Applied Mathematics Letters 28 (2014) 14–19

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Boundary problems for fractional differential equations

Tadeusz Jankowski *

Gdansk University of Technology, Department of Differential Equations and Applied Mathematics, 11/12, G.Narutowicz Str., 80-233 Gdańsk, Poland

ARTICLE INFO

Article history:
Received 31 July 2013
Accepted 10 September 2013

Keywords:
Fractional differential equations with boundary conditions
Monotone iterative method
Lower and upper solutions
Existence of solutions
Mittag-Leffler functions

ABSTRACT

In this paper, the existence of solutions of fractional differential equations with nonlinear boundary conditions is investigated. The monotone iterative method combined with lower and upper solutions is applied. Fractional differential inequalities are also discussed. Two examples are added to illustrate the results.

1. Introduction

This paper discusses the boundary value problem

$$\begin{cases}(D_T^q x)(t) = f(t, x(t)) \equiv (Fx)(t), & t \in J_0 = [0, T),\ T > 0,\\ 0 = g(\bar{x}(0), \bar{x}(T)),\end{cases} \tag{1}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}),\ J = [0, T],\ g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R}),\ \bar{x}(0) = (T-t)^{1-q}x(t)_{|t=0}, \bar{x}(T) = (T-t)^{1-q}x(t)_{|t=T}$ and $D_T^q x$ denotes the right-handed Riemann–Liouville fractional derivative of x with $q \in (0, 1)$ defined by

$$(D_T^q x)(t) = -\frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_t^T (s-t)^{-q}x(s)ds, \quad t \in J_0,$$

and if $q = 1$, then $D_T^1 x(t) = -x'(t)$, see [1].

Let us introduce the right-sided fractional integral I_T^q of order $q > 0$ by

$$\left(I_T^q x\right)(t) = \frac{1}{\Gamma(q)}\int_t^T (s-t)^{q-1}x(s)ds, \quad t \in J_0,$$

see [1].

Function x is a solution of problem (1) if:

(i) $x \in C(J_0, \mathbb{R}),\ (T-t)^{1-q}u \in C(J, \mathbb{R})$ and its fractional integral $I_T^{1-q}x$ is continuously on J_0,
(ii) x satisfies problem (1).

Recently, much attention has been paid to study fractional differential problems with fractional derivatives $D^q x$, using the monotone iterative method, see for example, [1–11]. In this paper, we study fractional differential problems with derivatives

* Tel.: +48 583472262; fax: +48 5 83472 821.
E-mail address: tjank@mif.pg.gda.pl.

0893-9659/$ – see front matter
http://dx.doi.org/10.1016/j.aml.2013.09.004

$D^q_T x$ of order $q \in (0, 1]$. As we see later, we will discuss fractional differential equations with initial conditions at the endpoint of interval J. Therefore, we have to introduce the space C_{1-q} by

$$C_{1-q}(J, \mathbb{R}) = \{u \in C([0, T), \mathbb{R}) : \ (T - t)^{1-q} u \in C(J, \mathbb{R})\}, \quad q \in (0, 1)$$

and $C_0(J, \mathbb{R}) = C(J, \mathbb{R})$ if $q = 1$. Our first result concerns the existence of a unique solution of nonlinear fractional differential equations with initial condition at the point T, see Theorem 1. Theorem 2 gives the unique solution of linear fractional differential problems in terms of the Mittag-Leffler function. In Section 4, we discuss problem (1) giving sufficient conditions which guarantee that problem (1) has extremal solutions, Theorem 3. Two examples illustrate the results.

2. Nonlinear fractional differential equations

Let us consider the following problem

$$(D^q_T x)(t) = (Fx)(t), \quad t \in J_0, \ \bar{x}(T) = k. \tag{2}$$

For $u \in C_{1-q}(J, \mathbb{R})$, we define a weighted norm:

$$\|u\|_* = \max_{[0,T]} (T - t)^{1-q} e^{\lambda(t-T)} |x(t)|$$

with a corresponding fixed positive constant λ.

