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Combinatorics

## Bounds on the vertex–edge domination number of a tree



*Bornes sur le nombre de domination sommet–arête d'un arbre*

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### ABSTRACT

A vertex–edge dominating set of a graph  $G$  is a set  $D$  of vertices of  $G$  such that every edge of  $G$  is incident with a vertex of  $D$  or a vertex adjacent to a vertex of  $D$ . The vertex–edge domination number of a graph  $G$ , denoted by  $\gamma_{ve}(G)$ , is the minimum cardinality of a vertex–edge dominating set of  $G$ . We prove that for every tree  $T$  of order  $n \geq 3$  with  $l$  leaves and  $s$  support vertices, we have  $(n - l - s + 3)/4 \leq \gamma_{ve}(T) \leq n/3$ , and we characterize the trees attaining each of the bounds.

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### RÉSUMÉ

Un ensemble sommet–arête dominant d'un graphe  $G$  est un ensemble  $D$  de sommets de  $G$  tel que chaque arête de  $G$  soit incidente à un sommet de  $D$  ou à un sommet adjacent à un sommet de  $D$ . Le nombre de domination sommet–arête d'un graphe  $G$ , noté  $\gamma_{ve}(G)$ , est le cardinal minimum d'un ensemble sommet–arête dominant de  $G$ . Nous prouvons que, pour chaque arbre  $T$  d'ordre  $n \geq 3$  avec  $l$  feuilles et des sommets  $s$  de soutien, que nous avons  $(n - l - s + 3)/4 \leq \gamma_{ve}(T) \leq n/3$ , et nous caractérisons les arbres atteignant chacune des limites.

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## 1. Introduction

Let  $G = (V, E)$  be a graph. By the neighborhood of a vertex  $v$  of  $G$  we mean the set  $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ . The degree of a vertex  $v$ , denoted by  $d_G(v)$ , is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). We denote by  $P_n$  the path on  $n$  vertices. Let  $T$  be a tree, and let  $v$  be a vertex of  $T$ . We say that  $v$  is adjacent to a path  $P_n$  if there is a neighbor of  $v$ , say  $x$ , such that the subtree

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resulting from  $T$  by removing the edge  $vx$  and which contains the vertex  $x$  as a leaf, is a path  $P_n$ . By a star we mean a connected graph in which exactly one vertex has degree greater than one.

A subset  $D \subseteq V(G)$  is a dominating set of  $G$  if every vertex of  $V(G) \setminus D$  has a neighbor in  $D$ . The domination number of  $G$  is the minimum cardinality of a dominating set of  $G$ . For a comprehensive survey of domination in graphs, see [2,3].

An edge  $e \in E(G)$  is vertex-edge dominated by a vertex  $v \in V(G)$  if  $e$  is incident to  $v$ , or  $e$  is adjacent to an edge incident to  $v$ . A subset  $D \subseteq V(G)$  is a vertex-edge dominating set, abbreviated VEDS, of  $G$  if every edge of  $G$  is vertex-edge dominated by a vertex of  $D$ . The vertex-edge domination number of  $G$ , denoted by  $\gamma_{ve}(G)$ , is the minimum cardinality of a vertex-edge dominating set of  $G$ . A vertex-edge dominating set of  $G$  of minimum cardinality is called a  $\gamma_{ve}(G)$ -set. Vertex-edge domination in graphs was introduced in [7], and further studied in [6].

Chellali and Haynes [1] established the following lower bound on the total domination number of a tree. For every tree  $T$  of order  $n$  with  $l$  leaves, we have  $\gamma_t(T) \geq (n - l + 2)/2$ . They also characterized the extremal trees. In [4] a lower bound on the total outer-independent domination number of a tree was given together with the characterization of the extremal trees. Lemańska [5] proved that the domination number of a tree is bounded below by  $(n - l + 2)/3$ .

We prove the following bounds on the vertex-edge domination number of a tree  $T$  of order  $n \geq 3$  with  $l$  leaves and  $s$  support vertices,  $(n - l - s + 3)/4 \leq \gamma_{ve}(T) \leq n/3$ . We also characterize the trees attaining each of the bounds.

## 2. Results

We begin with the following straightforward observation.

