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Bounded solutions of odd nonautonomous ODE

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Abstract

We use a Borsuk-Ulam type argument in order to prove existence of nontrivial bounded solutions to some nonautonomous differential equations, which are odd with respect to the spatial variable.

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1. Introduction

The main purpose of this short note is to present a simple topological argument for the existence of fully bounded solutions of some nonautonomous differential equations satisfying symmetry conditions. We can consider a process defined by the equation in the extended phase space. Since invariant sets are noncompact in general, we propose to define another topologically equivalent dynamical system which can be extended to a compact space. To this end, we use an old idea coming from H. Poincaré [9] which has been applied to some planar systems. This technique has been used by E. González-Velasco in [5] in order to study the properties of polynomial vector fields at infinity.

Simple symmetry properties (like oddness) of a vector field (the right-hand side of a system of differential equations) yield analogous symmetries of the induced flow in phase space. Our main topological argument here is the Borsuk-Ulam theorem in the version, which asserts that there are no continuous equivariant maps between the spheres S^n and S^k with the standard free antipodal actions of $\mathbb{Z}_2$ whenever $n > k$. We consider only this simplest case of the theorem here, because such symmetry of a vector field is almost trivial to verify. However, some more sophisticated properties can be used in connection with the antipodal type actions (comp. e.g. [4]). Those properties have been applied successfully to some PDE's, see e.g. [14].

In this article we assume some basic knowledge of the theory of dynamical systems and their properties which can be found in many textbooks (see e.g. [8]). We briefly recall only some necessary notation.

A *dynamical system* (a flow) in a space X is a continuous map $g : X \times \mathbb{R} \to X$ such that $g(x,0) = x$ for all $x \in X$ and $g(g(x,t),s) = g(x,t+s)$ for all $t, s \in \mathbb{R}$, $x \in X$. Usually (in $\mathbb{R}^n$) a flow is induced by a vector field by means of solving Cauchy initial problems, which satisfy standard existence and uniqueness assumptions. Subset $A \subset X$ is called *invariant* if for every $x \in A$ also $g(x,t) \in A$ for all $t \in \mathbb{R}$ (one can consider also either positively or negatively invariant sets defined in a natural way). The set $\{g(x,t) \mid t \in \mathbb{R}\}$ is called the *orbit* of the point x. An *ω-limit set* of x is the set of all limits of convergent sequences $g(x,t_n)$ with $t_n \to +\infty$. Similarly one defines an α-limit set with $t_n \to -\infty$. Two dynamical systems on X and Y are *topologically equivalent* if there exists a homeomorphism $h : X \to Y$ which maps orbits onto orbits (with the same direction of time).

2. Poincaré compactification

In this section we recall briefly a procedure described in [1], [5], following the ideas of H. Poincaré [9], which shows how a polynomial vector field in $\mathbb{R}^n$ gives rise to a vector field in a sphere $S^n \subset \mathbb{R}^{n+1}$ by means of central projection.

Let us consider a polynomial vector field $X = (P_1, P_2, ..., P_n)$ generating a flow in $\mathbb{R}^n$. We can identify $\mathbb{R}^n$ with the hyperplane $\Pi = \{y \in \mathbb{R}^{n+1} \mid y_{n+1} = 1\}$ in $\mathbb{R}^{n+1}$, tangent to the unit sphere $S^n = \{y \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} y_i^2 = 1\}$ at the north pole $(0, ..., 0, 1)$. Denote by S^n_+ and S^n_- the open northern and southern hemisphere respectively. Define the following two diffeomorphisms $\Phi^+ : \mathbb{R}^n \to S^n_+$ and $\Phi^- : \mathbb{R}^n \to S^n_-$ by the formula $\Phi^\pm(x) = \pm\frac{1}{\Delta(x)}(x_1, x_2, ..., x_n, 1)$, where $\Delta(x) = (1 + \sum_{i=1}^n x_i^2)^{\frac{1}{2}}$. Thus we can define a vector field Y in $S^n_+ \cup S^n_-$ by $Y(y) = D\Phi^\pm_x X(x)$ whenever $y = \Phi^\pm(x)$. Now let k denote the maximum of the degrees of the polynomials P_i. The following theorem has been proved in [5](see also [1]).