Theorem 1. *Let $q \in (0, 1)$, $f \in C(J \times \mathbb{R}, \mathbb{R})$. In addition, we assume that:*
H_1: there exists a nonnegative constant K such that

$$|f(t, u_1) - f(t, v_1)| \le K|v_1 - u_1|.$$

Then problem (2) has a unique solution.

Proof. Using Lemma 2.6 [1], it is easy to show that problem (2) is equivalent to the following integral equation:

$$x(t) = k(T - t)^{q-1} + \frac{1}{\Gamma(q)} \int_t^T (s - t)^{q-1} Fx(s) ds. \tag{3}$$

We write Eq. (3) in the form $u = Au$, where A is defined by the right-hand-side of (3). Now, we have to show that operator A has a fixed point. To do it we shall show that A is a contraction map.

Let us choose constants m, r such that $1 < m < \frac{1}{1-q}$ and $r = \frac{m}{m-1}$. Now, we use the norm $\| \cdot \|_*$ with a positive λ such that:

$$\lambda^{\frac{1}{r}} > \frac{K}{\Gamma(q)} T^{q-1+\frac{1}{m}} \left(\frac{\Gamma^2(m(q-1)+1)}{\Gamma(2[m(q-1)+1])} \right)^{\frac{1}{m}} \left(\frac{1}{r} \right)^{\frac{1}{r}} \equiv \rho. \tag{4}$$

Note that

$$B(t) = \int_t^T (s - t)^{m(q-1)} (T - s)^{m(q-1)} ds = \frac{\Gamma^2(m(q-1)+1)}{\Gamma(2[m(q-1)+1])} (T - t)^{2m(q-1)+1},$$

$$D(t) = \int_t^T e^{-\lambda rs} ds < \frac{1}{\lambda r} e^{-\lambda rt}.$$

Then using the Hőlder inequality for integrals

$$\int_t^T |a(s)||b(s)| ds \le \left(\int_t^T |a(s)|^m ds \right)^{\frac{1}{m}} \left(\int_t^T |b(s)|^r ds \right)^{\frac{1}{r}},$$

and assumption H_1, for $x, y \in C_{1-q}(J, \mathbb{R})$ we obtain

$$\begin{aligned}
\|Ax - Ay\|_* &\le \frac{1}{\Gamma(q)} \max_{t\in J} (T - t)^{1-q} e^{\lambda(t-T)} \int_t^T (s - t)^{q-1} |(Fx)(s) - (Fy)(s)| ds \\
&\le \frac{K}{\Gamma(q)} \|x - y\|_* \max_{t\in J} (T - t)^{1-q} e^{\lambda t} \int_t^T (s - t)^{q-1} (T - s)^{q-1} e^{-\lambda s} ds \\
&\le \frac{K}{\Gamma(q)} \|x - y\|_* \max_{t\in J} (T - t)^{1-q} e^{\lambda t} [B(t)]^{\frac{1}{m}} [D(t)]^{\frac{1}{r}} \\
&\le \frac{K}{\Gamma(q)} \left(\frac{\Gamma^2(m(q-1)+1)}{\Gamma(2[m(q-1)+1])} \right)^{\frac{1}{m}} \frac{1}{(\lambda r)^{\frac{1}{r}}} \|x - y\|_* \max_{t\in J} (T - t)^{q-1+\frac{1}{m}} \le \frac{\rho}{\lambda^{\frac{1}{r}}} \|x - y\|_*.
\end{aligned}$$

This and condition (4) prove that problem (2) has a unique solution, by the Banach fixed point theorem. This ends the proof. ■

Remark 1. If $q = 1$, then $(D_T^1 x)(t) = -x'(t)$, and operator A has the form

$$(Ax)(t) = k + \int_t^T (Fx)(s)ds.$$

In this case, $\|x\|_* = \max_{t\in J} e^{\lambda(t-T)}|x(t)|$, so choosing $\lambda > K$, we can show that operator A has a unique fixed point.

