Conjectured strong complementary-correlations tradeoff

Andrzej Grudka,[1] Michał Horodecki,[2] Paweł Horodecki,[3,4] Ryszard Horodecki,[2,4] Waldemar Kłobus,[1] and Łukasz Pankowski[2]
[1]*Faculty of Physics, Adam Mickiewicz University, 61-614 Poznań, Poland*
[2]*Institute for Theoretical Physics and Astrophysics, University of Gdańsk, 80-952 Gdańsk, Poland*
[3]*Faculty of Technical Physics and Applied Mathematics, Gdańsk University of Technology, 80-952 Gdańsk, Poland*
[4]*National Quantum Information Centre in Gdańsk, 81-824 Sopot, Poland*

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We conjecture uncertainty relations that restrict correlations between the results of measurements performed by two separate parties on a shared quantum state. The first uncertainty relation bounds the sum of two mutual informations when one party measures a single observable and the other party measures one of two observables. The uncertainty relation does not follow from the Maassen-Uffink uncertainty relation and is much stronger than the Hall uncertainty relation derived from the latter. The second uncertainty relation bounds the sum of two mutual informations when each party measures one of two observables. We provide numerical evidence for the validity of the conjectured uncertainty relations, and we prove them for large classes of states and observables.

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I. INTRODUCTION

Uncertainty relations impose fundamental limitations on our ability to simultaneously predict the outcomes of measurements of different observables. They are widely known in the form given by Robertson [1], which relates variances of two operators with their commutator by inequality:

$$\Delta A \Delta B \geqslant \tfrac{1}{2}|\langle\psi|[A,B]|\psi\rangle|. \tag{1}$$

A special example of such an uncertainty relation is the Heisenberg uncertainty relation for position and momentum measurements, which states that $\Delta x \Delta p \geqslant \hbar/2$. However, for many operators, the right-hand side (RHS) of (1) depends on a state $|\psi\rangle$ and can be equal to 0, although both variances on the left-hand side (LHS) of (1) cannot simultaneously be equal to 0.

Several authors recognized that one can express uncertainty relations in terms of entropies (see [2,3] for review and [4–11] for recent developments). In particular, Maassen and Uffink [12], inspired by the work of Deutsch [13], derived the following entropic uncertainty relation:[1]

$$S(B^{(1)}) + S(B^{(2)}) \geqslant -\log a. \tag{2}$$

Here $S(B^{(s)})$ is the entropy of measurement outcomes when a measurement of observable $B^{(s)}$ is performed on a state ρ, and $a = \max_{i,j} |\langle b_i^{(1)}|b_j^{(2)}\rangle|^2$ is the square of maximum overlap between eigenvectors $|b_i^{(1)}\rangle$ of the observable $B^{(1)}$ and eigenvectors $|b_j^{(2)}\rangle$ of the observable $B^{(2)}$. This uncertainty relation does not suffer from the previous criticism because the right-hand side of (2) does not depend on ρ.

The result of Maassen and Uffink was extended by Hall [14] to derive a bound on accessible information about a quantum system represented by an ensemble of states. Let us suppose that Alice prepares a state ρ_i with probability p_i and Bob performs a measurement on it of an observable $B^{(1)}$ or $B^{(2)}$. The Hall uncertainty principle states that the sum of two accessible information satisfies the inequality

$$I(B^{(1)}|\mathcal{E}) + I(B^{(2)}|\mathcal{E}) \leqslant 2\log d + \log a, \tag{3}$$

where d is the dimension of Bob's Hilbert space. Here $I(B^{(s)}|\mathcal{E}) = S(B^{(s)})_\rho - \sum_i p_i S(B^{(s)})_{\rho_i}$ is accessible information about the ensemble of states $\mathcal{E}$ when a measurement of the observable $B^{(s)}$ is performed on it, and $\rho = \sum_i p_i \rho_i$.

