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Comparative analysis of numerical with optical soliton solutions of stochastic Gross–Pitaevskii equation in dispersive media

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ABSTRACT

This article deals with the stochastic Gross–Pitaevskii equation (SGPE) perturbed with multiplicative time noise. The numerical solutions of the governing model are carried out with the proposed stochastic non-standard finite difference (SNSFD) scheme. The stability of the scheme is proved by using the Von-Neumann criteria and the consistency is shown in the mean square sense. To seek exact solutions, we applied the Sardar subequation (SSE) and modified exponential rational functional (MERF) techniques. The exact solutions are constructed in the form of exponential, hyperbolic, and trigonometric forms. Finally, the comparison of the exact solutions with numerical solutions is drawn in the 3D and line plots for the different values of parameters.

Introduction

In this modern era of research, seeking the exact solutions of nonlinear partial differential equations (NLPDEs) is a very important topic, because NLPDEs have physical complex phenomena in different fields such as physics, biology, mechanics, biomathematics, mechanics, etc. [1–4]. The Gross–Pitaevskii equation (GPE) is a classical nonlinear evolution [5]. It is the type of famous nonlinear Schrödinger equation. The NLSE is a universal governing model that evolution of complex fields that are used in dispersive media [6,7]. A Bose–Einstein condensate (BEC) made up of N can be modeled using the GPE, which is a classical field equation with applications to the propagation of light in optical fibre, planar waveguides, and Bose–Einstein condensates constrained to highly anisotropic sigar-shaped traps in the mean-field regime of dilute gases, as noted in [8–10]. This equation can also be found in the research of Langmuir waves in hot plasmas, small wave-amplitude gravity waves on the surface of deep inviscid (zero-viscosity) water, plane-diffracted wave beams in the focusing regions of the ionosphere, Davydov's alpha-helix solitons, which are responsible for energy transfer along molecular chains, and many other phenomena.

At the micro level, every mathematical model physically does not appear in the linear motion but has a random motion. So, for the sake of randomness, many researchers use the multiplicative white noise in the mathematical model [11]. By appearing this term differential equations called the stochastic differential equation.

When we see most of the physical phenomena at the micro-scale and magnify them, then these phenomena are stochastic or random phenomena. It is very natural to consider the differential equation which has some kind of randomness involved. So, if this randomness is going bounded in the solution of the differential equations, such problems are stochastic differential equations. The numerical solutions of the stochastic NLPDEs are estimated by some scientists [12, 13]. They developed the numerical schemes and the simulations and also gives the stability and consistency of schemes [14]. The "Good" Boussinesq equation have been studied numerically by using different techniques [15–17]. For more literature on SPDE's [18–27]. In this current era of research many researchers are working on the different stochastic NLPDEs, numerically and analytically but their is a gape

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of the comparison of the results. So, in this study we compare the numerical result with exact solitary wave solutions by simulations. For the simulations of result we use the MATLAB 2015a. For the sake of exact solutions, many researchers also use multiplicative white noise and show the randomness in their solutions [28,29]. But in this study, we use the both numerical and analytical solutions to compare the graphical simulations with the noise effect. The aim of finding the exact solutions vast variety of powerful and direct methods are used named as, He's variational iteration [30–33], new modified extended direct algebraic [34,35], G'/G-expansion [36,37], ϕ^6-model expansion [38,39], hirota bilinear [40,41], modified exponential rational functional method [42,43], etc. The numerical approximation of the some partial differential equations are carried out such as [44–50].

The comparison of numerical solutions of the stochastic differential equation with analytical or solitary wave solutions is not a simple job, it becomes more difficult when our governing equation is a nonlinear stochastic differential equation and we have tried to overcome such issues. But in this study, the underlying SGPE model is investigated numerically and analytically. But the main purpose of this study is the comparison of exact and numerical solutions under the effect of time noise.

The main contribution and novelty of the current study is as follow:

- The Gross–Pitaevskii equation under the influence of the time noise is under consideration.
- The numerical solutions are derived by proposed stochastic NSFD scheme.
- The analysis of the scheme is proved in mean square sense.
- The solitary wave solutions are extracted by two techniques namely Sardar subequation (SSE) and modified exponential rational (MERF).
- The comparison of some solitary wave solutions with numerical solutions are represented by simulations.

Governing model

The stochastic Gross–Pitaevskii equation (SGPE) suitable units, in one-dimension can be taken as [1–8],

$$i\Phi_t = -\frac{1}{2}\Phi_{xx} + V(x)\Phi + g|\Phi|^2\Phi + \sigma \dot{W}(t)\Phi, \tag{1}$$

where $\Phi(x,t)$ represents the macroscopic wave function of the condensate, $V(x)$ is a real-valued function that generated macroscopic potential experimentally. The parameter g encapsulates the strength of the atom–atom interactions. Which determine whether this model is repulsive when the value of g is 1 (defocusing nonlinearity) or shows attraction when g is -1, (focusing nonlinearity) [51]. Here, σ is represents the strength of the Brownian motion and $\dot{W}(t)$ is the random function.

Proposed stochastic non-standard finite difference (NSFD) scheme

The Φ_t and Φ_{xx} in Eq. (12) are approximated as follows,

$$\Phi_t = \frac{\Phi_p^{q+1} - \Phi_p^q}{k}, \qquad \Phi_{xx} = \frac{\Phi_{p+1}^q - 2\Phi_p^{q+1} + \Phi_{p-1}^q}{h^2},$$

here, space and time stepsizes are take as $h = \Delta x^2$ and $k = \Delta t$ respectively. The state variable Φ_p^q is the approximation of $\Phi(x,t)$ at $\Phi(p\Delta x, q\Delta t)$. Now,by replacing the values of Φ_t and Φ_{xx} in Eq. (12) and after some basic arithmetics, we get

$$\begin{aligned}(i+2\alpha)\Phi_p^{q+1} &= \alpha\Phi_{p+1}^q + \alpha\Phi_{p-1}^q + (i + kV_p)\Phi_p^q + gk|\Phi_p^q|^2\Phi_p^q \\ &\quad + k\sigma\Phi_p^q(W^{(q+1)k} - W^{qk}),\end{aligned} \tag{2}$$

where $\alpha = \frac{-k}{2h^2}$. The Eq. (2) is required proposed finite difference scheme for Eq. (12).

Consistency of scheme

The consistency of the scheme is proved in the mean square sense [13,14].

Theorem 1. *The proposed stochastic NSFD scheme for Φ in (2) is consistent with (12), in mean square sense.*

Proof. We suppose the smooth function $\Phi(x,t)$ and using the operator $Z(\Phi) = \int_{q\Delta t}^{(q+1)\Delta t}(\Phi)ds$. Apply this on Eq. (12) and get as follows;

$$\begin{aligned}Z(\Phi)_p^q &= \Phi(p\Delta x,(q+1)\Delta t) - \Phi(p\Delta x, q\Delta t) + \frac{d}{2}\int_{q\Delta t}^{(q+1)\Delta t}\Phi_{xx}(p\Delta x, s)ds \\ &\quad - V(x)\int_{q\Delta t}^{(q+1)\Delta t}\Phi(p\Delta x, s)ds \\ &\quad - g\int_{q\Delta t}^{(q+1)\Delta t}|\Phi(p\Delta x, s)|^2\Phi(p\Delta x, s)ds \\ &\quad - \sigma\int_{q\Delta t}^{(q+1)\Delta t}\Phi(p\Delta x, s)dW|_s,\end{aligned} \tag{3}$$

$$\begin{aligned}Z|_p^q(\Phi) &= \Phi(p\Delta x,(q+1)\Delta t) - \Phi(p\Delta x, q\Delta t) \\ &\quad + \frac{d}{2}\Delta t\frac{\Phi((p+1)\Delta x, q\Delta t) - 2\Phi(p\Delta x,(q+1)\Delta t) + \Phi((p-1)\Delta x, q\Delta t)}{\Delta x^2} \\ &\quad - V(x)\Delta t\Phi(p\Delta x, q\Delta t) - g\Delta t|\Phi(p\Delta x, q\Delta t)|^2\Phi(p\Delta x, q\Delta t) \\ &\quad - \sigma\Phi(p\Delta x, q\Delta t)(W^{(q+1)\Delta t} - W^{q\Delta t}),\end{aligned} \tag{4}$$

in the mean square sense the above relations take form as follows;

$$\begin{aligned}E|Z(\Phi)_p^q - Z|_p^q(\Phi)|^2 &\le d^2E|\int_{q\Delta t}^{(q+1)\Delta t}(\Phi_{xx}(p\Delta x, s) \\ &\quad - \frac{\Phi((p+1)\Delta x, q\Delta t) - 2\Phi(p\Delta x,(q+1)\Delta t) + \Phi((p-1)\Delta x, q\Delta t)}{\Delta x^2})ds|^2 \\ &\quad + 4(V(x))^2E\left|\int_{q\Delta t}^{(q+1)\Delta t}(-\Phi(p\Delta x, s) + \Phi(p\Delta x, q\Delta t))ds\right|^2 \\ &\quad + 4g^2E\left|\int_{q\Delta t}^{(q+1)\Delta t}(-|\Phi(p\Delta x, s)|^2\Phi(p\Delta x, s) \right. \\ &\quad \left. + |\Phi(p\Delta x, q\Delta t)|^2\Phi(p\Delta x, q\Delta t))ds\right|^2 \\ &\quad + 4\sigma^2E\left|\int_{q\Delta t}^{(q+1)\Delta t}(-\Phi(p\Delta x, s) + \Phi(q\Delta x, p\Delta t))\,dW|_s\right|^2,\end{aligned}$$

square property of the Itô's integral gives us,

$$\begin{aligned}E|Z(\Phi)_p^q - Z|_p^q(\Phi)|^2 &\le d^2E|\int_{q\Delta t}^{(q+1)\Delta t}(\Phi_{xx}(p\Delta x, s) \\ &\quad - \frac{\Phi((p+1)\Delta x, q\Delta t) - 2\Phi(p\Delta x,(q+1)\Delta t) + \Phi((p-1)\Delta x, q\Delta t)}{\Delta x^2})ds|^2 \\ &\quad + 4(V(x))^2E\left|\int_{q\Delta t}^{(q+1)\Delta t}(-\Phi(p\Delta x, s) + \Phi(p\Delta x, q\Delta t))ds\right|^2 \\ &\quad + 4g^2E\left|\int_{q\Delta t}^{(q+1)\Delta t}(-|\Phi(p\Delta x, s)|^2\Phi(p\Delta x, s) \right. \\ &\quad \left. + |\Phi(p\Delta x, q\Delta t)|^2\Phi(p\Delta x, q\Delta t))ds\right|^2 \\ &\quad + 4\sigma^2\int_{q\Delta t}^{(q+1)\Delta t}E\,|-\Phi(p\Delta x, s) + \Phi(q\Delta x, p\Delta t)|^2\,ds,\end{aligned}$$

$E|Z(\Phi)_p^q - Z|_p^q(\Phi)|^2 \to 0$ as $p \to \infty, q \to \infty$, so this proposed scheme for Φ is consistent with stochastic PDE (12).