Theorem 2.1. *The vector field Y can be extended analytically to the whole sphere S^n after multiplication by the factor y_{n+1}^{k-1} and in such a way that the equator $S^{n-1} = \{y \in S^n \mid y_{n+1} = 0\}$ is invariant.*

The above theorem has been used to investigate the behavior of the polynomial vector field at infinity. An important information from the proof of Theorem 2.1 in [5] is that we can consider also a smooth vector field with polynomial growth and we obtain a smooth vector field in the sphere. Let us make this observation more precise.

Definition 2.2. *A C^1-smooth vector field $X : \mathbb{R}^n \to \mathbb{R}^n$ is said to be asymptotically polynomial of degree k if there exists a polynomial vector field $P = (P_1, P_2, ..., P_n) : \mathbb{R}^n \to \mathbb{R}^n$ with $k = \max(degP_i)$ and such that for every $\varepsilon > 0$ there exists a constant $K > 0$ such that*

$$\sup_{||x|| \geq K} ||X(x) - P(x)|| + \sup_{||x|| \geq K} ||DX(x) - DP(x)|| \leq \varepsilon.$$

The second norm in the above definition can be understood as the Euclidean norm of the Jacobi matrix.

Theorem 2.3. *The vector field Y obtained by the above procedure applied to asymptotically polynomial C^1-smooth vector field X of degree k can be extended smoothly to the whole sphere S^n after multiplication by the factor y_{n+1}^{k-1} and in such a way that the equator $S^{n-1} = \{y \in S^n \mid y_{n+1} = 0\}$ is invariant.*

Proof: We use Definition 2.2 for a given vector field X and a fixed polynomial vector field P. In the first step we observe that for a given constant K we can find $L > K$ and a smooth partition of unity $\{p_1, p_2\}$ subordinate to the open covering $U_1 = \{t \in \mathbb{R} \mid |t| < L\}, U_2 = \{t \in \mathbb{R} \mid |t| > K\}$ of $\mathbb{R}$. We can assume that the derivatives are bounded $|p_i'(t)| \leq 1$ for all $t \in \mathbb{R}$. Then we define a mixed vector field $\overline{X}$ given by $\overline{X}(x) = p_1(||x||)X(x) + p_2(||x||)P(x)$.

The following formula has been carefully calculated in [5] for the induced vector field Y in the unit sphere: We have $2(n+1)$ coordinate neighborhoods on the sphere $V_i = \{y \in S^n \mid y_i > 0\}$, $V_i' = \{y \in S^n \mid y_i < 0\}$, and the corresponding coordinate maps $\varphi_i : V_i \to \mathbb{R}^n$ are given by

$$\varphi_i(y) = \frac{1}{y_i}(y_1, ..., \check{y}_i, ..., y_{n+1}),$$

where the symbol $\check{y}_i$ indicates that the ith element is deleted, and similarly for V_i'. Thus for $y \in V_i$ we calculate that for j, s with $1 \leq j \leq i, i \leq s \leq k$,

$$Y(y) = D\Phi_x^{\pm} X(x) = \left(\frac{y_{n+1}}{y_i}\right)^2 \left[..., \frac{y_i}{y_{n+1}}\overline{X}_j(\tilde{y}) - \frac{y_j}{y_{n+1}}\overline{X}_i(\tilde{y}), ...,\right.$$

$$\frac{y_i}{y_{n+1}}\overline{X}_s(\tilde{y}) - \frac{y_{s+1}}{y_{n+1}}\overline{X}_i(\tilde{y}), ..., -\overline{X}_i(\tilde{y})\Bigg],$$