**Observation 1.** For every connected graph  $G$  of diameter at least two, there exists a  $\gamma_{ve}(G)$ -set that contains no leaf.

First we show that if  $T$  is a nontrivial tree of order  $n$  with  $l$  leaves and  $s$  support vertices, then  $\gamma_{ve}(T)$  is bounded below by  $(n - l - s + 3)/4$ . For the purpose of characterizing the trees attaining this bound, we introduce a family  $\mathcal{T}$  of trees  $T = T_k$  that can be obtained as follows. Let  $T_1$  be a path  $P_5$ . If  $k$  is a positive integer, then  $T_{k+1}$  can be obtained recursively from  $T_k$  by one of the following operations.

- Operation  $\mathcal{O}_1$ : Attach a vertex by joining it to any support vertex of  $T_k$ .
- Operation  $\mathcal{O}_2$ : Attach a path  $P_2$  by joining one of its vertices to a vertex of  $T_k$ , which is not a leaf and is adjacent to a support vertex of degree two.
- Operation  $\mathcal{O}_3$ : Attach a path  $P_4$  by joining one of its leaves to a leaf of  $T_k$  adjacent to a weak support vertex.

We now prove that for every tree  $T$  of the family  $\mathcal{T}$  we have  $\gamma_{ve}(T) = (n - l - s + 3)/4$ .

**Lemma 2.** If  $T \in \mathcal{T}$ , then  $\gamma_{ve}(T) = (n - l - s + 3)/4$ .

**Proof.** We use the induction on the number  $k$  of operations performed to construct the tree  $T$ . If  $T = T_1 = P_5$ , then  $(n - l - s + 3)/4 = (5 - 2 - 2 + 3)/4 = 1 = \gamma_{ve}(T)$ . Let  $k$  be a positive integer. Assume that the result is true for every tree  $T' = T_k$  of the family  $\mathcal{T}$  constructed by  $k - 1$  operations. Let  $n'$  be the order of the tree  $T'$ ,  $l'$  the number of its leaves, and  $s'$  the number of support vertices. Let  $T = T_{k+1}$  be a tree of the family  $\mathcal{T}$  constructed by  $k$  operations.

First assume that  $T$  is obtained from  $T'$  by operation  $\mathcal{O}_1$ . We have  $n = n' + 1$ ,  $l = l' + 1$  and  $s = s'$ . It is straightforward to see that any  $\gamma_{ve}(T')$ -set is a VEDS of the tree  $T$ . Thus  $\gamma_{ve}(T) \leq \gamma_{ve}(T')$ . Obviously,  $\gamma_{ve}(T') \leq \gamma_{ve}(T)$ . This implies that  $\gamma_{ve}(T) = \gamma_{ve}(T')$ . We now get  $\gamma_{ve}(T) = \gamma_{ve}(T') = (n' - l' - s' + 3)/4 = (n - 1 - l + 1 - s + 3)/4 = (n - l - s + 3)/4$ .

Now assume that  $T$  is obtained from  $T'$  by operation  $\mathcal{O}_2$ . We have  $n = n' + 2$ ,  $l = l' + 1$  and  $s = s' + 1$ . It is straightforward to see that any  $\gamma_{ve}(T')$ -set is a VEDS of the tree  $T$ . Thus  $\gamma_{ve}(T) \leq \gamma_{ve}(T')$ . This implies that  $\gamma_{ve}(T) = \gamma_{ve}(T')$ . We now get  $\gamma_{ve}(T) = \gamma_{ve}(T') = (n' - l' - s' + 3)/4 = (n - 2 - l + 1 - s + 1 + 3)/4 = (n - l - s + 3)/4$ .