Stability

The stability of the current scheme is shown with the help of the Von-Neumann criteria.

Von-Neumann criteria

By this method, $\Phi_{q,p}$ is replaced in the differential equation as follows,

$$\Phi_{q,p} = g(t)e^{i(\beta x)}, \tag{5}$$

and by doing some basic calculation, one get the amplification factor as follow, [13]

$$E\left|\frac{g(t+\Delta t)}{g(t)}\right|^2 \leq 1+\chi\Delta t, \tag{6}$$

where χ is a constant.

It is a necessary and sufficient condition of stability.

Theorem 2. *The given scheme for Φ is unconditionally stable with $(q+1)\Delta t = T$ [52].*

Proof. To check the stability of the scheme, the Von-Neumann technique is used, so by linearizing Eq. (2) as follows,

$$(\imath+2\alpha)\Phi_p^{q+1} = \alpha\Phi_{p+1}^q + \alpha\Phi_{p-1}^q + (\imath+kV_p)\Phi_p^q + \sigma\Phi_p^q(W^{(q+1)\Delta t} - W^{q\Delta t}),$$

$$(\imath+2\alpha)g(t+\Delta t)e^{i(\beta x)} = (\alpha(e^{i(\beta\Delta x)} + e^{-i(\beta\Delta x)}) + (\imath+kV_p)) + \sigma(W^{(q+1)\Delta t} - W^{q\Delta t})g(t)e^{i(\beta x)},$$

$$E\left|\frac{g(t+\Delta t)}{g(t)}\right|^2 = \left|\frac{2\alpha - 4\alpha\sin^2(\frac{\beta\Delta x}{2}) + \imath + kV_p}{\imath+2\alpha}\right|^2 + \left|\frac{\sigma}{\imath+2\alpha}\right|^2 (W^{(q+1)\Delta t} - W^{q\Delta t}),$$

as maximum value of $\sin^2(\frac{\beta\Delta x}{2}) = 1$ and using Φ as the independent from the state of Winer process

$$E\left|\frac{g(t+\Delta t)}{g(t)}\right|^2 \leq \left|\frac{(\imath-2\alpha+kV_p)}{\imath+2\alpha}\right|^2 + \left|\frac{\sigma}{\imath+2\alpha}\right|^2 \Delta t,$$

and $\frac{(\imath-2\alpha+kV_p)}{\imath+2\alpha} \leq 1$, $\frac{\sigma}{\imath+2\alpha} = \chi$, above inequality reduces to

$$E\left|\frac{g(t+\Delta t)}{g(t)}\right|^2 \leq 1+|\chi|^2\Delta t.$$

So, the given scheme is stable.

Convergence of the proposed scheme

The convergence of the scheme is discussed in the mean square sense.

Theorem 3. *The proposed scheme given by Eq. (2) is convergent in the mean square sense.*

Proof.

$$E\left|\Phi|_{p,q} - \Phi\right|^2 = E\left|(Z_{p,q})^{-1}(Z_{p,q}\Phi|_{p,q} - Z_{p,q}\Phi)\right|^2,$$

as proposed scheme (2) is consistent in the mean square sense i.e., $Z_{p,q}\Phi|_{p,q} \to Z_{p,q}\Phi$ as $\Delta x \to 0, \Delta t \to 0$ and $(p\Delta x, q\Delta t) \to (x,t)$,

$$E\left|(Z_{p,q})^{-1}(Z_{p,q}\Phi|_{p,q} - Z_{p,q}\Phi)\right|^2 \to 0,$$

also, as scheme is stable, then $(Z_{p,q})^{-1}$ is bounded. So, $E\left|\Phi|_{p,q}-\Phi\right|^2 \to 0$. Hence proposed scheme (2) for Φ is convergent in the mean square sense.

Remark 1. The second order accurate (in time) numerical scheme can be designed and its corresponding analysis can be carried out.

Theoretical analysis of method's

The proposed NSFD scheme is used to find the numerical approximation of the underlying model. The scheme is consistent with given equation and stability analysis is shown with the help of Von-Neumann criteria. The convergence analysis is also carried out in the mean square sense. Further for the sake of exact solutions we use Sardar subequation (SSE) and modified exponential rational (MERF) techniques that will gives us the different types of wave structures in the form of dark, bright, mixed, trigonometric, and exponential function solutions as well.

Extraction of exact solutions

Proceeding to find the exact solutions of Eq. (12), by converting PDE into ODE for by choosing the transformation $\Phi(x,t) = \Omega(\eta)e^{i(lx+mt)}$, where $\eta = \alpha x - ct$ for more detail see [53–56]. So, substituting this transformation into Eq. (12), we get the ODE form and compare real and imaginary values as follows:

$$\alpha^2\Omega''(\eta) + (m - l^2 + V(x))\Omega(\eta) + g\Omega^3(\eta) + \sigma\dot{W}(t)\Omega(\eta) = 0, \tag{7}$$

$$c\Omega'(\eta) + 2l\alpha\Omega'(\eta) = 0, \tag{8}$$

where Ω is a polynomial and $' = \frac{d}{d\eta}$.

Solutions via sardar subequation approach [57–60]

Consider only the real part of ODE eq. (real) which has the solution in the following form;

$$\Omega(\eta) = \sum_{j=0}^{N} \kappa_j\ \Theta^j(\eta), \tag{9}$$

where κ_j $(0 \leq j \leq N)$ are the constants that are found later, and $\Theta'(\eta)$ is satisfy the Eq. (7) and taking as;

$$\Theta'(\eta) = \sqrt{\gamma + \chi\Theta(\eta)^2 + \Theta(\eta)^4}, \tag{10}$$

where γ and χ are real constants. The value of N is obtained with the help of homogeneous balancing between $\Omega^3(\eta)$ and $\Omega''(\eta)$ in Eq. (7), which gives us $N = 1$. Thus, Eq. (26) takes the following form;

$$\Omega(\eta) = \kappa_0 + \kappa_1\Theta(\eta). \tag{11}$$

By substituting the Eq. (26) with its derivatives into the Eq. (7) with Eq. (27) and gathering all the terms of the same power of $\Theta^j(\eta)$. The coefficients of these polynomials equate to zero and get an algebraic system of equations. With the help of *mathematica*11.1 solving this system, we obtained the different families of solutions for the values of constant and parameters as follows:

Family-1.

$$\left\{ \kappa_0 = 0; \quad \kappa_1 = \pm\frac{i\sqrt{2}\alpha}{\sqrt{g}}; \quad \chi = \frac{l^2-m-V-\sigma W}{\alpha^2}; \right.$$

The different families' solutions of Eq. (12) are extracted as;

Type-1: For $\chi = \frac{l^2-m-V-\sigma W}{\alpha^2} > 0$ and $\gamma = 0$, then we obtained the hyperbolic solutions of Eq. (12) as follows;

$$\Phi_1^{\pm}(x,t) = \left[\pm\frac{i\sqrt{2}\alpha}{\sqrt{g}}\sqrt{-\frac{pq(l^2-m-V-\sigma W)}{\alpha^2}} \times \operatorname{sech}_{pq}\left(\sqrt{\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - ct)\right)\right] e^{i(lx+mt)}. \tag{12}$$

$$\Phi_2^{\pm}(x,t) = \left[\pm\frac{i\sqrt{2}\alpha}{\sqrt{g}}\sqrt{\frac{pq(l^2-m-V-\sigma W)}{\alpha^2}} \times \operatorname{csch}_{pq}\left(\sqrt{\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - ct)\right)\right] e^{i(lx+mt)}. \tag{13}$$

where $\text{sech}_{pq} = \frac{2}{pe^{\eta}+qe^{-\eta}}$ and $\text{csch}_{pq} = \frac{2}{pe^{\eta}-qe^{-\eta}}$.

Type-2: For $\chi = \frac{l^2-m-V-\sigma W}{\alpha^2} < 0$ and $\gamma = 0$, then we attained the trigonometric solutions of Eq. (12) as follows;

$$\Phi_3^{\pm}(x,t) = \left[\pm\frac{i\sqrt{2}\alpha}{\sqrt{g}}\sqrt{-\frac{pq(l^2-m-V-\sigma W)}{\alpha^2}} \times \sec_{pq}\left(\sqrt{-\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - ct)\right)\right] e^{i(lx+mt)}. \tag{14}$$

$$\Phi_4^{\pm}(x,t) = \left[\pm\frac{i\sqrt{2}\alpha}{\sqrt{g}}\sqrt{-\frac{pq(l^2-m-V-\sigma W)}{\alpha^2}} \times \csc_{pq}\left(\sqrt{-\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - ct)\right)\right] e^{i(lx+mt)}. \tag{15}$$

where $\sec_{pq} = \frac{2}{pe^{i\eta}+qe^{-i\eta}}$ and $\csc_{pq} = \frac{2}{pe^{i\eta}-qe^{-i\eta}}$.

Type-3: For $\chi = \frac{l^2-m-V-\sigma W}{\alpha^2} < 0$ and $\gamma = \sqrt{\frac{\chi^2}{4}}$, then we obtained the singular and dark solution of Eq. (12) as follows;

$$\Phi_5^{\pm}(x,t) = \left[\pm\frac{i\alpha}{\sqrt{g}}\sqrt{-\frac{l^2-m-V-\sigma W}{\alpha^2}} \times \tanh_{pq}\left(\frac{\sqrt{-\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - ct)}{\sqrt{2}}\right)\right] e^{i(lx+mt)}. \tag{16}$$

$$\Phi_6^{\pm}(x,t) = \left[\pm\frac{i\alpha}{\sqrt{g}}\sqrt{-\frac{l^2-m-V-\sigma W}{\alpha^2}} \times \coth_{pq}\left(\frac{\sqrt{-\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - ct)}{\sqrt{2}}\right)\right] e^{i(lx+mt)}. \tag{17}$$

We obtained the complex combined dark-bright solution of Eq. (12) as follows;

$$\Phi_7^{\pm}(x,t) = \left[\pm\frac{i\alpha}{\sqrt{g}}\sqrt{-\frac{l^2-m-V-\sigma W}{\alpha^2}} \times \left(\tanh_{pq}\left(\sqrt{2}\sqrt{-\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - ct)\right) + i\sqrt{pq}\,\text{sech}_{pq}\left(\sqrt{2}\sqrt{-\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - ct)\right)\right)\right] \times e^{i(lx+mt)}. \tag{18}$$

We obtained the mixed singular solution of Eq. (12) as follows;

$$\Phi_8^{\pm}(x,t) = \left[\pm\frac{i\alpha}{\sqrt{g}}\sqrt{-\frac{l^2-m-V-\sigma W}{\alpha^2}} \times \left(\coth_{pq}\left(\sqrt{2}\sqrt{-\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - ct)\right) + \sqrt{pq}\,\text{csch}_{pq}\left(\sqrt{2}\sqrt{-\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - ct)\right)\right)\right] \times e^{i(lx+mt)}. \tag{19}$$

We obtained the solitary wave solution of Eq. (12) as follows;

$$\Phi_9^{\pm}(x,t) = \left[\pm\frac{i\alpha}{2\sqrt{g}}\sqrt{-\frac{l^2-m-V-\sigma W}{\alpha^2}} \times \left(\tanh_{pq}\left(\frac{\sqrt{-\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - ct)}{2\sqrt{2}}\right) + \coth_{pq}\left(\frac{\sqrt{-\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - ct)}{2\sqrt{2}}\right)\right)\right] e^{i(lx+mt)}. \tag{20}$$

where $\tanh_{pq} = \frac{pe^{\eta}-qe^{-\eta}}{pe^{\eta}+qe^{-\eta}}$ and $\coth_{pq} = \frac{pe^{\eta}+qe^{-\eta}}{pe^{\eta}-qe^{-\eta}}$.