3. Linear fractional differential equations

Let us consider the linear fractional differential problem

$$(D_T^q x)(t) = \lambda x(t) + \sigma(t), \quad t \in J_0, \ \bar{x}(T) = k, \tag{5}$$

where λ is a real number and $\sigma \in C_{1-q}(J, T)$.

Theorem 2. *Let $q \in (0, 1]$, $\lambda \in \mathbb{R}$, $\sigma \in C_{1-q}(J, \mathbb{R})$. Then problem (5) has a unique solution given by formula*

$$x(t) = k\Gamma(q)(T-t)^{q-1}E_{q,q}(\lambda(T-t)^q) + \int_t^T (s-t)^{q-1}E_{q,q}(\lambda(s-t)^q)\sigma(s)ds, \tag{6}$$

where $E_{q,q}(\zeta) = \sum_{r=0}^{\infty} \frac{\zeta^r}{\Gamma(q(r+1))}$ is the Mittag-Leffler function.

Proof. Indeed, problem (5) is equivalent in the space $C_{1-q}(J, \mathbb{R})$ to the following fractional integral equation

$$x(t) = x_0(t) + \frac{\lambda}{\Gamma(q)}\int_t^T (s-t)^{q-1}x(s)ds + \frac{1}{\Gamma(q)}\int_t^T (s-t)^{q-1}\sigma(s)ds, \quad t \in J_0, \tag{7}$$

or

$$x(t) = x_0(t) + \lambda(I_T^q x)(t) + (I_T^q \sigma)(t),$$

where

$$x_0(t) = \frac{\bar{k}}{\Gamma(q)}(T-t)^{q-1}, \quad \bar{k} = k\Gamma(q).$$

We apply the method of successive approximations to find the solution of problem (7), so for $n = 0, 1, \ldots,$ we have

$$x_{n+1}(t) = x_0(t) + \lambda(I_T^q x_n)(t) + (I_T^q \sigma)(t).$$

Hence,

$$\begin{aligned} x_1(t) &= x_0(t) + \lambda(I_T^q x_0)(t) + (I_T^q \sigma)(t) \\ &= x_0(t) + \frac{\lambda\bar{k}}{\Gamma^2(q)}\int_t^T (s-t)^{q-1}(T-s)^{q-1}ds + \frac{1}{\Gamma(q)}\int_t^T (s-t)^{q-1}\sigma(s)ds \\ &= \frac{\bar{k}}{\Gamma(q)}(T-t)^{1-q} + \frac{\lambda\bar{k}}{\Gamma(2q)}(T-t)^{2q-1} + \frac{1}{\Gamma(q)}\int_t^T (s-t)^{q-1}\sigma(s)ds, \end{aligned}$$

using the formula

$$\int_t^T (s-t)^{rq-1}(T-s)^{mq-1}ds = \frac{\Gamma(rq)\Gamma(mq)}{\Gamma((r+m)q)}(T-t)^{(r+m)q-1}, \quad r, m \in \mathbb{N} = \{1, 2, \ldots\}.$$

Using this x_1, we find the next approximation x_2 as

$$\begin{aligned} x_2(t) &= x_0(t) + \lambda(I_T^q x_1)(t) + (I_T^q \sigma)(t) \\ &= x_0(t) + \lambda(I_T^q[x_0 + \lambda I_T^q x_0 + I_T^q \sigma])(t) + (I_T^q \sigma)(t) \\ &= x_0(t) + \lambda(I_T^q x_0)(t) + \lambda^2(I_T^{2q} x_0)(t) + \lambda(I_T^{2q}\sigma)(t) + (I_T^q \sigma)(t) \\ &= x_0(t) + \frac{\lambda\bar{k}}{\Gamma(2q)}(T-t)^{2q-1} + \frac{\lambda^2\bar{k}}{\Gamma(3q)}(T-t)^{3q-1} + \frac{1}{\Gamma(q)}\int_t^T (s-t)^{q-1}\sigma(s)ds \\ &\quad + \frac{\lambda}{\Gamma(2q)}\int_t^T (s-t)^{2q-1}\sigma(s)ds \\ &= \bar{k}(T-t)^{q-1}\left[\frac{1}{\Gamma(q)} + \frac{\lambda}{\Gamma(2q)}(T-t)^q + \frac{\lambda^2}{\Gamma(3q)}(T-t)^{2q}\right] \\ &\quad + \int_t^T (s-t)^{q-1}\left[\frac{1}{\Gamma(q)} + \frac{\lambda}{\Gamma(2q)}(s-t)^q\right]\sigma(s)ds, \end{aligned}$$

