tau_0)(e,e,1)\neq 0$.

Then $(0,\tau_0)$ is a bifurcation point of the equation

$$F(x,\tau)=0.$$

In fact, the solution set of this equation in a certain neighborhood of $(0, \tau_0)$ *consists of the curve* $\Gamma_1 = \{(0, \tau) : \tau \in \mathbb{R}_\delta(\tau_0)\}$ *and a* C^{r-2}*-curve* Γ_2*, intersecting only at* $(0, \tau_0)$*. Moreover, if* $r \geq 3$*, the curve* Γ_2 *can be parametrized by a variable* t*,* $|t| \leq \varepsilon$*, as*

$$\Gamma_2 = \{(x(t), \tau(t)) : t \in \mathbb{R}_\varepsilon(0)\},$$

where $x(0) = 0$*,* $\tau(0) = \tau_0$ *and* $x'(0) = e$*.*

The operator $\mathcal{F}$ is easily seen to be C^∞-smooth. What is more, we check at once that

$$\mathcal{F}'_x(0, p)(h, h_1, h_2) = \left(\tilde{F}'_x(0, p)(h, h_1, h_2), \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\cos(\theta)d\theta, \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta\right),$$

where

$$\tilde{F}'_x(0, p)(h, h_1, h_2) = \hat{F}'_\varrho(0, p)h + h_1\cos(\theta) + h_2$$

and

$$\hat{F}'_\varrho(0, p)h = -\frac{1}{R_p}h'' - \omega^2 mh$$

for $p \in (\mathbb{R}_+)_\delta(p_0)$ and $h \in X, h_1, h_2 \in \mathbb{R}$.

Our goal now is to prove that $\mathcal{F}$ satisfies the assumptions of Theorem 3.5 for $p_0 = p_k = \frac{k^2}{4} - 1$, $k \geq 4$, $k \in 2\mathbb{N}$.

Lemma 3.6. *For each* $p \in (\mathbb{R}_+)_\delta(p_0)$*,* $\mathcal{F}'_x(0, p) : X \times \mathbb{R}^2 \to Y \times \mathbb{R}^2$ *is a Fredholm map of index zero.*

Proof. Fix $p \in (\mathbb{R}_+)_\delta(p_0)$. The map

$$\mathcal{F}'_x(0, p) : X \times \mathbb{R}^2 \to Y \times \mathbb{R}^2$$

may be written as the sum

$$\mathcal{F}'_x(0, p)(h, h_1, h_2) = A(h, h_1, h_2) + B(h, h_1, h_2),$$

where $A : X \times \mathbb{R}^2 \to Y \times \mathbb{R}^2$ and $B : X \times \mathbb{R}^2 \to Y \times \mathbb{R}^2$ are defined as follows:

$$A(h, h_1, h_2) = \left(-\frac{1}{R_p}h'', 0, 0\right)$$

and

$$B(h, h_1, h_2) = \left(-\omega^2 mh + h_1\cos(\theta) + h_2, \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\cos(\theta)d\theta, \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta\right).$$

To finish the proof it suffices to show that A is a Fredholm map of index 0, and B is completely continuous. It is easily seen that A is a linear continuous map with

$$\ker A = \left\{(C, h_1, h_2) \in X \times \mathbb{R}^2 : C, h_1, h_2 \in \mathbb{R}\right\},$$

and so $\dim \ker A = 3$. Moreover, we can see that

$$Y \times \mathbb{R}^2 = (Y_1 \oplus Y_2) \times \mathbb{R}^2,$$

where

$$Y_2 \approx \mathrm{im}A = \left\{(y, 0, 0) \in Y \times \mathbb{R}^2 : \int_0^{2\pi} y(\theta)d\theta = 0\right\},$$

and $Y_1 \times \mathbb{R}^2 = \ker A$. Therefore codim im$A = \dim\ker A = 3$, which implies A is a Fredholm map of index 0.

Now let us consider the map B. It is easy to check that B is linear and continuous. We prove that B is completely continuous. Let $\{(h_n, h_{n1}, h_{n2})\}_{n=1}^{\infty} \subset X \times \mathbb{R}^2$ be a bounded sequence. We show that $\{B(h_n, h_{n1}, h_{n2})\}_{n=1}^{\infty} \subset Y \times \mathbb{R}^2$ is relatively compact.

Let $B = (B_1, B_2, B_3)$ and $\{B_1(h_n, h_{n1}, h_{n2})\}_{n=1}^{\infty} \subset Y$.

It is sufficient to show that $\{B_1(h_n, h_{n1}, h_{n2})\}_{n=1}^{\infty}$ is a sequence of uniformly bounded and equicontinuous functions. By assumption there is $M > 0$ such that for all $n \in \mathbb{N}$,

$$\|h_n\|_X \leq M, \ |h_{n1}| \leq M, \ |h_{n2}| \leq M.$$

Therefore

$$\|B_1(h_n, h_{n1}, h_{n2})\|_Y \leq \omega^2 m \|h_n\|_Y + |h_{n1}| \|\cos(\cdot)\|_Y + |h_{n2}| \leq (\omega^2 m + 2)M.$$

Let $\theta_1, \theta_2 \in [0, 2\pi]$ and $n \in \mathbb{N}$. We have

$$\begin{aligned}
&| - \omega^2 m h_n(\theta_1) + h_{n1}\cos(\theta_1) + h_{n2} - (-\omega^2 m h_n(\theta_2) + h_{n1}\cos(\theta_2) + h_{n2})| \\
&\leq \omega^2 m |h_n(\theta_1) - h_n(\theta_2)| + |h_{n1}| \, |\cos(\theta_1) - \cos(\theta_2)| \\
&\leq \omega^2 m |h_n'(\xi_{1,2}^n)| \, |\theta_1 - \theta_2| + M|\theta_1 - \theta_2| \leq (\omega^2 m + 1)M|\theta_1 - \theta_2|.
\end{aligned}$$