Type-4: For $\chi = \frac{l^2-m-V-\sigma W}{\alpha^2} > 0$ and $\gamma = \sqrt{\frac{\chi^2}{4}}$, then we obtained the trigonometric solution of Eq. (12) as follows;

$$\Phi_{10}^{\pm}(x,t) = \left[\pm\frac{i\alpha}{\sqrt{g}}\sqrt{\frac{l^2-m-V-\sigma W}{\alpha^2}} \times \tan_{pq}\left(\frac{\sqrt{\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - ct)}{\sqrt{2}}\right)\right] e^{i(lx+mt)}. \tag{21}$$

$$\Phi_{11}^{\pm}(x,t) = \left[\pm\frac{i\alpha}{\sqrt{g}}\sqrt{\frac{l^2-m-V-\sigma W}{\alpha^2}} \times \cot_{pq}\left(\frac{\sqrt{\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - ct)}{\sqrt{2}}\right)\right] e^{i(lx+mt)}. \tag{22}$$

$$\Phi_{12}^{\pm}(x,t) = \left[\pm\frac{i\alpha}{\sqrt{g}}\sqrt{\frac{l^2-m-V-\sigma W}{\alpha^2}} \times \left(\tan_{pq}\left(\sqrt{2}\sqrt{\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - vt)\right) + \sqrt{pq}\sec_{pq}\left(\sqrt{2}\sqrt{\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - vt)\right)\right)\right] \times e^{i(lx+mt)}. \tag{23}$$

$$\Phi_{13}^{\pm}(x,t) = \left[\pm\frac{i\alpha}{\sqrt{g}}\sqrt{\frac{l^2-m-V-\sigma W}{\alpha^2}} \times \left(\cot_{pq}\left(\sqrt{2}\sqrt{\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - vt)\right) + \sqrt{pq}\csc_{pq}\left(\sqrt{2}\sqrt{\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - vt)\right)\right)\right] \times e^{i(lx+mt)}. \tag{24}$$

$$\Phi_{14}^{\pm}(x,t) = \left[\pm\frac{i\alpha}{\sqrt{2g}}\sqrt{\frac{l^2-m-V-\sigma W}{\alpha^2}}\left(\tan_{pq}\left(\frac{\sqrt{\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - ct)}{2\sqrt{2}}\right) + \cot_{pq}\left(\frac{\sqrt{\frac{l^2-m-V-\sigma W}{\alpha^2}}(\alpha x - ct)}{2\sqrt{2}}\right)\right)\right] e^{i(lx+mt)}. \tag{25}$$

where $\tan_{pq} = -i\frac{pe^{i\eta}-qe^{-i\eta}}{pe^{i\eta}+qe^{-i\eta}}$ and $\cot_{pq} = i\frac{pe^{i\eta}+qe^{-i\eta}}{pe^{i\eta}-qe^{-i\eta}}$.

Solutions via modified exponential rational function method (MERFM) [61–63]

Consider only the real part of ODE eq. (real) which has the solution in the following form;

$$\Omega(\eta) = \frac{\alpha_0 + \alpha_1\Theta(\eta) + \cdots + \alpha_N\Theta^N(\eta)}{\beta_0 + \beta_1\Theta(\eta) + \cdots + \beta_N\Theta^N(\eta)}, \tag{26}$$

where

$$\Theta(\eta) = \frac{\tau_1 e^{\lambda_1\eta} + \tau_2 e^{\lambda_2\eta}}{\tau_3 e^{\lambda_3\eta} + \tau_4 e^{\lambda_4\eta}}, \tag{27}$$

where α_j, β_j $(0 \le j \le N)$ are real unknown constants and τ_1, λ_i $(1 \le j \le 4)$ are arbitrary constants and substituting the value of N

Eq. (26) takes the following form;

$$\Omega(\eta) = \frac{\alpha_0 + \alpha_1 \Theta(\eta)}{\beta_0 + \beta_1 \Theta(\eta)}. \tag{28}$$

By substituting the Eq. (26) with its derivatives into the Eq. (7) with Eq. (27) and gathering all the terms of the same power of $e^{\lambda_j \eta}$ with (j = 1,2,3,4). The coefficients of these polynomials equate to zero and get an algebraic system of equations. With the help of $mathematica 11.1$ solving this system, we obtained the different families of solutions for the values of constant and parameters as follows:

Case-1. When $\tau = [-1, -1, 1-1]$ and $\lambda = [1, -1, 1, -1]$, then we obtained

$$\Omega(\eta) = -\frac{\cosh(\eta)}{\sinh(\eta)}. \tag{29}$$

The dark solitons solution Eq. (12) are derived as follows;

Family-1.

$$\left\{ \alpha_0 = -\frac{i\sqrt{2}\alpha\beta_1}{\sqrt{g}}; \quad \alpha_1 = -\frac{i\sqrt{2}\alpha\beta_0}{\sqrt{g}}; \quad m = 2\alpha^2 + l^2 - V - \sigma W; \right.$$

$$\Phi_{15}(x,t) = \left(\frac{\left(-\frac{i\sqrt{2}\alpha\beta_0 \coth(ct-\alpha x)}{\sqrt{g}} - \frac{i\sqrt{2}\alpha\beta_1}{\sqrt{g}} \right)}{\beta_0 + \beta_1 \coth(ct - \alpha x)} \right) e^{i\left(t\left(2\alpha^2+l^2-V-\sigma W\right)+lx\right)}, \tag{30}$$

Family-2.

$$\left\{ \alpha_0 = 0; \quad \alpha_1 = -\frac{i\sqrt{2}\alpha\beta_0}{\sqrt{g}}; \quad \beta_0 = \beta_0; \quad \beta_1 = 0; \quad m = 2\alpha^2 + l^2 - V - \sigma W; \right.$$

$$\Phi_{16}(x,t) = \left(\frac{i\sqrt{2}\alpha \coth(\eta)}{\sqrt{g}} \right) e^{i\left(t\left(2\alpha^2+l^2-V-\sigma W\right)+lx\right)}, \tag{31}$$

Case-2. When $\tau = [2, 0, 1-1]$ and $\lambda = [1, 0, 1, -1]$, then we obtained

$$\Omega(\eta) = \frac{\sinh(\eta) + \cosh(\eta)}{\sinh(\eta)}. \tag{32}$$

The wave solutions of Eq. (12) are derived as follows;

Family-1.

$$\left\{ \alpha_0 = \frac{i\sqrt{2}\alpha\beta_0}{\sqrt{g}}; \quad \alpha_1 = -\frac{i\alpha\beta_0}{\sqrt{2}\sqrt{g}}; \quad \beta_1 = -\frac{\beta_0}{2}; \quad m = 2\alpha^2 + l^2 - V - \sigma W; \right.$$

$$\Phi_{17}(x,t) = \left(\frac{\left(\frac{i\alpha\beta_0 \operatorname{csch}(ct-\alpha x)(\cosh(ct-\alpha x)-\sinh(ct-\alpha x))}{\sqrt{2}\sqrt{g}} + \frac{i\sqrt{2}\alpha\beta_0}{\sqrt{g}} \right)}{\beta_0 + \frac{1}{2}\beta_0 \operatorname{csch}(ct-\alpha x)(\cosh(ct-\alpha x) - \sinh(ct-\alpha x))} \right) \times e^{i\left(t\left(2\alpha^2+l^2-V-\sigma W\right)+lx\right)}, \tag{33}$$

Family-2.

$$\left\{ \alpha_0 = \frac{i\sqrt{2}\alpha\beta_0}{\sqrt{g}}; \quad \alpha_1 = -\frac{i\sqrt{2}\alpha(\beta_0+\beta_1)}{\sqrt{g}}; \quad m = 2\alpha^2 + l^2 - V - \sigma W; \right.$$

$$\Phi_{18}(x,t) = \left(\frac{\left(\frac{i\sqrt{2}\alpha(\beta_0+\beta_1)\operatorname{csch}(ct-\alpha x)(\cosh(ct-\alpha x)-\sinh(ct-\alpha x))}{\sqrt{g}} + \frac{i\sqrt{2}\alpha\beta_0}{\sqrt{g}} \right)}{\beta_0 - \beta_1 \operatorname{csch}(ct-\alpha x)(\cosh(ct-\alpha x) - \sinh(ct-\alpha x))} \right) \times e^{i\left(t\left(2\alpha^2+l^2-V-\sigma W\right)+lx\right)}, \tag{34}$$

Family-3.

$$\left\{ \alpha_1 = -\frac{\alpha_0}{2}; \quad \beta_1 = -\frac{\beta_0}{2}; \quad m = \frac{\alpha_0^2(-g)+\beta_0^2 l^2 - \beta_0^2 V - \beta_0^2 \sigma W}{\beta_0^2}; \right.$$

$$\Phi_{19}(x,t) = \left(\frac{\left(\alpha_0 + \frac{1}{2}\alpha_0 \operatorname{csch}(ct-\alpha x)(\cosh(ct-\alpha x) - \sinh(ct-\alpha x)) \right)}{\beta_0 + \frac{1}{2}\beta_0 \operatorname{csch}(ct-\alpha x)(\cosh(ct-\alpha x) - \sinh(ct-\alpha x))} \right) \times e^{i\left(\frac{t\left(\alpha_0^2(-g)+\beta_0^2 l^2-\beta_0^2 V-\beta_0^2 \sigma W\right)}{\beta_0^2}+lx\right)}, \tag{35}$$