where $\tilde{y} = (1/y_{n+1})(y_1, ..., y_n)$. The same formula is valid for $y \in V_i'$. Observe that the above formula makes sense only outside of the equator $S^{n-1} = \{y \in S^n \mid y_{n+1} = 0\}$. But in a sufficiently thin neighborhood of the equator (i.e. if y_{n+1} is small enough) the norm of $\tilde{y}$ becomes greater than L and therefore the components $\overline{X}_i(\tilde{y}) = P_i(\tilde{y})$ are polynomials. By Theorem 2.1 the vector field Y, when it is multiplied by the factor y_{n+1}^{k-1}, has an extension to the whole sphere S^n. It is given by the above formula because the terms with y_{n+1} disappear from all of the denominators in the reduction process. The extended vector field is therefore C^1-smooth as soon as $\overline{X}$ is C^1-smooth.

Now we can consider a sequence of constants $K_r \to \infty$ and repeat the first step. In this way we obtain a sequence of constants $L_r \to \infty$ and vector fields Y_r defined in the sphere. Observe that the polynomial vector field P is the same for all $r \in \mathbb{N}$ and thus all of the extended vector fields are equal on the equator. The desired C^1-smooth vector field Y is the point-limit of Y_r as $r \to \infty$. □

Corollary 2.4. *Given a flow in $\mathbb{R}^n$ defined by a C^1-smooth vector field X with a polynomial growth, there exists a flow Y defined on the closed unit disc $D^n = \{y \in \mathbb{R}^n \mid \|y| \leq 1\}$ such that the open disc and the boundary are invariant sets and the restriction of the flow generated by Y to the open disc is topologically equivalent to the original flow in $\mathbb{R}^n$.*

Proof: By means of the projection $\pi(y_1, y_2, ..., y_{n+1}) = (y_1, y_2, ..., y_n)$ we obtain a homeomorphism $h : S_+^n \to intD^n$, which gives the equivalence of the flows: the one induced by the vector field from Theorem 2.3 in the open northern hemisphere and the second induced by its projection onto the open unit disc in $\mathbb{R}^n$. On the other hand, the flow in the open hemisphere is topologically equivalent to the original flow on $\mathbb{R}^n$ because of the diffeomorphism Φ^+ and the multiplication by a positive analytic factor y_{n+1}^{k-1} (comp. e.g. [8]). This extends to the closed hemisphere because π is an identity on the equator. □

3. *G*-spaces and *G*-index

We use here the Čech cohomology functor H^* since it satisfies the continuity property (if a cohomology class vanishes on a closed set, then it vanishes on a neighborhood of this set). The group $\mathbb{Z}_2$ of integers mod 2 will be used as a coefficient group in cohomology.

Let G be the group $\mathbb{Z}_2$. Assume that G acts freely on a paracompact space X (it means $gx \neq x$ for all $x \in X$ if $g \neq e$). We call such a space X a *G-space*. Every such G-space admits an equivariant map $h : X \to EG$ into a classifying space EG; moreover every such two maps are homotopic (see [2] Thm 8.12 and Thm 6.14). The map h induces a map of the orbit spaces $\hat{h} : X/G \to BG := EG/G$. Consequently we obtain a uniquely determined homomorphism of their cohomology groups

$$\hat{h}^* : H^*(BG, \mathbb{Z}_2) \to H^*(X/G, \mathbb{Z}_2).$$

This schedule works in general, but in our special case $G = \mathbb{Z}_2$ the space EG can be identified with the infinite-dimensional sphere S^∞ and the standard free antipodal action of G in it. The orbit space BG is the infinite-dimensional real projective space P^∞.

The concept of G-index for a free $\mathbb{Z}_2$-space has been introduced by C.T. Yang in [15]. Then it was extended to other groups (see e.g. [12], [6]). We recall the definition for $\mathbb{Z}_2$ from [12].