by the property

$$(I_T^{\alpha} I_T^{\beta} f)(t) = (I_T^{\alpha+\beta} f)(t), \quad \alpha, \beta > 0.$$

Thus, in general, we get by induction x_n as follows

$$x_n(t) = \bar{k}(T-t)^{q-1} \sum_{r=0}^{n} \frac{\lambda^r}{\Gamma(q(r+1))}(T-t)^{rq} + \int_t^T (s-t)^{q-1} \left(\sum_{r=0}^{n-1} \frac{\lambda^r}{\Gamma(q(r+1))}(s-t)^{rq} \right) \sigma(s)ds, \quad n = 0, 1, \ldots,$$

where $\sum_0^{-1} = 0$. Taking the limit as $n \to \infty$, we obtain the solution x in terms of Mittag-Leffler's function given by formula (6). ■

Remark 2. Let $q = 1$. Then problem (5) takes the form

$$-x'(t) = \lambda x(t) + \sigma(t), \quad t \in J_0, \; x(T) = k.$$

Since $E_{1,1}(t) = e^t$, then, in view of (6), the solution of this problem is given by

$$\begin{aligned} x(t) &= ke^{\lambda(T-t)} + \int_t^T e^{\lambda(s-t)} \sigma(s)ds \\ &= e^{\lambda(T-t)} \left[k + \int_t^T e^{-\lambda(T-s)} \sigma(s)ds \right], \quad t \in J. \end{aligned}$$

Example 1. For $q \in (0, 1]$, let us consider the following problem

$$\begin{cases} (D_T^q x)(t) = x(t) + (T-t)^{2-q} \left[\dfrac{\Gamma(3)}{\Gamma(3-q)} - (T-t)^q \right], & t \in J_0 = [0, T), \\ \bar{x}(T) = 0. \end{cases} \tag{8}$$

Comparing this problem with (5) we see that

$$\lambda = 1, \qquad \sigma(t) = (T-t)^{2-q} \left[\frac{\Gamma(3)}{\Gamma(3-q)} - (T-t)^q \right], \qquad k = 0.$$

In view of Theorem 2, problem (8) has a unique solution given by

$$x(t) = \int_t^T (s-t)^{q-1} E_{q,q}((s-t)^q) \sigma(s)ds,$$

so

$$\begin{aligned} x(t) &= \int_t^T (s-t)^{q-1} \sum_{n=0}^{\infty} \frac{(s-t)^{qn}}{\Gamma((n+1)q)} (T-s)^{2-q} \left[\frac{\Gamma(3)}{\Gamma(3-q)} - (T-s)^q \right] ds \\ &= \frac{\Gamma(3)}{\Gamma(3-q)} \sum_{n=0}^{\infty} \frac{1}{\Gamma((n+1)q)} \int_t^T (s-t)^{q(n+1)-1} (T-s)^{2-q} ds \\ &\quad - \sum_{n=0}^{\infty} \frac{1}{\Gamma((n+1)q)} \int_t^T (s-t)^{q(n+1)-1} (T-s)^2 ds \\ &= \frac{\Gamma(3)}{\Gamma(3-q)} \sum_{n=1}^{\infty} \frac{1}{\Gamma((n+1)q)} (T-t)^{qn+2} \frac{\Gamma((n+1)q)\Gamma(3-q)}{\Gamma(qn+3)} \\ &\quad - \sum_{n=0}^{\infty} \frac{1}{\Gamma((n+1)q)} (T-t)^{q(n+1)+2} \frac{\Gamma((n+1)q)\Gamma(3)}{\Gamma(q(n+1)+3)} \\ &= \Gamma(3) E_{q,3}((T-t)^q)(T-t)^2 - (T-t)^2 \left[E_{q,3}((T-t)^q) - \frac{1}{\Gamma(3)} \right] \Gamma(3) \\ &= (T-t)^2, \end{aligned}$$