Case-3. When $\tau = [-1-i, 1-i, -1, 1]$ and $\lambda = [i, -i, i, -i]$, then we obtained

$$\Omega(\eta) = \frac{\sin(\eta) + \cos(\eta)}{\sin(\eta)}. \tag{36}$$

The wave solutions of Eq. (12) are derived as follows;

Family-1.

$$\left\{ \alpha_0 = (-1-i)\alpha_1; \quad \beta_0 = (-1-i)\beta_1; \quad m = \frac{\alpha_1^2(-g)+\beta_1^2 l^2-\beta_1^2 V-\beta_1^2 \sigma W}{\beta_1^2}; \right.$$

$$\Phi_{20}(x,t) = \left(\frac{(-\alpha_1 \csc(ct-\alpha x)(\cos(ct-\alpha x) - \sin(ct-\alpha x)) + (-1-i)\alpha_1)}{-\beta_1 \csc(ct-\alpha x)(\cos(ct-\alpha x) - \sin(ct-\alpha x)) + (-1-i)\beta_1} \right) \times e^{i\left(\frac{t\left(\alpha_1^2(-g)+\beta_1^2 l^2-\beta_1^2 V-\beta_1^2 \sigma W\right)}{\beta_1^2}+lx\right)}, \tag{37}$$

Family-2.

$$\left\{ \alpha_0 = \frac{i\sqrt{2}\alpha\beta_0}{\sqrt{g}}; \quad \alpha_1 = -\frac{i\sqrt{2}\alpha\beta_0}{\sqrt{g}}; \quad \beta_1 = 0; \quad m = -2\alpha^2 + l^2 - V - \sigma W; \right.$$

$$\Phi_{21}(x,t) = \left(\frac{\left(\frac{i\sqrt{2}\alpha\beta_0 \csc(ct-\alpha x)(\cos(ct-\alpha x)-\sin(ct-\alpha x))}{\sqrt{g}} + \frac{i\sqrt{2}\alpha\beta_0}{\sqrt{g}} \right)}{\beta_0} \right) \times e^{i\left(t\left(-2\alpha^2+l^2-V-\sigma W\right)+lx\right)}, \tag{38}$$

Family-3.

$$\left\{ \alpha_0 = -\frac{i\sqrt{2}\alpha\beta_1}{\sqrt{g}}; \quad \alpha_1 = 0; \quad \beta_0 = -\beta_1; \quad m = -2\alpha^2 + l^2 - V - \sigma W; \right.$$

$$\Phi_{22}(x,t) = \left(-\frac{i\sqrt{2}\alpha\beta_1}{\sqrt{g}\left(\beta_1(-\csc(ct-\alpha x))(\cos(ct-\alpha x) - \sin(ct-\alpha x)) - \beta_1\right)} \right) \times e^{i\left(t\left(-2\alpha^2+l^2-V-\sigma W\right)+lx\right)}. \tag{39}$$

Case-4. When $\tau = [2, 0, 1, 1]$ and $\lambda = [-1, 0, 1, -1]$, then we obtained

$$\Omega(\eta) = \frac{\cosh(\eta) + \sinh(\eta)}{\cosh(\eta)}. \tag{40}$$

The soliton solutions of Eq. (12) are derived as follows;

Family-1.

$$\left\{ \alpha_0 = \frac{i\sqrt{2}\alpha\beta_0}{\sqrt{g}}; \quad \alpha_1 = -\frac{i\alpha\beta_0}{\sqrt{2}\sqrt{g}}; \quad \beta_1 = -\frac{\beta_0}{2}; m = 2\alpha^2 + l^2 - V - \sigma W; \right.$$

$$\Phi_{23}(x,t) = \left(\frac{\frac{i\sqrt{2}\alpha\beta_0}{\sqrt{g}} - \frac{i\alpha\beta_0 \operatorname{sech}(ct-\alpha x)(\sinh(ct-\alpha x)+\cosh(ct-\alpha x))}{\sqrt{2}\sqrt{g}}}{\beta_0 - \frac{1}{2}\beta_0 \operatorname{sech}(ct-\alpha x)(\sinh(ct-\alpha x) + \cosh(ct-\alpha x))} \right) \times e^{i\left(t\left(-2\alpha^2+l^2-V-\sigma W\right)+lx\right)}, \tag{41}$$

Family-2.

$$\left\{ \alpha_0 = -\frac{i\sqrt{2}\alpha\beta_0}{\sqrt{g}}; \alpha_1 = \frac{i\sqrt{2}\alpha\beta_0}{\sqrt{g}}; \beta_1 = 0; m = 2\alpha^2 + l^2 - V - \sigma W; \right.$$

$$\Phi_{24}(x,t) = \left(\frac{\left(\frac{i\sqrt{2}\alpha\beta_0 \operatorname{sech}(ct-\alpha x)(\sinh(ct-\alpha x)+\cosh(ct-\alpha x))}{\sqrt{g}} - \frac{i\sqrt{2}\alpha\beta_0}{\sqrt{g}} \right)}{\beta_0} \right) \times e^{i\left(t\left(2\alpha^2+l^2-V-\sigma W\right)+lx\right)}. \tag{42}$$

Case-5. When $\tau = [-3, -2, 1, 1]$ and $\lambda = [0, 1, 0, 1]$, then we obtained

$$\Omega(\eta) = \frac{-2e^{\eta} - 3}{e^{\eta} + 1}. \tag{43}$$

The wave solutions of Eq. (12) are derived as follows;
Family-1.

$$\left\{ \alpha_1 = \frac{\alpha_0}{3}; \quad \beta_1 = \frac{\beta_0}{3}; \quad m = \frac{\alpha_0^2(-g)+\beta_0^2 l^2-\beta_0^2 V-\beta_0^2\sigma W}{\beta_0^2}; \right.$$

$$\Phi_{25}(x,t) = \left(\frac{\left(\alpha_0 + \frac{\alpha_0(-2e^{\alpha x-ct}-3)}{3(e^{\alpha x-ct}+1)}\right)}{\beta_0 + \frac{\beta_0(-2e^{\alpha x-ct}-3)}{3(e^{\alpha x-ct}+1)}} \right) e^{i\left(\frac{t\left(\alpha_0^2(-g)+\beta_0^2 l^2-\beta_0^2 V-\beta_0^2\sigma W\right)}{\beta_0^2}+lx\right)}, \tag{44}$$

Family-2.

$$\left\{ \alpha_0 = \frac{6i\sqrt{2}\alpha\beta_1}{\sqrt{g}}; \quad \alpha_1 = \frac{5i\alpha\beta_1}{\sqrt{2}\sqrt{g}}; \quad \beta_0 = 0; \quad m = \frac{1}{2}\left(\alpha^2+2l^2-2V-2\sigma W\right); \right.$$

$$\Phi_{26}(x,t) = \left(\frac{\left(e^{\alpha x-ct}+1\right)\left(\frac{5i\alpha\beta_1(-2e^{\alpha x-ct}-3)}{\sqrt{2}\sqrt{g}(e^{\alpha x-ct}+1)} + \frac{6i\sqrt{2}\alpha\beta_1}{\sqrt{g}}\right) e^{i\left(\frac{1}{2}t(\alpha^2+2l^2-2V-2\sigma W)+lx\right)}}{\beta_1\left(-2e^{\alpha x-ct}-3\right)} \right), \tag{45}$$

Family-3.

$$\left\{ \alpha_0 = 0; \quad \alpha_1 = -\frac{i\alpha\beta_0}{12\sqrt{2}\sqrt{g}}; \quad \beta_1 = \frac{5\beta_0}{12}; \quad m = \frac{1}{2}\left(\alpha^2+2l^2-2V-2\sigma W\right); \right.$$

$$\Phi_{27}(x,t) = \left(-\frac{i\alpha\beta_0\left(-2e^{\alpha x-ct}-3\right)}{12\sqrt{2}\sqrt{g}\left(e^{\alpha x-ct}+1\right)\left(\beta_0 + \frac{5\beta_0(-2e^{\alpha x-ct}-3)}{12(e^{\alpha x-ct}+1)}\right)} \right) \times e^{i\left(\frac{1}{2}t(\alpha^2+2l^2-2V-2\sigma W)+lx\right)}. \tag{46}$$

Case-6. When $\tau = [3,2,1,1]$ and $\lambda = [0,1,0,1]$, then we obtained

$$\Omega(\eta) = \frac{3e^\eta+2}{e^\eta+1}. \tag{47}$$

The soliton-like solutions of Eq. (12) are derived as follows;
Family-1.

$$\left\{ \begin{array}{l} \alpha_0 = -\frac{6i\sqrt{2}\alpha\beta_1}{\sqrt{g}}; \quad \alpha_1 = \frac{5i\alpha\beta_1}{\sqrt{2}\sqrt{g}}; \quad \beta_0 = 0; \\ m = \frac{1}{2}\left(\alpha^2+2l^2-2V-2\sigma W\right); \end{array} \right.$$

$$\Phi_{28}(x,t) = \left(\frac{\left(e^{\alpha x-ct}+1\right)\left(\frac{5i\alpha\beta_1(3e^{\alpha x-ct}+2)}{\sqrt{2}\sqrt{g}(e^{\alpha x-ct}+1)} - \frac{6i\sqrt{2}\alpha\beta_1}{\sqrt{g}}\right)}{\beta_1\left(3e^{\alpha x-ct}+2\right)} \right) \times e^{i\left(\frac{1}{2}t(\alpha^2+2l^2-2V-2\sigma W)+lx\right)}, \tag{48}$$

Family-2.

$$\left\{ \alpha_1 = -\frac{1}{12}\left(5\alpha_0\right); \quad \beta_1 = -\frac{1}{12}\left(5\beta_0\right); \quad m = \frac{\alpha_0^2(-g)+\beta_0^2 l^2-\beta_0^2 V-\beta_0^2\sigma W}{\beta_0^2}; \right.$$

$$\Phi_{29}(x,t) = \left(\frac{\left(\alpha_0 - \frac{5\alpha_0(3e^{\alpha x-ct}+2)}{12(e^{\alpha x-ct}+1)}\right)}{\beta_0 - \frac{5\beta_0(3e^{\alpha x-ct}+2)}{12(e^{\alpha x-ct}+1)}} \right) e^{i\left(\frac{t\left(\alpha_0^2(-g)+\beta_0^2 l^2-\beta_0^2 V-\beta_0^2\sigma W\right)}{\beta_0^2}+lx\right)}, \tag{49}$$

Comparison of plots

In this section, the graphical comparison of numerical solutions and exact solutions are drawn,

Example 1. The test problem is considered as follows,

$$i\Phi_t = -\frac{1}{2}\Phi_{xx} + V(x)\Phi + g|\Phi|^2\Phi + \sigma\dot{W}(t)\Phi, \tag{50}$$

using the initial condition as

$$\Phi(x,0) = (-3.35819+0.i)e^{i(2.95x+0.)}\mathrm{sech}(0.793915(2.991x+0.)), \tag{51}$$

and BCs as follows,

$$\Phi(0,t) = (-3.35819+0.i)e^{i(1.91t+0.)}\mathrm{sech}(0.793915(0.-0.09t)), \tag{52}$$

$$\Phi(1,t) = (-3.35819+0.i)e^{i(1.91t+2.95)}\mathrm{sech}(0.793915(2.991-0.09t)). \tag{53}$$

We construct the graphs of NFDS from Eq. (2) using the above IC and BCs and compare them with the exact solution from the Eq. (12). So, Fig. 1 shows the graphical comparison of NSFD with exact solution $\Phi_1(x,t)$.