Definition 3.1. *We say that the G-index of X is not less than k if the homomorphism $\hat{h}^k : H^k(BG, \mathbb{Z}_2) \to H^k(X/G, \mathbb{Z}_2)$ is a monomorphism.*

Most of the properties of the G-index are immediate consequences of the definition. In particular, we have:

- (*Monotonicity*):

  If X, Y are free G-spaces and $f : X \to Y$ is an equivariant map, then $\mathrm{ind}_G Y \geq \mathrm{ind}_G X$.

- (*Dimension*):

  If $\dim X < m$, then $\mathrm{ind}_G X < m$, where dim denotes the topological covering dimension.

  An important special case of the above is the following:

- If $\mathrm{ind}_G X \geq 0$, then $X \neq \emptyset$.

  The next property is a version of the classical Borsuk-Ulam theorem:

- (*Borsuk-Ulam property*):

  A sphere S^n with the standard antipodal action is a G-space and moreover $\mathrm{ind}_G S^n = n$.

  The following one is a consequence of the continuity property of the Čech cohomology theory:

- (*Continuity property*):

  Let G act freely on X and let $A \subset X$ be a compact G-subspace. Then there is an open neighborhood U of A in X which is a G-space and $\mathrm{ind}_G U = \mathrm{ind}_G A$.

4. Nonautonomous systems

Since we would like to illustrate only the simple geometric idea involved, we do not try to formulate the most general possible result here.

Let us consider a nonautonomous asymptotically linear system of ordinary differential equations in $\mathbb{R}^n$:

$$x'(t) = A(t)(x(t)) + F(x,t), \tag{1}$$

where $A : \mathbb{R} \to M_{n\times n}$ is a continuous map to the space of square matrices, $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a nonlinear map which satisfies appropriate conditions for the uniqueness and existence of global solutions to the Cauchy initial problems. Here we assume for simplicity that F is smooth.

Additionally we assume the following:

- (A1) F is *odd* in x, i.e. $F(-x,t) = -F(x,t)$ for all $x \in, \mathbb{R}^n, t \in \mathbb{R}$;
- (A2) $\lim_{t\to\pm\infty} F(x,t) = 0$ uniformly in x;
- (A3) the growth of F in x is at most polynomial (of the same degree for all t for simplicity);
- (A4) $\lim_{t\to\pm\infty} A(t) = A_{\pm}$;
- (A5) the matrices A_+, A_- are hyperbolic, i.e. they have no eigenvalues with 0 real part.

Let us denote by k the number of eigenvalues λ of A_- with negative real part and similarly let l be the number of eigenvalues of A_+ with negative real part. It is clear that $x \equiv 0$ is a (trivial) solution of the equation (1), because it follows from (A1) that $F(0,t) = 0$. Our main result is the following:

Theorem 4.1. *Assume (A1)-(A5) and let $k < l$. Then the equation (1) has nontrivial bounded solutions.*

Proof: It is well known that the equation (1) determines a process in the extended phase space $\mathbb{R}^n \times \mathbb{R}$, which is induced by the vector field $X(x,t) = (A(t)x + F(x,t), 1)$. In the terminology of [10] and [13] it is called a skew-product flow.

Now we can apply the procedure from Section 2 in order to obtain a flow in a compact space. We can do this in two steps. First we apply it to a "frozen" vector field in $\mathbb{R}^n$ given by $x \mapsto A(t)x + F(x,t)$ for every fixed t. In this manner we obtain a vector field in the closed unit disc D^n in $\mathbb{R}^n$ for every t. Then for each fixed x we apply the similar procedure to the constant vector field on $\mathbb{R}$. More explicitly, we can multiply this constant vector field by a smooth, even and monotone in $\mathbb{R}^+$, positive-valued function $k(t)$ such that $k(0) = 1$ and $\lim_{t\to\infty} k(t) = 0$ (e.g. $k(t) = \exp(-|t|)$), and then apply the procedure from Section 2. In this way we obtain a skew-product flow defined in a solid cylinder $D^n \times [-1, +1]$. This flow φ, as considered in the interior of the cylinder, is topologically equivalent to the original one in $\mathbb{R}^{n+1}$.