where $E_{q,r}$ is the Mittag-Leffler function defined by

$$E_{q,r}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(qn+r)}.$$

It proves that $x(t) = (T-t)^2$ is the unique solution of problem (8).

4. Extremal solutions to problem (1)

Let us introduce the following definition.

We say that $u \in C_{1-q}(J, \mathbb{R})$ is a lower solution of problem (1) if

$$(D_T^q u)(t) \le (Fu)(t), \quad t \in J, \qquad g(u(0), u(T)) \ge 0,$$

and it is an upper solution of (1) if the above inequalities are reversed.

Now, we give a result for fractional differential inequalities.

Lemma 1. *Let $q \in (0, 1]$, $M \in \mathbb{R}$. Assume that $p \in C_{1-q}(J, \mathbb{R})$ satisfies problem*

$$\begin{cases} (D_T^q p)(t) \le -Mp(t), & t \in J_0, \\ \bar{p}(T) \le 0. \end{cases}$$

Then $p(t) \le 0$, $t \in J$.

Proof. Let $(D_T^q p)(t) = -Mp(t) + \sigma(t)$, $t \in J_0$, $\bar{p}(T) = c$, where $\sigma(t) \le 0$, $c \le 0$. Then, in view of formula (6), we have $p(t) \le 0$, $t \in J$. This ends the proof. ■

The next result concerns the case when problem (1) has extremal solutions.

Theorem 3. *Assume that $q \in (0, 1]$, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. Let $y_0, z_0 \in C_{1-q}(J, \mathbb{R})$ be lower and upper solutions of (1), respectively and $y_0(t) \le z_0(t)$, $t \in J$. In addition, we assume that*

H_3: there exists a constant M such that

$$f(t, u_1) - f(t, v_1) \le M[v_1 - u_1]$$

if $y_0(t) \le u_1 \le v_1 \le z_0(t)$,

H_4: there exists a constant $\nu > 0$ such that

$$g(v, u) - g(v_0, u_0) \le \nu(u_0 - u)$$

if $\bar{y}_0(T) \le u \le u_0 \le \bar{z}_0(T)$, $\bar{y}_0(0) \le v \le v_0 \le \bar{z}_0(0)$.

Then problem (1) has extremal solutions in the sector

$$[y_0, z_0]_* = \{w \in C_{1-q}(J, \mathbb{R}) : y_0(t) \le w(t) \le z_0(t),\ t \in J\}.$$

Proof. For $n = 0, 1, \ldots,$ let us define

$$\begin{aligned} &(D_T^q y_{n+1})(t) = (Fy_n)(t) - M[y_{n+1}(t) - y_n(t)], \quad t \in J_0, \\ &0 = g(\bar{y}_n(0), \bar{y}_n(T)) - \nu[\bar{y}_{n+1}(T) - \bar{y}_n(T)], \\ &(D_T^q z_{n+1})(t) = (Fz_n)(t) - M[z_{n+1}(t) - z_n(t)], \quad t \in J_0, \\ &0 = g(\bar{z}_n(0), \bar{z}_n(T)) - \nu[\bar{z}_{n+1}(T) - \bar{z}_n(T)]. \end{aligned}$$