Example 2. The test problem is considered as follows,

$$i\Phi_t = -\frac{1}{2}\Phi_{xx} + V(x)\Phi + g|\Phi|^2\Phi + \sigma\dot{W}(t)\Phi, \tag{54}$$

using the initial condition as

$$\Phi(x,0) = (-3.00688+0.i)e^{i(2.95x+0.)}\mathrm{sech}((0.710861+0.i)(2.991x+0.)), \tag{55}$$

and BCs as follows,

$$\Phi(0,t) = (-3.00688+0.i)e^{i(2.91t+0.)}\mathrm{sech}((0.710861+0.i)(0.-0.9t)), \tag{56}$$

$$\Phi(1,t) = (-3.00688+0.i)e^{i(2.91t+2.95)}\mathrm{sech}((0.710861+0.i)(2.991-0.9t)). \tag{57}$$

We construct the graphs of NFDS from Eq. (2) using the above IC and BCs and compare them with the exact solution from the Eq. (14). So, Fig. 2 shows the graphical comparison of NSFD with exact solution $\Phi_3(x,t)$.

Example 3. The test problem is considered as follows,

$$i\Phi_t = -\frac{1}{2}\Phi_{xx} + V(x)\Phi + g|\Phi|^2\Phi + \sigma\dot{W}(t)\Phi, \tag{58}$$

using the initial condition as

$$\Phi(x,0) = (0.+1.52948i)e^{i(0.905x+0.)}\tanh(0.566234(1.91x+0.)), \tag{59}$$

and BCs as follows,

$$\Phi(0,t) = (0.+1.52948i)e^{i(1.9091t+0.)}\tanh(0.566234(0.-1.9t)), \tag{60}$$

$$\Phi(1,t) = (0.+1.52948i)e^{i(1.9091t+0.905)}\tanh(0.566234(1.91-1.9t)). \tag{61}$$

We construct the graphs of NFDS from Eq. (2) using the above IC and BCs and compare them with the exact solution from the Eq. (28). So, Fig. 3 shows the graphical comparison of NSFD with exact solution $\Phi_5(x,t)$.

Example 4. The test problem is considered as follows,

$$i\Phi_t = -\frac{1}{2}\Phi_{xx} + V(x)\Phi + g|\Phi|^2\Phi + \sigma\dot{W}(t)\Phi, \tag{62}$$

using the initial condition as

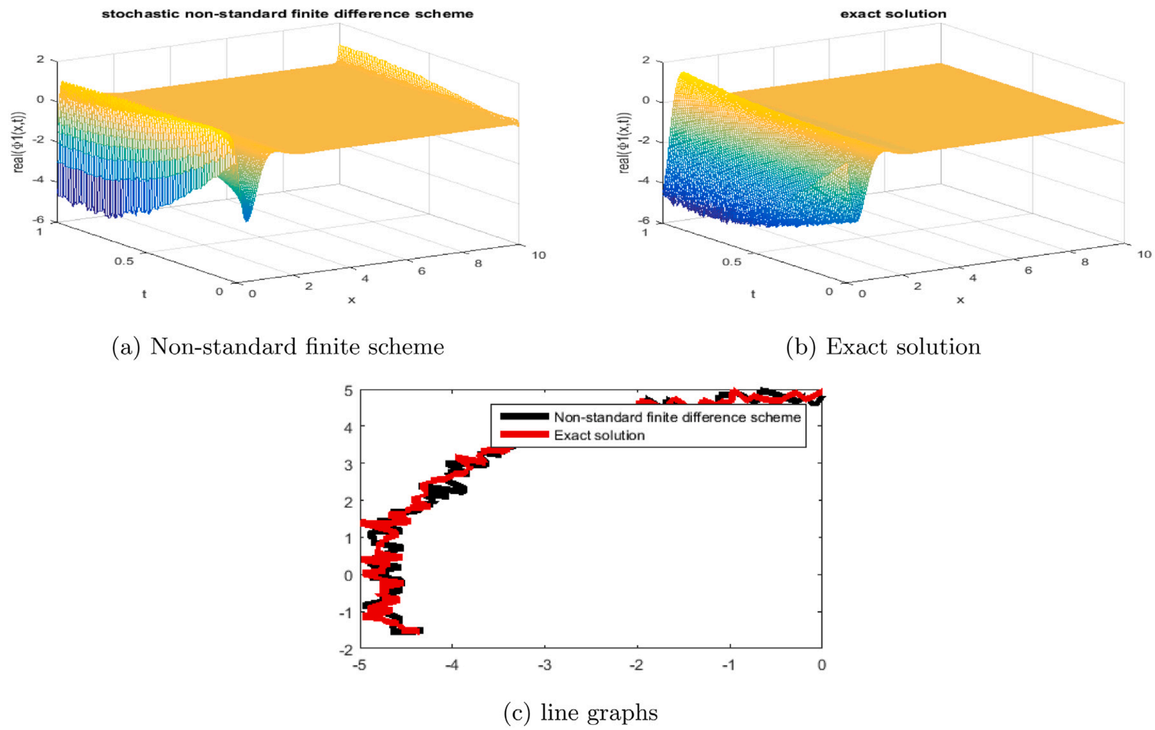
$$\Phi(x,0) = (0.+2.08475i)e^{i(2.5x+0.)}\tan(0.506577(2.91x+0.)), \tag{63}$$

and BCs as follows,

$$\Phi(0,t) = (0.+2.08475i)e^{i(0.91t+0.)}\tan(0.506577(0.-1.9t)), \tag{64}$$

$$\Phi(1,t) = (0.+2.08475i)e^{i(0.91t+2.5)}\tan(0.506577(2.91-1.9t)). \tag{65}$$

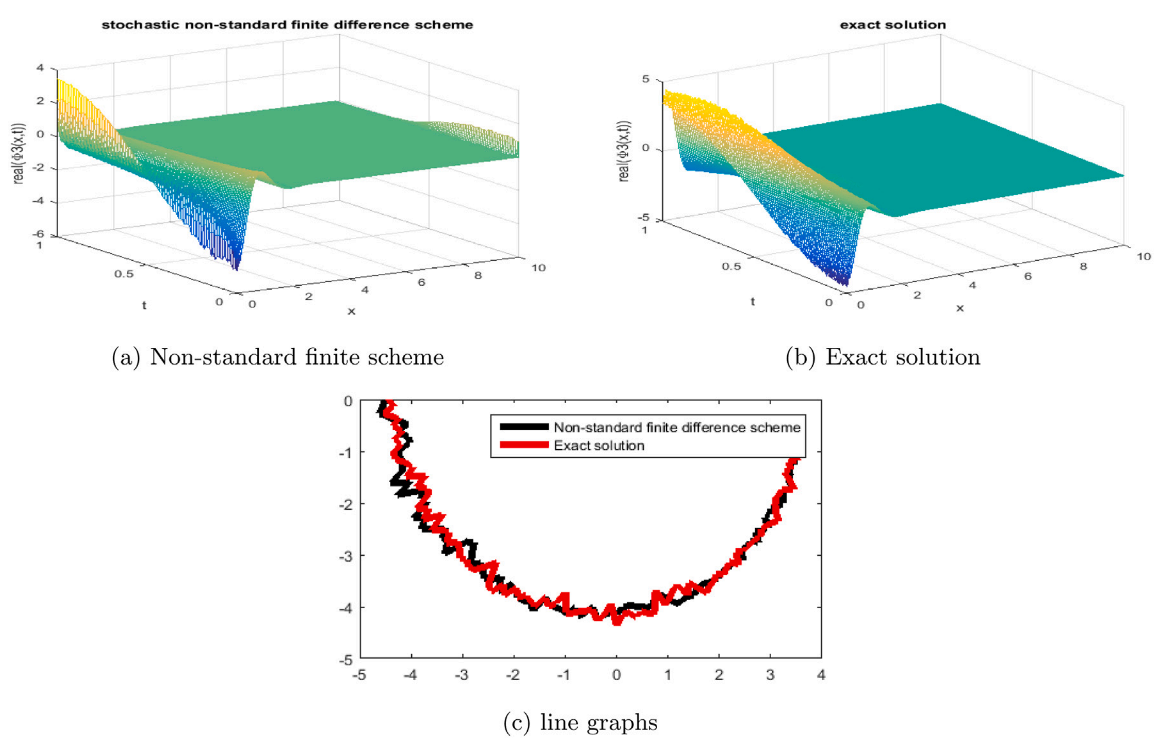
We construct the graphs of NFDS from Eq. (2) using the above IC and BCs and compare them with the exact solution from the Eq. (21). So, Fig. 4 shows the graphical comparison of NSFD with exact solution $\Phi_{10}(x,t)$.