Now assume that  $T$  is obtained from  $T'$  by operation  $\mathcal{O}_3$ . We have  $n = n' + 4$ ,  $l = l'$  and  $s = s'$ . We denote by  $x$  the leaf to which  $P_4$  is attached. Let  $v_1 v_2 v_3 v_4$  be the attached path. Let  $v_1$  be joined to  $x$ . Let  $D'$  be any  $\gamma_{ve}(T')$ -set. It is easy to see that  $D' \cup \{v_2\}$  is a VEDS of the tree  $T$ . Thus  $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$ . Now let us observe that there exists a  $\gamma_{ve}(T)$ -set that does not contain the vertices  $v_4$ ,  $v_3$ , and  $v_1$ . Let  $D$  be such a set. To dominate the edge  $v_3 v_4$ , we have  $v_2 \in D$ . Observe that  $D \setminus \{v_2\}$  is a VEDS of the tree  $T'$ . Therefore  $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$ . We now conclude that  $\gamma_{ve}(T) = \gamma_{ve}(T') + 1$ . We get  $\gamma_{ve}(T) = \gamma_{ve}(T') + 1 = (n' - l' - s' + 3)/4 + 1 = (n - 4 - l - s + 7)/4 = (n - l - s + 3)/4$ .  $\square$

We now give a lower bound on the vertex-edge domination number of a tree together with the characterization of the extremal trees.

**Theorem 3.** If  $T$  is a nontrivial tree of order  $n$  with  $l$  leaves and  $s$  support vertices, then  $\gamma_{ve}(T) \geq (n - l - s + 3)/4$  with equality if and only if  $T \in \mathcal{T}$ .

**Proof.** If  $\text{diam}(T) = 1$ , then  $T = P_2$ . We have  $(n - l - s + 3)/4 = (2 - 2 - 2 + 3)/4 < 1 = \gamma_{ve}(T)$ . If  $\text{diam}(T) = 2$ , then  $T$  is a star. We have  $l = n - 1$  and  $s = 1$ . Consequently,  $(n - l - s + 3)/4 = (n - n + 1 - 1 + 3)/4 = 3/4 < 1 = \gamma_{ve}(T)$ .

Now assume that  $\text{diam}(T) \geq 3$ . Thus the order  $n$  of the tree  $T$  is at least four. We obtain the result by the induction on the number  $n$ . Assume that the theorem is true for every tree  $T'$  of order  $n' < n$  with  $l'$  leaves and  $s'$  support vertices.

First assume that some support vertex of  $T$ , say  $x$ , is strong. Let  $y$  be a leaf adjacent to  $x$ . Let  $T' = T - y$ . We have  $n' = n - 1$ ,  $l' = l - 1$  and  $s' = s$ . Obviously,  $\gamma_{ve}(T') \leq \gamma_{ve}(T)$ . We get  $\gamma_{ve}(T) \geq \gamma_{ve}(T') \geq (n' - l' - s' + 3)/4 = (n - 1 - l + 1 - s + 3)/4 = (n - l - s + 3)/4$ . If  $\gamma_{ve}(T) = (n - l - s + 3)/4$ , then obviously  $\gamma_{ve}(T') = (n' - l' - s' + 3)/4$ . By the inductive hypothesis, we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by operation  $\mathcal{O}_1$ . Thus  $T \in \mathcal{T}$ . Henceforth, we can assume that every support vertex of  $T$  is weak.

We now root  $T$  at a vertex  $r$  of maximum eccentricity  $\text{diam}(T)$ . Let  $t$  be a leaf at maximum distance from  $r$ ,  $v$  be the parent of  $t$ , and  $u$  be the parent of  $v$  in the rooted tree. If  $\text{diam}(T) \geq 4$ , then let  $w$  be the parent of  $u$ . If  $\text{diam}(T) \geq 5$ , then let  $d$  be the parent of  $w$ . If  $\text{diam}(T) \geq 6$ , then let  $e$  be the parent of  $d$ . By  $T_x$  we denote the subtree induced by a vertex  $x$  and its descendants in the rooted tree  $T$ .

Assume that some child of  $u$ , say  $x$ , is a leaf. Let  $T' = T - x$ . We have  $n' = n - 1$ ,  $l' = l - 1$  and  $s' = s - 1$ . We get  $\gamma_{ve}(T) \geq \gamma_{ve}(T') \geq (n' - l' - s' + 3)/4 = (n - 1 - l + 1 - s + 1 + 3)/4 > (n - l - s + 3)/4$ .