In view of Theorem 1, functions y_1, z_1 are well defined. First, we show that

$$z_0(t) \le z_1(t) \le y_1(t) \le y_0(t), \quad t \in J. \tag{9}$$

Put $p = y_0 - y_1$. This and the assumption that y_0 is a lower solution of problem (1) yield

$$\begin{aligned} &(D_T^q p)(t) \le (Fy_0)(t) - (Fy_0)(t) + M[y_1(t) - y_0(t)] = -Mp(t), \\ &0 = g(\bar{y}_0(0), \bar{y}_0(T)) - \nu[\bar{y}_1(T) - \bar{y}_0(T)] \ge \nu\bar{p}(T). \end{aligned}$$

Hence, $y_0(t) \le y_1(t)$, $t \in J$, by Lemma 1. By a similar way, we can show that $z_1(t) \le z_0(t)$, $t \in J$. Now, we put $p = y_1 - z_1$. Hence, in view of assumptions H_4, H_3 we have

$$\begin{aligned} &0 = g(\bar{y}_0(0), \bar{y}_0(T)) - g(\bar{z}_0(0), \bar{z}_0(T)) - \nu[\bar{y}_1(T) - \bar{y}_0(T) - \bar{z}_1(T) + \bar{z}_0(T)] \le -\nu\bar{p}(T), \\ &(D_T^q p)(t) = (Fy_0)(t) - (Fz_0)(t) - M[y_1(t) - y_0(t) - z_1(t) + z_0(t)] \le -Mp(t). \end{aligned}$$

This and Lemma 1 prove that $y_1(t) \le z_1(t)$, $t \in J$, so relation (9) holds.

In the next step, we show that y_1, z_1 are lower and upper solutions of problem (1), respectively. Note that

$$\begin{aligned} (D_T^q y_1)(t) &= (Fy_0)(t) - (Fy_1)(t) + (Fy_1)(t) - M[y_1(t) - y_0(t)] \\ &\le M[y_1(t) - y_0(t)] - M[y_1(t) - y_0(t)] + (Fy_1)(t) = (Fy_1)(t), \end{aligned}$$

$$\begin{aligned} 0 &= g(\bar{y}_0(0), \bar{y}_0(T)) - g(\bar{y}_1(0), \bar{y}_1(T)) + g(\bar{y}_1(0), \bar{y}_1(T)) - \nu[\bar{y}_1(T) - \bar{y}_0(T)] \\ &\le \nu[\bar{y}_1(T) - \bar{y}_0(T)] + g(\bar{y}_1(0), \bar{y}_1(T)) - \nu[\bar{y}_1(T) - \bar{y}_0(T)] = g(\bar{y}_1(0), \bar{y}_1(T)) \end{aligned}$$

by assumptions H_3, H_4. This proves that y_1 is a lower solution of problem (1). Similarly, we can show that z_1 is an upper solution of (1).

Using the mathematical induction, we can show that

$$y_0(t) \le y_1(t) \le \cdots \le y_n(t) \le y_{n+1}(t) \le z_{n+1}(t) \le z_n(t) \le \cdots \le z_1(t) \le z_0(t)$$

for $t \in J$ and $n = 1, 2, \ldots$. Employing standard arguments we see that the sequences $\{y_n, z_n\}$ converge to their limit functions y, z, respectively. Indeed, y and z are solutions of problem (1) and $y_0(t) \le y(t) \le z(t) \le z_0(t)$ on J.

To show that y, z are the minimum and maximum solutions of (1) we have to prove that if $u \in [y_0, z_0]_*$ is any solution of (1), then $y(t) \le u(t) \le z(t)$ on J. To do this, we assume that $y_m(t) \le u(t) \le z_m(t), t \in J$ for some m. Let $p = y_{m+1} - u,\ P = u - z_{m+1}$. Then, in view of assumptions H_3, H_4, we can prove that:

$$\begin{aligned}
&(D_T^q p)(t) = (Fy_m)(t) - (Fu)(t) - M[y_{m+1}(t) - y_m(t)] \le -Mp(t),\\
&(D_T^q P)(t) = (Fu)(t) - (Fz_m)(t) + M[z_{m+1}(t) - z_m(t)] \le -MP(t),\\
&0 = g(\bar{y}_m(0), \bar{y}_m(T)) - \nu[\bar{y}_{m+1}(T) - \bar{y}_m(T)] - g(\bar{u}(0), \bar{u}(T)) \le -\nu p(T),\\
&0 = g(\bar{u}(0), \bar{u}(T)) - g(\bar{z}_m(0), \bar{z}_m(T)) + \nu[\bar{z}_{m+1}(T) - \bar{z}_m(T)] \le -\nu P(T).
\end{aligned}$$