(a) Non-standard finite scheme (b) Exact solution

(c) line graphs

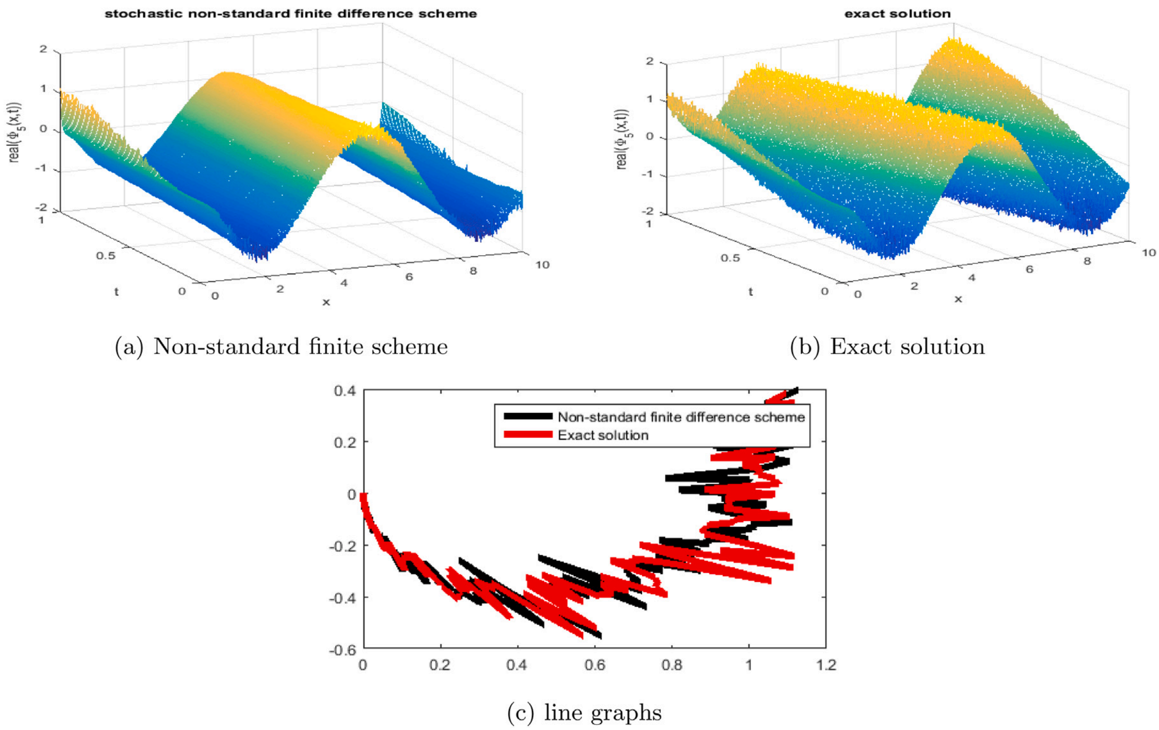
Fig. 1. The subfig (a) for non-standard finite difference scheme, subfig (b) for the exact solution $\Phi_1(x,t)$ and subfig (c) shows the line comparison for both (a) and (b) subfigs using the parameters values that are given in Appendix "Graphics parameters".



(a) Non-standard finite scheme (b) Exact solution

(c) line graphs

Fig. 2. The subfig (a) for non-standard finite difference scheme, subfig (b) for the exact solution $\Phi_3(x,t)$ and subfig (c) shows the line comparison for both (a) and (b) subfigs using the parameters values that are given in Appendix "Graphics parameters".

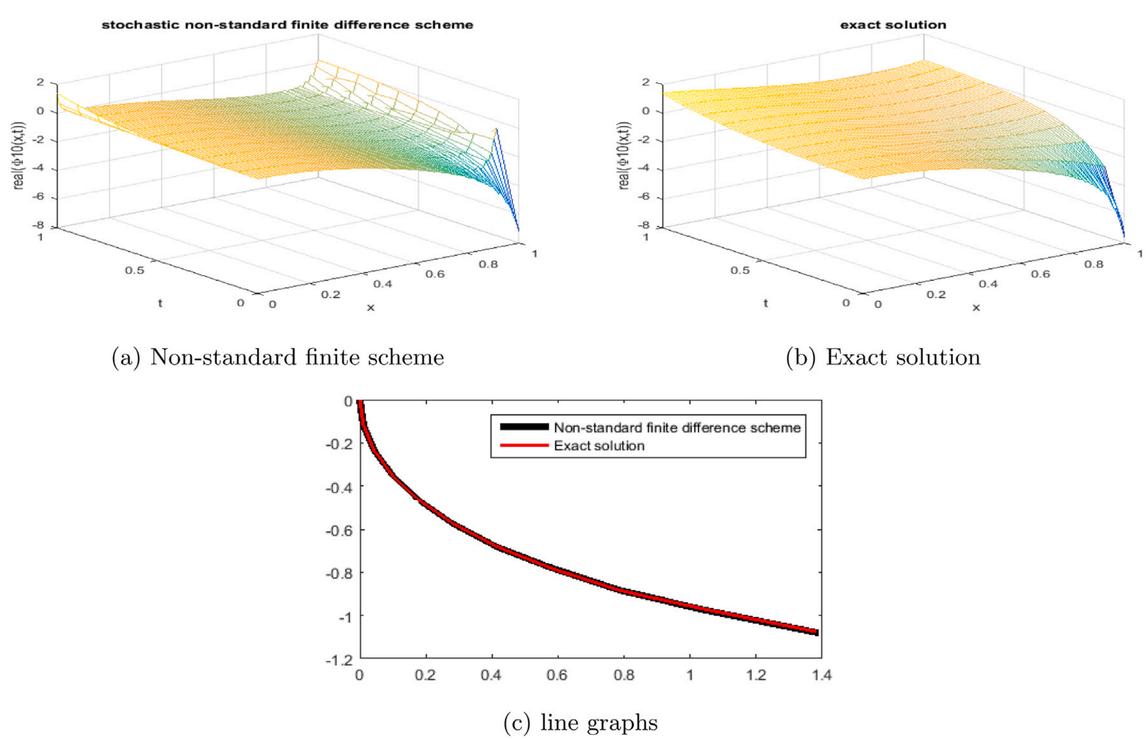


(a) Non-standard finite scheme

(b) Exact solution

(c) line graphs

Fig. 3. The subfig (a) for non-standard finite difference scheme, subfig (b) for the exact solution $\Phi_5(x,t)$ and subfig (c) shows the line comparison for both (a) and (b) subfigs using the parameters' values that are given in Appendix "Graphics parameters".



(a) Non-standard finite scheme

(b) Exact solution

(c) line graphs

Fig. 4. The subfig (a) for non-standard finite difference scheme, subfig (b) for the exact solution $\Phi_{10}(x,t)$ and subfig (c) shows the line comparison for both (a) and (b) subfigs using the parameters values that are given in Appendix "Graphics parameters".

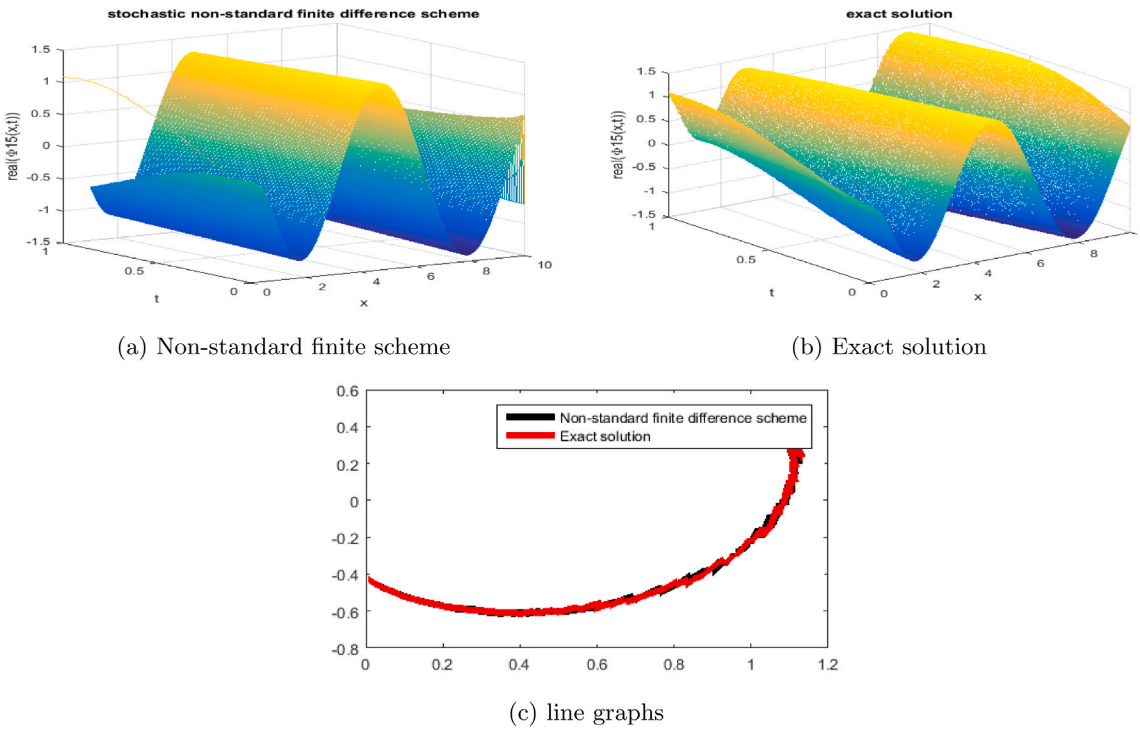


Fig. 5. The subfig (a) for non-standard finite difference scheme, subfig (b) for the exact solution $\Phi_{15}(x,t)$ and subfig (c) shows the line comparison for both (a) and (b) subfigs using the parameters values that are given in Appendix "Graphics parameters".