Now assume that among the children of  $u$  there is a support vertex other than  $v$ . Let  $T' = T - T_v$ . We have  $n' = n - 2$ ,  $l' = l - 1$  and  $s' = s - 1$ . We get  $\gamma_{ve}(T) \geq \gamma_{ve}(T') \geq (n' - l' - s' + 3)/4 = (n - 2 - l + 1 - s + 1 + 3)/4 = (n - l - s + 3)/4$ . If  $\gamma_{ve}(T) = (n - l - s + 3)/4$ , then obviously  $\gamma_{ve}(T') = (n' - l' - s' + 3)/4$ . By the inductive hypothesis, we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by operation  $\mathcal{O}_2$ . Thus  $T \in \mathcal{T}$ .

Now assume that  $d_T(u) = 2$ . Assume that  $d_T(w) \geq 3$ . First assume that some child of  $w$ , say  $x$ , is a leaf. Let  $T' = T - x$ . We have  $n' = n - 1$ ,  $l' = l - 1$  and  $s' = s - 1$ . We get  $\gamma_{ve}(T) \geq \gamma_{ve}(T') \geq (n' - l' - s' + 3)/4 = (n - 1 - l + 1 - s + 1 + 3)/4 > (n - l - s + 3)/4$ .

Now assume that no child of  $w$  is a leaf. Let  $T' = T - T_u$ . We have  $n' = n - 3$ ,  $l' = l - 1$  and  $s' = s - 1$ . Let us observe that there exists a  $\gamma_{ve}(T)$ -set that does not contain the vertices  $t$  and  $v$ . Let  $D$  be such a set. To dominate the edge  $vt$ , we have  $u \in D$ . Let us observe that  $D \setminus \{u\}$  is a VEDS of the tree  $T'$ . Therefore  $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$ . We now get  $\gamma_{ve}(T) \geq \gamma_{ve}(T') + 1 \geq (n' - l' - s' + 3)/4 + 1 = (n - 3 - l + 1 - s + 1 + 7)/4 > (n - l - s + 3)/4$ .

If  $d_T(w) = 1$ , then  $T = P_4$ . We have  $(n - l - s + 3)/4 = (4 - 2 - 2 + 3)/4 < 1 = \gamma_{ve}(T)$ . Now assume that  $d_T(w) = 2$ . First assume that  $d_T(d) \geq 3$ . Let  $T' = T - T_w$ . We have  $n' = n - 4$ ,  $l' = l - 1$  and  $s' = s - 1$ . Let us observe that there exists a  $\gamma_{ve}(T)$ -set that does not contain the vertices  $t$ ,  $v$  and  $w$ . Let  $D$  be such a set. To dominate the edge  $vt$ , we have  $u \in D$ . Observe that  $D \setminus \{u\}$  is a VEDS of the tree  $T'$ . Therefore  $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$ . We now get  $\gamma_{ve}(T) \geq \gamma_{ve}(T') + 1 \geq (n' - l' - s' + 3)/4 + 1 = (n - 4 - l + 1 - s + 1 + 7)/4 > (n - l - s + 3)/4$ .

Now assume that  $d_T(d) = 2$ . First assume that some child of  $e$  is a leaf. Let  $T' = T - T_w$ . We have  $n' = n - 4$ ,  $l' = l$  and  $s' = s - 1$ . Similarly as in the previous possibility, we conclude that  $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$ . We get  $\gamma_{ve}(T) \geq \gamma_{ve}(T') + 1 \geq (n' - l' - s' + 3)/4 + 1 = (n - 4 - l - s + 1 + 7)/4 > (n - l - s + 3)/4$ .

Now assume that no child of  $e$  is a leaf. Let  $T' = T - T_w$ . We have  $n' = n - 4$ ,  $l' = l$  and  $s' = s$ . If  $n' = 1$ , then  $T = P_5 = T_1 \in \mathcal{T}$ . Assume that  $n' \geq 2$ . Similarly as earlier, we conclude that  $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$ . We now get  $\gamma_{ve}(T) \geq \gamma_{ve}(T') + 1 \geq (n' - l' - s' + 3)/4 + 1 = (n - 4 - l - s + 7)/4 = (n - l - s + 3)/4$ . If  $\gamma_{ve}(T) = (n - l - s + 3)/4$ , then obviously  $\gamma_{ve}(T') = (n' - l' - s' + 3)/4$ . By the inductive hypothesis, we have  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by operation  $\mathcal{O}_3$ .  $\square$