For some reason, this latter direction was not further developed, even though it involves a central quantity for communication, which is mutual information. At the same time, one can see that Hall inequalities are actually very far from being tight, unlike the original entropic inequalities. Secondly, once we deal with mutual information, it is tempting to consider two subsystems. This is the feature of the newest uncertainty principle conjectured in [4] and proved in [5] involving conditional entropy. However, once there are two systems at play, it is natural to ask about the principle that is symmetric with respect to the subsystems. None of the existing uncertainty principles has this feature.

In this paper, we aim to overcome these two drawbacks. First, we propose an uncertainty relation based on mutual information that is stronger than Hall's uncertainty relation. Second, we introduce an alternative uncertainty relation that, being not yet fully symmetric with respect to two subsystems, exhibits symmetry of the following sort: it involves measurements of two observables on each of two subsystems. We provide numerical evidence for the validity of conjectured relations, and we prove them for large classes of states and observables.

More specifically, we consider two parties—Alice and Bob—who share a quantum state, and we assume that Alice performs a measurement on her part of the state in some basis and Bob performs a measurement on his part of the state in one of two bases. We will be interested in accessible correlations between measurement outcomes when Alice and Bob measure one pair of their observables rather than in accessible information about one party's measurement. We will derive an uncertainty relation which bounds the sum of two mutual informations—the first one between the results of Alice's measurement and those of Bob's first measurement,

[1]All logarithms in this paper are of base 2.

and the second one between the results of Alice's measurement and those of Bob's second measurement. Although we proved our uncertainty relation only for certain states, we suppose that it holds in general. This relation, called here the *mutual uncertainty relation,* does not follow from the Maassen-Uffink uncertainty relation and is much stronger than the Hall uncertainty relation derived from the latter. As a special case, we present a much stronger bound on accessible information about certain ensembles of states $\mathcal{E}$ when measurements of certain observables $B^{(s)}$ are performed on it. Moreover, we derive a mutual uncertainty relation in the case in which Alice and Bob share a maximally entangled state and each party performs a measurement in one of two bases.

The paper is organized as follows. In Sec. II we formulate an uncertainty relation for a case in which Alice measures one observable and Bob measures one of two observables. In Sec. III we formulate uncertainty for a case in which both parties measure one of two observables. In Sec. IV we present analytical results for certain states and observables. Finally, in Sec. V we present our conclusions.

II. MUTUAL UNCERTAINTY RELATION FOR ONE VERSUS TWO OBSERVABLES

Suppose that Alice performs a measurement in a basis $\mathcal{A} = \{|a_k\rangle\}$ and Bob performs a measurement in one of two bases $\mathcal{B}^{(s)} = \{|b_j^{(s)}\rangle\}$. We conjecture that the following uncertainty relation for the sum of two mutual informations—the first one between Alice's results of the measurement and Bob's results of the measurement when he performs a measurement in the *first* basis, and the second one between Alice's results of the measurement and Bob's results of the measurement when he performs a measurement in the *second* basis—holds:

$$I(A : B^{(1)}) + I(A : B^{(2)}) \leqslant \log d + \log c, \tag{4}$$

where d is the dimension of each party's Hilbert space and c is the sum of d largest coefficients c_{ij} with

$$c_{ij} = \left|\langle b_i^{(1)}|b_j^{(2)}\rangle\right|^2. \tag{5}$$

We have tested this uncertainty relation numerically for the dimension of each party's subsystem up to $d = 4$ and have found no violation (details of numerical calculations are given in the Appendix). Moreover, we prove in Theorem 1 that if after Alice's measurement the state is diagonal in a basis $|a_k\rangle \otimes |b_j^{(1)}\rangle$, then the uncertainty relation (4) holds. However, we were not able to prove it in a general case.