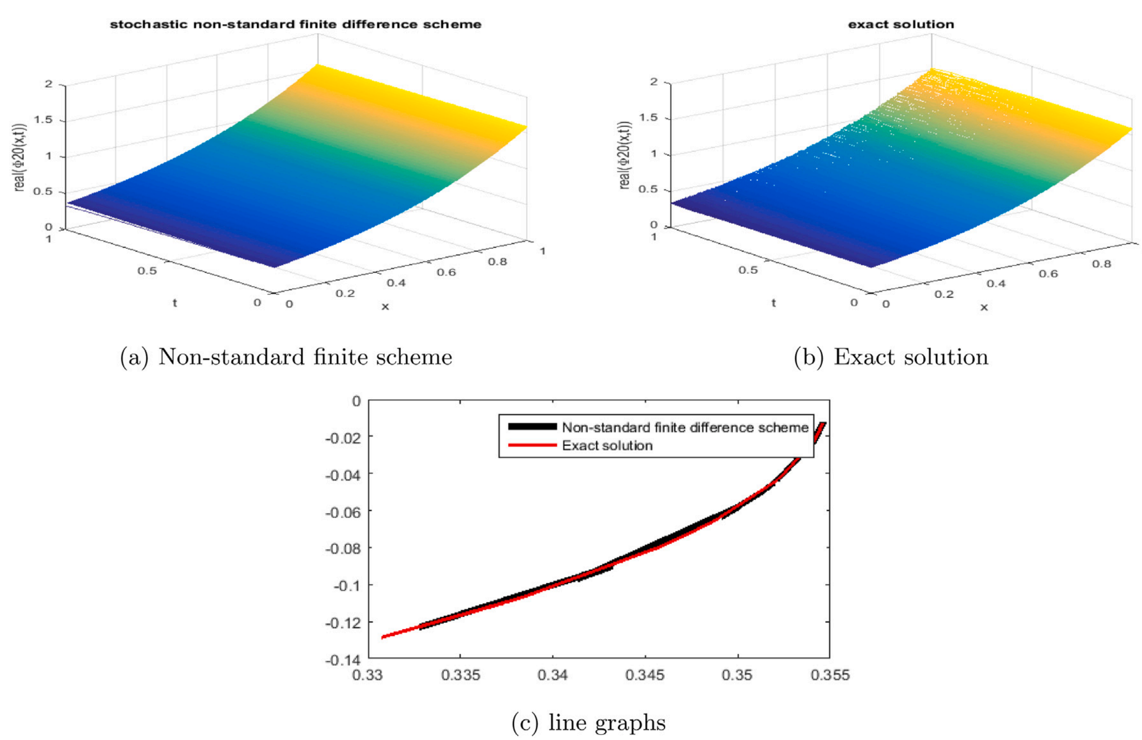
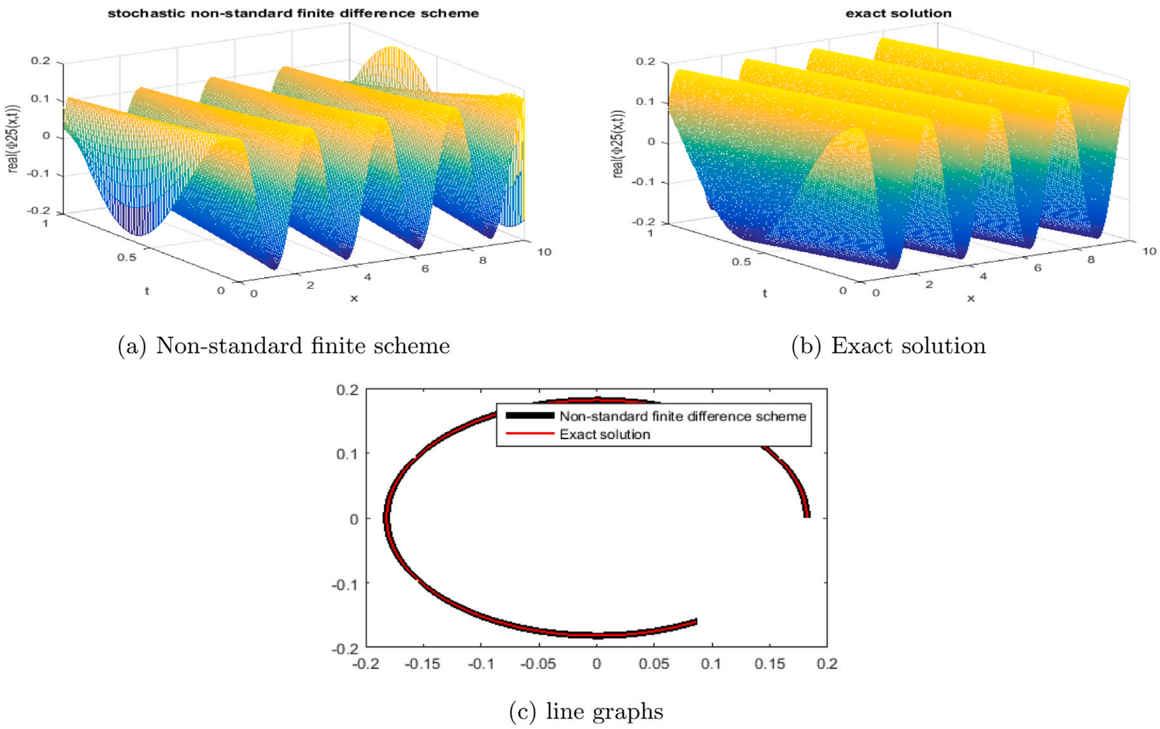


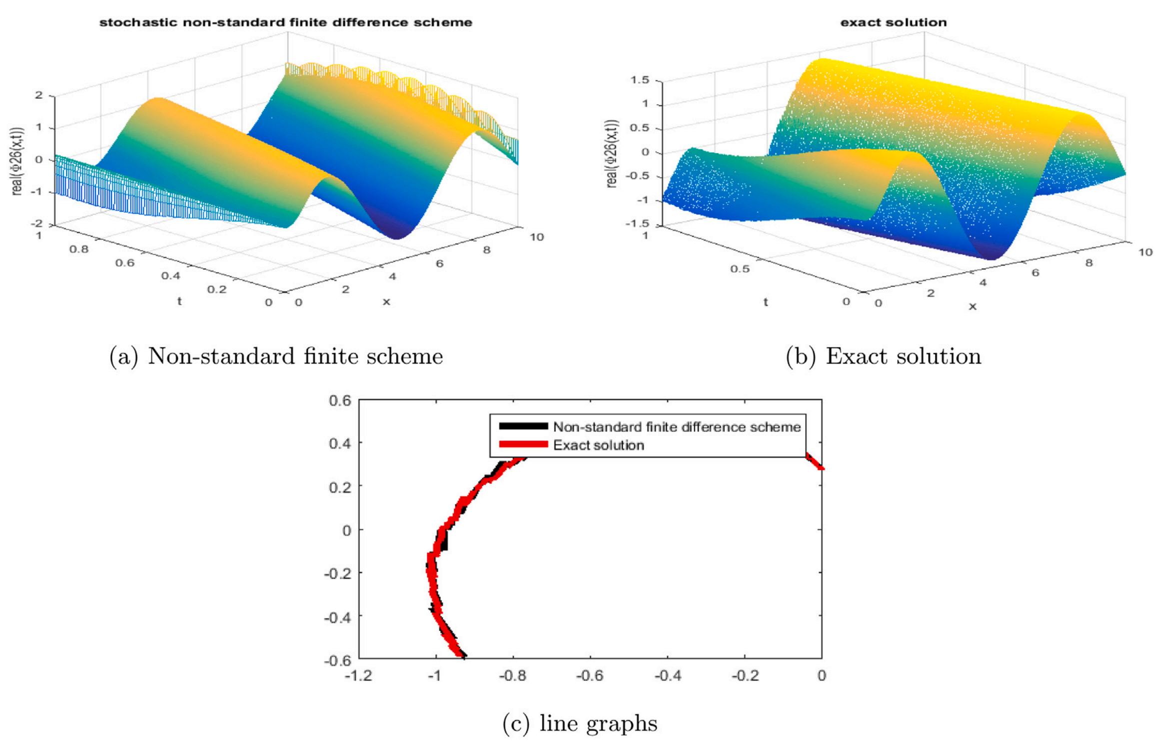
Fig. 6. The subfig (a) for non-standard finite difference scheme, subfig (b) for the exact solution $\Phi_{20}(x,t)$ and subfig (c) shows the line comparison for both (a) and (b) subfigs using the parameters values that are given in Appendix "Graphics parameters".



(a) Non-standard finite scheme (b) Exact solution

(c) line graphs

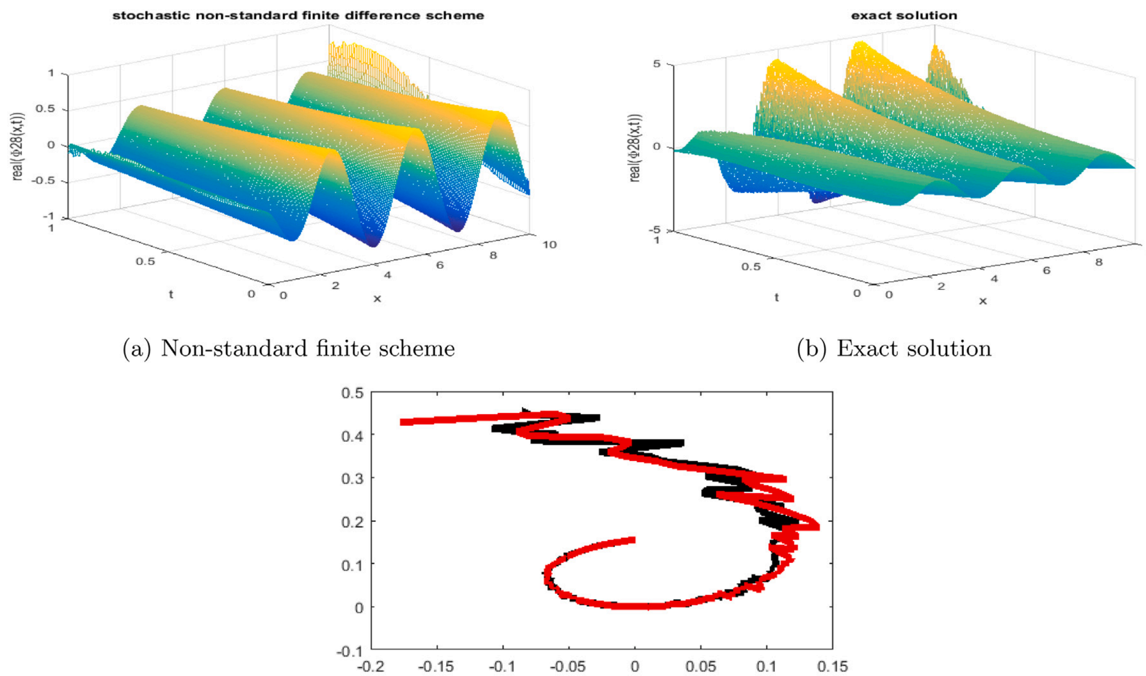
Fig. 7. The subfig (a) for non-standard finite difference scheme, subfig (b) for the exact solution $\Phi_{25}(x,t)$ and subfig (c) shows the line comparison for both (a) and (b) subfigs using the parameters values that are given in Appendix "Graphics parameters".



(a) Non-standard finite scheme (b) Exact solution

(c) line graphs

Fig. 8. The subfig (a) for non-standard finite difference scheme, subfig (b) for the exact solution $\Phi_{26}(x,t)$ and subfig (c) shows the line comparison for both (a) and (b) subfigs using the parameters values that are given in Appendix "Graphics parameters".



(a) Non-standard finite scheme

(b) Exact solution

(c) line graphs

Fig. 9. The subfig (a) for non-standard finite difference scheme, subfig (b) for the exact solution $\Phi_{28}(x,t)$ and subfig (c) shows the line comparison for both (a) and (b) subfigs using the parameters values that are given in Appendix "Graphics parameters".

Example 5. The test problem is considered as follows,

$$i\Phi_t = -\frac{1}{2}\Phi_{xx} + V(x)\Phi + g|\Phi|^2\Phi + \sigma \dot{W}(t)\Phi, \tag{66}$$

using the initial condition as

$$\Phi(x,0) = \frac{e^{i(0.995x+0.)}((0.\ -4.20446i) - (0.\ +1.26134i)\coth(0.\ -0.991x))}{3\coth(0.\ -0.991x)+0.9}, \tag{67}$$

and BCs as follows,

$$\Phi(0,t) = \frac{e^{i(2.01991t+0.)}((0.\ -4.20446i) - (0.\ +1.26134i)\coth(0.9t+0.))}{3\coth(0.9t+0.)+0.9}, \tag{68}$$

$$\Phi(1,t) = \frac{e^{i(2.01991t+0.995)}((0.\ +1.26134i)\coth(0.991\ -0.9t) + (0.\ -4.20446i))}{0.9\ -3\coth(0.991\ -0.9t)}. \tag{69}$$

We construct the graphs of NFDS from Eq. (2) using the above IC and BCs and compare them with the exact solution from the Eq. (30). So, Fig. 5 shows the graphical comparison of NSFD with exact solution $\Phi_{15}(x,t)$.