We have a natural antipodal action of the group $\mathbb{Z}_2$ on the cylinder, given by $(x,t) \mapsto (-x,t)$. Since the central projection used in Section 2 is an odd map, the constructed flow φ is equivariant, i.e. $\varphi(-x,t) = -\varphi(x,t)$ for all $x \in D^n, t \in [-1,1]$.

One easily observes that the sets

$$D^n \times \{-1\},\ D^n \times \{1\},\ \partial D^n \times [-1,1],\ \{0\} \times [-1,1]$$

are invariant sets with respect to the flow φ, which has been described above. Observe also that the ω-limit set of every point (x,t) with $t > -1$ is contained in the set $D^n \times \{1\}$ because of our assumptions.

Suppose that all nontrivial solutions of (1) are unbounded.

Considering the flow restricted to the invariant set $D^n \times \{-1\}$, the dimension of the unstable subspace of the hyperbolic stationary point is equal to $n-k$. Thus we consider a sphere $S_\varepsilon = S^{n-k-1}$ of a small positive radius ε, centered at 0, in the unstable subspace.

Now let $S^{n-k-1} \times \{-1+\delta\}$ be a small sphere on a bit higher level. An orbit of every point from this set has to approach a neighborhood V of a sphere $S^{n-l-1} \times \{1\} \subset \partial D^n \times \{1\}$ (a sphere in the unstable subspace) as soon as t is sufficiently close to 1. Since the sphere is compact, we can find the common value of t_0 such that for all points $x \in S^{n-k-1}$ we have $\varphi(x, t_0) \in V$. Of course we can choose a $\mathbb{Z}_2$-invariant neighborhood V which is arbitrarily close to $S^{n-l-1} \times \{1\}$. Therefore the formula $\beta(x, -1+\delta) = \varphi(x, t_0)$ defines an equivariant map

$$\beta : S^{n-k-1} \times \{-1+\delta\} \to V.$$

By the continuity property of the G-index, $ind_G V = n - l - 1$. Thus from the monotonicity we obtain the following inequality

$$n - k - 1 \leq n - l - 1,$$

and therefore $l \leq k$, contrary to our assumption. This contradiction ends the proof. □

Let us remark that in the special case of linear nonautonomous systems the result has been proved in [3] and it seems to be complementary to some results from [10].

As an example of application of the Conley index techniques in [7] (comp. also [11]) the following planar Fourier–Taylor polynomial equations were considered

$$z' = \frac{1}{p+1} iz + e^{it}\overline{z}^p + e^{ikt}\overline{z}^q$$

for $z \in \mathbb{C}$. The authors in [7] gave some sufficient conditions for existence of periodic solutions as well as for merely bounded ones.

Observe that if both p and q are odd, then the right-hand side vector field is odd. However, our assumptions A1-A5 are not satisfied here. Let us modify the equation.

Example 4.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\lim_{t\to-\infty} f(t) = 0, \lim_{t\to\infty} f(t) = 1$ and let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function of compact support. Then the planar equation*

$$z' = f(t)z + f(-t)\overline{z} + g(t)(e^{it}\overline{z}^p + e^{ikt}\overline{z}^q)$$

satisfies the assumptions of Theorem 4.1 whenever p, q are odd.

In fact, one can also consider a function g such that $\lim_{t\to\pm\infty} g(t) = 0$. Examples in higher dimensional spaces can be easily produced. Let us observe that the schedule of our proof is rather straightforward. It seems possible to repeat it in the case of differential inclusions whenever the solution sets of Cauchy problems are good enough to produce a generalized (multivalued) dynamical system. Some versions of the Borsuk-Ulam theorem has been proved to be true for admissible multivalued maps in the sense of Górniewicz (see e.g. [4]) and may be applicable in this context. We do not claim that our arguments are the most powerful. A kind of the Conley index theory may be also used here (see [13]). Typical assumptions are that the considered system of equations is some kind of perturbation of an autonomous linear system and then a continuation property of the Conley index is being applied.

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