Next we show that if  $T$  is a tree of order  $n \geq 3$ , then  $\gamma_{ve}(T)$  is bounded above by  $n/3$ . For the purpose of characterizing the trees attaining this bound, we introduce a family  $\mathcal{F}$  of trees  $T = T_k$  that can be obtained as follows. Let  $T_1$  be a path  $P_3$ . If  $k$  is a positive integer, then  $T_{k+1}$  can be obtained recursively from  $T_k$  by attaching a path  $P_3$  by joining one of its leaves to a vertex of  $T_k$  adjacent to a path  $P_2$  or  $P_3$ .

We now prove that for every tree  $T$  of the family  $\mathcal{F}$  we have  $\gamma_{ve}(T) = n/3$ .

**Lemma 4.** *If  $T \in \mathcal{F}$ , then  $\gamma_{ve}(T) = n/3$ .*

**Proof.** We use the induction on the number  $k$  of operations performed to construct the tree  $T$ . If  $T = T_1 = P_3$ , then  $\gamma_{ve}(T) = 1 = n/3$ . Let  $k$  be a positive integer. Assume that the result is true for every tree  $T' = T_k$  of the family  $\mathcal{F}$  constructed by  $k - 1$  operations. Let  $n'$  be the order of the tree  $T'$ . Let  $T = T_{k+1}$  be a tree of the family  $\mathcal{F}$  constructed by  $k$  operations. We have  $n = n' + 3$ . We denote by  $x$  the vertex to which is attached  $P_3$ . Let  $v_1 v_2 v_3$  be the attached path. Let  $v_1$  be adjacent to  $x$ . Let  $D'$  be any  $\gamma_{ve}(T')$ -set. It is easy to see that  $D' \cup \{v_1\}$  is a VEDS of the tree  $T$ . Thus  $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$ . If  $x$  is adjacent to a path  $P_2$ , then let us observe that there exists a  $\gamma_{ve}(T)$ -set that contains the vertices  $v_1$  and  $x$ . Let  $D$  be such a set. The set  $D$  is minimal, thus  $v_2, v_3 \notin D$ . It is easy to observe that  $D \setminus \{v_1\}$  is a VEDS of the tree  $T'$ . If  $x$  is adjacent to a path  $P_3$  different from  $v_1 v_2 v_3$ , say  $abc$ , then let  $a$  and  $x$  be adjacent. Let us observe that there exists a  $\gamma_{ve}(T)$ -set that contains the vertices  $v_1$  and  $a$ . Let  $D$  be such a set. The set  $D$  is minimal, thus  $v_2, v_3 \notin D$ . Let us observe that  $D \setminus \{v_1\}$  is a VEDS of the tree  $T'$ . We now conclude that  $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$ , and consequently,  $\gamma_{ve}(T) = \gamma_{ve}(T') + 1$ . We get  $\gamma_{ve}(T) = \gamma_{ve}(T') + 1 = n'/3 + 1 = (n - 3)/3 + 1 = n/3$ .  $\square$

We now give an upper bound on the vertex-edge domination number of a tree together with the characterization of the extremal trees.

**Theorem 5.** *If  $T$  is a tree of order  $n \geq 3$ , then  $\gamma_{ve}(T) \leq n/3$  with equality if and only if  $T \in \mathcal{F}$ .*

**Proof.** First assume that  $\text{diam}(T) = 2$ . Thus  $T$  is a star. If  $T = P_3$ , then  $T = T_1 \in \mathcal{F}$ . If  $T$  is a star different from  $P_3$ , then we get  $n/3 > 1 = \gamma_{ve}(T)$ .

Now assume that  $\text{diam}(T) \geq 3$ . Thus the order  $n$  of the tree  $T$  is at least four. We obtain the result by the induction on the number  $n$ . Assume that the theorem is true for every tree  $T'$  of order  $n' < n$ .