As $B_2\colon X \times \mathbb{R}^2 \to \mathbb{R}$ and $B_3\colon X \times \mathbb{R}^2 \to \mathbb{R}$ transform bounded sets in $X \times \mathbb{R}^2$ into bounded sets in $\mathbb{R}$, $\{B(h_n, h_{n1}, h_{n2})\}$ is relatively compact, which completes the proof. $\square$

In order to find critical values of bifurcation parameter we have to solve the equation

$$\mathcal{F}_x'(0,p)(h, h_1, h_2) = 0. \tag{19}$$

This equation is equivalent to the following system of three equations:

$$\begin{cases}
-\dfrac{1}{R_p} h''(\theta) - \omega^2 m h(\theta) + h_1\cos(\theta) + h_2 = 0, \\
\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} h(\theta)\cos(\theta) d\theta = 0, \\
\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} h(\theta) d\theta = 0,
\end{cases} \tag{20}$$

where $h \in X$, $h_1, h_2 \in \mathbb{R}$ are unknowns. The evenest of $h(\theta)$ and the second and third equations of the system (20) exclude $h(\theta) = const$, $h(\theta) = \cos(\theta)$. Therefore

$$h(\theta) = \sum_{n=2}^{\infty} a_n \cos(n\theta). \tag{21}$$

Substituting (21) to the first equation of the system (20) and using linear independence of functions $\cos(n\theta)$, $n \in \mathbb{N} \cup \{0\}$, we conclude that $h_1 = 0$, $h_2 = 0$ and

$$-\frac{1}{R_p} h''(\theta) - \omega^2 m h(\theta) = 0.$$

Solving the latter we obtain

$$h(\theta) = C\cos(\theta\sqrt{p+1}), \quad C \in \mathbb{R}, \ \sqrt{p+1} \in \mathbb{N}. \tag{22}$$

Substituting (22) to the second and third equation of (20) we get

$$p = p_k = \frac{k^2}{4} - 1, \quad k \geq 4, \ k \in 2\mathbb{N},$$

and hence

$$h(\theta) = C\cos\left(\frac{k\theta}{2}\right).$$

Set $e_k(\theta) = \left(\sqrt{2}\cos\left(\frac{k\theta}{2}\right), 0, 0\right) \in X \times \mathbb{R}^2$. Then $\langle e_k, e_k\rangle = 1$. By the above,

$$\ker \mathcal{F}'_x(0, p_k) = \left\{\left(C\cos\left(\frac{k\theta}{2}\right), 0, 0\right) : C \in \mathbb{R}\right\} = \{Ce_k \colon C \in \mathbb{R}\},$$

and so

$$\dim\ker \mathcal{F}'_x(0, p_k) = 1. \tag{23}$$

From Lemma 3.6 we conclude that

$$\operatorname{codim}\ \operatorname{im}\ \mathcal{F}'_x(0, p_k) = 1. \tag{24}$$

To finish the proof of Theorem 3.4 it is sufficient to show the lemma below.

Lemma 3.7. *The functional* $\varepsilon \colon X_\delta(0) \times \mathbb{R}^2 \times (\mathbb{R}_+)_\delta(p_k) \to \mathbb{R}$, $k \geq 4$, $k \in 2\mathbb{N}$, *satisfies the condition*

$$\varepsilon'''_{xxp}(0, p_k)e_k e_k \neq 0.$$

Proof. By Lemma 3.3 we get

$$\varepsilon'_x(x, p)(h, h_1, h_2) = \langle \mathcal{F}(x, p), (h, h_1, h_2)\rangle,$$

hence

$$\varepsilon''_{xx}(x, p)(h, h_1, h_2)(g, g_1, g_2) = \langle \mathcal{F}'_x(x, p)(h, h_1, h_2), (g, g_1, g_2)\rangle,$$

and so

$$\varepsilon'''_{xxp}(x, p)(h, h_1, h_2)(g, g_1, g_2) = \left\langle \mathcal{F}''_{xp}(x, p)(h, h_1, h_2), (g, g_1, g_2)\right\rangle,$$

where $x = (\varrho, \lambda_1, \lambda_2) \in X_\delta(0) \times \mathbb{R}^2$, $(h, h_1, h_2), (g, g_1, g_2) \in X \times \mathbb{R}^2$ and $p \in (\mathbb{R}_+)_\delta(p_k)$. Differentiating $\mathcal{F}'_x(0, p)(h, h_1, h_2)$ with respect to p we get

$$\mathcal{F}''_{xp}(0, p)(h, h_1, h_2) = \left(\frac{m\omega^2}{(1+p)^2}h'', 0, 0\right),$$

Finally, substituting $(h, h_1, h_2) = (g, g_1, g_2) = e_k$ and $p = p_k$ to the above, we have

$$\begin{aligned}\varepsilon'''_{xxp}(0, 0, p_k)e_k e_k &= \left\langle \mathcal{F}''_{xp}(0, p_k)e_k, e_k\right\rangle \\ &= \frac{1}{2\pi}\int_0^{2\pi} -\frac{8m\omega^2}{k^2}\cos^2\left(\frac{k\theta}{2}\right)d\theta \\ &= -\frac{4m\omega^2}{k^2} \neq 0,\end{aligned}$$

which completes the proof of Lemma 3.7. ☐

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