Example 6. The test problem is considered as follows,

$$i\Phi_t = -\frac{1}{2}\Phi_{xx} + V(x)\Phi + g|\Phi|^2\Phi + \sigma \dot{W}(t)\Phi, \tag{70}$$

using the initial condition as

$$\Phi(x,0) = \frac{e^{i(1.5x+0.)}(-1.1\csc(0.\ -3.91x)(\cos(0.\ -3.91x) - \sin(0.\ -3.91x)) + (-1.1-1.1i))}{-3.1\csc(0.\ -3.91x)(\cos(0.\ -3.91x) - \sin(0.\ -3.91x)) + (-3.1-3.1i)}, \tag{71}$$

and BCs as follows,

$$\Phi(0,t) = \frac{e^{i(1.15725t+0.)}(-1.1\csc(1.9t+0.)(\cos(1.9t+0.) - \sin(1.9t+0.)) + (-1.1-1.1i))}{-3.1\csc(1.9t+0.)(\cos(1.9t+0.) - \sin(1.9t+0.)) + (-3.1-3.1i)}, \tag{72}$$

$$\Phi(1,t) = \frac{e^{i(1.15725t+1.5)}(1.1\csc(3.91\ -1.9t)(\sin(3.91\ -1.9t) + \cos(3.91\ -1.9t)) + (-1.1-1.1i))}{3.1\csc(3.91\ -1.9t)(\sin(3.91\ -1.9t) + \cos(3.91\ -1.9t)) + (-3.1-3.1i)}. \tag{73}$$

We construct the graphs of NFDS from Eq. (2) using the above IC and BCs and compare them with the exact solution from the Eq. (37). So, Fig. 6 shows the graphical comparison of NSFD with exact solution $\Phi_{20}(x,t)$.

Example 7. The test problem is considered as follows,

$$i\Phi_t = -\frac{1}{2}\Phi_{xx} + V(x)\Phi + g|\Phi|^2\Phi + \sigma \dot{W}(t)\Phi, \tag{74}$$

using the initial condition as

$$\Phi(x,0) = 0.181818e^{(0.+2.5i)x}, \tag{75}$$

and BCs as follows,

$$\Phi(0,t) = 0.181818e^{(0.+5.20733i)t}, \tag{76}$$

$$\Phi(1,t) = 0.181818e^{(0.+5.20733i)t+(0.+2.5i)}. \tag{77}$$

We construct the graphs of NFDS from Eq. (2) using the above IC and BCs and compare them with the exact solution from the Eq. (44). So, Fig. 7 shows the graphical comparison of NSFD with exact solution $\Phi_{25}(x,t)$.

Example 8. The test problem is considered as follows,

$$i\Phi_t = -\frac{1}{2}\Phi_{xx} + V(x)\Phi + g|\Phi|^2\Phi + \sigma \dot{W}(t)\Phi, \tag{78}$$

using the initial condition as

$$\Phi(x,0) = -\frac{(0. + 1.35057i)\left(1.e^{1.91x} - 1.5\right) e^{(0.+0.95i)x}}{1.5 + 1.e^{1.91x}}, \tag{79}$$

and BCs as follows,

$$\Phi(0,t) = \frac{(0. + 1.35057i)\left(1.e^{1.9t} - 0.666667\right) e^{(0.+1.73868i)t}}{0.666667 + 1.e^{1.9t}}, \tag{80}$$

$$\Phi(1,t) = \frac{(0. + 1.35057i)\left(1.e^{1.9t} - 4.50206\right) e^{(0.+1.73868i)t+(0.+0.95i)}}{4.50206 + 1.e^{1.9t}}. \tag{81}$$

We construct the graphs of NFDS from Eq. (2) using the above IC and BCs and compare with the exact solution from the Eq. (45). So, Fig. 8 shows the graphical comparison of NSFD with exact solution $\Phi_{26}(x,t)$.

Example 9. The test problem is considered as follows,

$$i\Phi_t = -\frac{1}{2}\Phi_{xx} + V(x)\Phi + g|\Phi|^2\Phi + \sigma\dot{W}(t)\Phi, \tag{82}$$

using the initial condition as

$$\Phi(x,0) = \frac{(0. + 0.777817i)\left(1.e^{1.1x} - 0.666667\right) e^{(0.+1.95i)x}}{0.666667 + 1.e^{1.1x}}, \tag{83}$$

and BCs as follows,

$$\Phi(0,t) = -\frac{(0. + 0.777817i)\left(1.e^{0.9t} - 1.5\right) e^{(0.+3.43782i)t}}{1.5 + 1.e^{0.9t}}, \tag{84}$$

$$\Phi(1,t) = -\frac{(0. + 0.777817i)\left(1.e^{0.9t} - 4.50625\right) e^{(0.+3.43782i)t+(0.+1.95i)}}{4.50625 + 1.e^{0.9t}}. \tag{85}$$

We construct the graphs of NFDS from Eq. (2) using the above IC and BCs and compare them with the exact solution from the Eq. (48). So, Fig. 9 shows the graphical comparison of NSFD with exact solution $\Phi_{28}(x,t)$.

Results and discussion

In this study, the stochastic Gross–Pitaevskii equation (SGPE) is investigated analytically and numerically with multiplicative time noise. For the approximate solutions, the proposed stochastic NSFD scheme is constructed. The given numerical scheme is unconditionally stable and consistent with the given equation in the mean square sense. Meanwhile, for the sake of exact or analytical solutions, we choose two novel techniques namely, Sardar subequation (SSE) and modified exponential rational functional (MERF). So, solutions are successfully gained in the form of exponential, hyperbolic, and trigonometric forms. The results achieved are amazing and different from what has previously been published. Mainly, the comparison of graphical behavior of numerical solutions with some exact solutions that are extracted successfully drawn by using the initial and boundary conditions are given in the above section. Moreover, the Figs. 1–9 shows the wave structure or solitary wave solutions with the noise effect that appears in the literature for the sake of stochastic behavior. The SPGE model is applicable in the propagation of light in optical fiber, planar waveguides, and Bose–Einstein condensates confined to highly anisotropic sigar-shaped traps in the mean-field regime of dilute gases. The stochastc behavior is due to time noise and σ is the noisy strength which is a Boral function, These results are very useful for the physical appearance of the optical fibers and dilute gases. Mainly contribution is that the result shows a much similar behavior of numerical and exact solutions. So, our results will significantly contribute to the understanding of optical waves. The graphical structures of the earned solutions are successfully drawn under the suitable values of parameters that are given in Appendix “Graphics parameters”. This unique work is successfully gained on Matlab 2015 and Mathematica 11.1 as well.

Conclusion

In this article, the stochastic Gross–Pitaevskii equation (SGPE) perturbed with multiplicative time noise in under investigation. The proposed stochastic non-standard finite difference (SNSFD) scheme is developed for the numerical solutions of the governing model. The stability of the scheme is proved by using the Von-Neumann criteria and the consistency is shown in the mean square sense. To seek exact solutions, we applied the Sardar subequation (SSE) and modified exponential rational functional (MERF) techniques. The exact solutions are constructed in the form of exponential, hyperbolic, and trigonometric forms. Additionally, we select the unique physical problem for the comparison of results. Finally, the comparison of the exact solutions with numerical solutions is drawn in the 3D and line plots for the different values of parameters.

CRediT authorship contribution statement

Muhammad Zafarullah Baber: Analysis. **Nauman Ahmed:** Analysis. **Muhammad Waqas Yasin:** Preparing figures. **Muhammad Sajid Iqbal:** Investigation. **Ali Akgül:** Supervision, Writing – editing. **Muhammad Bilal Riaz:** Supervision, Writing – editing. **Muhammad Rafiq:** Preparing figures. **Ali Raza:** Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix: Graphics parameters

The parameters that are used to drawn the Figs. 1–9 of some solutions are given below.

- For, Fig. 1 we use the parameters values as $\sigma = 0.3$, $n = 1000$, $N = 100$, $\alpha = 2.991$, $m = 1.91$, $l = 2.95$, $d = 0.01$, $g = 1$, $c = 0.09$, $p = 1$, $q = 2$ and $V(x) = 1$.
- For, Fig. 2 we use the parameters values as $\sigma = 0.3$, $n = 1000$, $N = 100$, $\alpha = 2.991$, $m = 2.91$, $l = 2.95$, $d = 0.01$, $g = 1$, $c = 0.9$, $p = 1$, $q = 2$ and $V(x) = 1$.
- For, Fig. 3 we use the parameters values as $\sigma = 0.3$, $n = 1000$, $N = 100$, $\alpha = 1.91$, $m = 1.9091$, $l = 0.905$, $d = 0.01$, $g = 1$, $c = 1.9$, and $V(x) = 1$.
- For, Fig. 4 we use the parameters values as $\sigma = 0.02$, $n = 100$, $N = 10$, $\alpha = 2.91$, $m = 0.9091$, $l = 2.5$, $d = 0.01$, $g = 1$, $c = 1.9$, and $V(x) = 1$.
- For, Fig. 5 we use the parameters values as $\sigma = 0.04$, $n = 1000$, $N = 100$, $\alpha = 0.991$, $\beta_1 = 3$, $\beta_0 = 0.9$, $l = 0.995$, $d = 0.3$, $g = 1$, $c = 0.9$, and $V(x) = 1$.
- For, Fig. 6 we use the parameters values as $\sigma = 0.04$, $n = 1000$, $N = 10$, $\alpha = 3.91$, $\beta_1 = 3.1$, $\alpha_1 = 1.1$, $l = 1.5$, $d = 0.01$, $g = 1$, $c = 1.9$, and $V(x) = 1$.
- For, Fig. 7 we use the parameters values as $\sigma = 0.01$, $n = 1000$, $N = 100$, $\alpha = 2.1$, $\beta_0 = 1.1$, $\alpha_0 = 0.2$, $l = 2.5$, $d = 0.01$, $g = 1$, $c = 3.9$, and $V(x) = 1$.
- For, Fig. 8 we use the parameters values as $\sigma = 0.04$, $n = 1000$, $N = 100$, $\alpha = 1.91$, $\beta_1 = 1.1$, $\alpha_1 = 1$, $l = 0.95$, $d = 0.01$, $g = 1$, $c = 1.9$, and $V(x) = 1$.
- For, Fig. 9 we use the parameters values as $\sigma = 0.1$, $n = 1000$, $N = 100$, $\alpha = 1.1$, $\beta_1 = 1$, $l = 1.95$, $d = 0.01$, $g = 1$, $c = 0.9$, and $V(x) = 1$.

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