This and Lemma 1 show $y_{m+1}(t) \le u(t) \le z_{m+1}(t),\ t \in J$, so by induction, $y_n(t) \le u(t) \le z_n(t)$ on J for all n. Taking the limit as $n \to \infty$, we conclude $y(t) \le u(t) \le z(t),\ t \in J$. ■

Example 2. Consider the following problem

$$\begin{cases}(D_1^q x)(t) = \dfrac{(1-t)^{-q}}{\Gamma(1-q)} + A[1 - t - x(t)]^3 \equiv (Fx)(t), \quad t \in J_0 = [0, 1),\\ 0 = \bar{x}(1)[1 - \bar{x}(0)] \equiv g(\bar{x}(0), \bar{x}(1)),\end{cases} \tag{10}$$

where $q \in (0, 1), A > 0, t \in J = [0, 1]$.

Let $y_0(t) = 0,\ z_0(t) = 2 - t,\ t \in J$. Then $\bar{y}_0(0) = \bar{y}_0(1) = 0,\ \bar{z}_0(0) = 2,\ \bar{z}_0(1) = 0$. Indeed, y_0 is a lower solution of (10). Moreover,

$$(Fz_0)(t) = \frac{(1-t)^{-q}}{\Gamma(1-q)} - A \le \frac{(1-t)^{-q}}{\Gamma(1-q)} + \frac{(1-t)^{1-q}}{\Gamma(2-q)} = (D_1^q z_0)(t), \qquad g(\bar{z}_0(0), \bar{z}_0(1)) = 0.$$

Note that $M = 3A$, $\nu = 1$. Obviously, z_0 is an upper solution of problem (10).

By Theorem 3, problem (10) has extremal solutions in the region $[y_0, z_0]_*$.

References

[1] A.A. Kilbas, H.R. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, in: North-Holland Mathematics Studies, vol. 204, Elsevier, Amsterdam, 2006.
[2] T. Jankowski, Fractional differential equations with deviating arguments, Dynam. Systems Appl. 17 (2008) 677–684.
[3] T. Jankowski, Fractional equations of Volterra type involving a Riemann–Liouville derivative, Appl. Math. Lett. 26 (2013) 344–350.
[4] T. Jankowski, Initial value problems for neutral fractional differential equations involving a Riemann Liouville derivative, Appl. Math. Comput. 219 (2013) 7772–7776.
[5] V. Lakshmikantham, S. Leela, J. Vasundhara, Theory of Fractional Dynamic Systems, Cambridge Academic Publishers, Cambridge, 2009.
[6] L. Lin, X. Liu, H. Fang, Method of upper and lower solutions for fractional differential equations, Electron. J. Differential Equations (100) (2012) 1–13.
[7] F.A. McRae, Monotone iterative technique and existence results for fractional differential equations, Nonlinear Anal. 71 (2009) 6093–6096.
[8] J.D. Ramirez, A.S. Vatsala, Monotone iterative technique for fractional differential equations with periodic boundary conditions, Opuscula Math. 29 (2009) 289–304.
[9] Z. Wei, G. Li, J. Che, Initial value problems for fractional differential equations involving Riemann–Liouville sequential fractional derivative, J. Math. Anal. Appl. 367 (2010) 260–272.
[10] G. Wang, Monotone iterative technique for boundary value problems of a nonlinear fractional differential equations with deviating arguments, J. Comput. Appl. Math. 236 (2012) 2425–2430.
[11] S. Zhang, Monotone iterative method for initial value problem involving Riemann–Liouville fractional derivatives, Nonlinear Anal. 71 (2009) 2087–2093.