First assume that some support vertex of  $T$ , say  $x$ , is strong. Let  $y$  be a leaf adjacent to  $x$ . Let  $T' = T - y$ . It is straightforward to see that any  $\gamma_{ve}(T')$ -set is a VEDS of the tree  $T$ . Thus  $\gamma_{ve}(T) \leq \gamma_{ve}(T')$ . We now get  $\gamma_{ve}(T) \leq \gamma_{ve}(T') \leq n'/3 < n/3$ .

We now root  $T$  at a vertex  $r$  of maximum eccentricity  $\text{diam}(T)$ . Let  $t$  be a leaf at maximum distance from  $r$ ,  $v$  be the parent of  $t$ , and  $u$  be the parent of  $v$  in the rooted tree. If  $\text{diam}(T) \geq 4$ , then let  $w$  be the parent of  $u$ . By  $T_x$  we denote the subtree induced by a vertex  $x$  and its descendants in the rooted tree  $T$ .

Assume that some child of  $u$ , say  $x$ , is a leaf. Let  $T' = T - x$ . Let  $D'$  be a  $\gamma_{ve}(T')$ -set that contains no leaf. It is easy to observe that  $D'$  is a VEDS of the tree  $T$ . Thus  $\gamma_{ve}(T) \leq \gamma_{ve}(T')$ . We now get  $\gamma_{ve}(T) \leq \gamma_{ve}(T') \leq n'/3 < n/3$ .

Now assume that among the children of  $u$  there is a support vertex other than  $v$ . Let  $T' = T - T_v$ . Let us observe that there exists a  $\gamma_{ve}(T')$ -set that contains the vertex  $u$ . Let  $D'$  be such a set. It is easy to see that  $D'$  is a VEDS of the tree  $T$ . Thus  $\gamma_{ve}(T) \leq \gamma_{ve}(T')$ . We now get  $\gamma_{ve}(T) \leq \gamma_{ve}(T') \leq n'/3 < n/3$ .

Now assume that  $d_T(u) = 2$ . First assume that  $w$  is adjacent to a leaf, say  $x$ . Let  $T' = T - x$ . Let us observe that there exists a  $\gamma_{ve}(T')$ -set that contains the vertex  $u$ . Let  $D'$  be such a set. It is easy to see that  $D'$  is a VEDS of the tree  $T$ . Thus  $\gamma_{ve}(T) \leq \gamma_{ve}(T')$ . We now get  $\gamma_{ve}(T) \leq \gamma_{ve}(T') \leq n'/3 < n/3$ .

Now assume that there is a child of  $w$  other than  $u$ , say  $x$ , such that the distance of  $w$  to the most distant vertex of  $T_x$  is two or three. It suffices to consider only the possibilities when  $T_x$  is a path  $P_2$  or  $P_3$ . Let  $T' = T - T_u$ . We have  $n' = n - 3$ . Let  $D'$  be any  $\gamma_{ve}(T')$ -set. It is easy to observe that  $D' \cup \{u\}$  is a VEDS of the tree  $T$ . Thus  $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$ . We now get  $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 \leq n'/3 + 1 = n/3$ . If  $\gamma_{ve}(T) = n/3$ , then obviously  $\gamma_{ve}(T') = n'/3$ . By the inductive hypothesis we have  $T' \in \mathcal{F}$ . The tree  $T$  can be obtained from  $T'$  by attaching a path  $P_3$  by joining one of its leaves to the vertex  $u$ . Thus  $T \in \mathcal{F}$ .

Now assume that  $d_T(w) = 2$ . Let  $T' = T - T_w$ . We have  $n' = n - 4$ . If  $n' = 1$ , then  $T = P_5$ . We have  $\gamma_{ve}(P_5) = 1 < 5/3$ . If  $n' = 2$ , then  $T = P_6$ . The path  $P_6$  can be obtained from two paths  $P_3$  by joining them through leaves. Thus  $T \in \mathcal{F}$ . Now assume that  $n' \geq 3$ . Let  $D'$  be any  $\gamma_{ve}(T')$ -set. Let us observe that  $D' \cup \{u\}$  is a VEDS of the tree  $T$ . Thus  $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$ . We now get  $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 \leq n'/3 + 1 = (n - 1)/3 < n/3$ .  $\square$

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