Let us now compare our mutual uncertainty relation with Hall's original uncertainty relation. Suppose that Alice and Bob share a state $\rho_{AB} = \sum_{i=0}^{d-1} p_i |i\rangle\langle i| \otimes |i\rangle\langle i|$. Alice performs a measurement in the basis $\mathcal{A} = \{|i\rangle, i = 0, \ldots, d-1\}$ while Bob performs a measurement in the basis $\mathcal{B}^{(1)} = \{|i\rangle, i = 0, \ldots, d-1\}$ or $\mathcal{B}^{(2)} = \{|0\rangle, |\tilde{j}\rangle, \tilde{j} = 1, \ldots, d-1\}$, where $|\tilde{j}\rangle = \frac{1}{\sqrt{d-1}} \sum_{k=1}^{d-1} \exp(\frac{2\pi i \tilde{j} k}{d-1})|k\rangle$. In such a case, $c = 2$, and, although two bases in which Bob performs the measurement have a common eigenvector, the sum of mutual informations is bounded by $\log d + 1$. In contrast, the Hall uncertainty relation gives a trivial bound $2 \log d$.

The Hall uncertainty relation follows directly from the Maassen-Uffink uncertainty relation. Hence, another possible improvement to the Hall uncertainty relation could be obtained by strengthening the Maassen-Uffink uncertainty relation. To see if this is possible, let us write the Maassen-Uffink uncertainty relation in the following way:

$$S(A) + S(B) \geqslant \min H_\infty(\{c_{ij}\}), \tag{6}$$

where $H_\infty(\{c_{ij}\})$ is the min-entropy of a row or a column of a bistochastic matrix (5) and the minimum is taken over all rows and columns of this matrix. The simplest generalization of the Maassen-Uffink uncertainty relation can be obtained by replacing in (6) min-entropy $H_\infty(\{c_{ij}\})$ by Renyi entropy $H_\alpha(\{c_{ij}\})$ with $\alpha < \infty$, where $H_\alpha(\{p_i\}) = \frac{1}{1-\alpha} \log \sum_i p_i^\alpha$, i.e., we can take the uncertainty relation in the form

$$S(A) + S(B) \geqslant \min\{\min_i H_\alpha(\{c_{ij}\}), \min_j H_\alpha(\{c_{ij}\})\}, \tag{7}$$

where $H_\alpha(\{c_{ij}\})$ is the Renyi entropy of a row (or a column) of a bistochastic matrix (5) and α is some constant to be determined. We recall that min-entropy is obtained as a limiting case of the Renyi entropy for $\alpha \to \infty$ and that Renyi entropies satisfy the inequality $H_\alpha(\{p_i\}) \geqslant H_\beta(\{p_i\})$ for $\alpha < \beta$. We have checked numerically if such strengthening of the Maassen-Uffink uncertainty relation is possible, and we found strong evidence that the uncertainty relation (7) does not hold in general for $\alpha < \infty$. More precisely, when we increase the dimension of the system, it is violated for larger α.

III. MUTUAL UNCERTAINTY RELATION FOR TWO VERSUS TWO OBSERVABLES

Suppose that Alice performs a measurement in one of two bases $\mathcal{A}^{(s)} = \{|a_k^{(s)}\rangle\}$ and Bob performs a measurement in one of two bases $\mathcal{B}^{(s)} = \{|b_j^{(s)}\rangle\}$. We conjecture that the following mutual uncertainty relation for the sum of two mutual informations—the first one between Alice's and Bob's results of measurements when both parties perform the measurements in the *first* bases, and the second one between Alice's and Bob's results of measurements when both parties perform the measurements in the *second* bases—holds:

$$I(A^{(1)} : B^{(1)}) + I(A^{(2)} : B^{(2)}) \leqslant 2 \log d + \log c', \tag{8}$$

where d is the dimension of each party's Hilbert space and $c' = \max_V \max_{i,j} |\langle b_i^{(1)}|V U^T V^\dagger |b_j^{(2)}\rangle|^2$. Here U is chosen in such a way that the relation

$$U^\dagger \left|a_k^{(2)}\right\rangle = \left|a_k^{(1)}\right\rangle \tag{9}$$

is satisfied for all k, and V is a unitary operator.

We note that the coefficient c' in the above uncertainty relation is analogous to the coefficient a in the Maassen-Uffink uncertainty relation. We have tested this uncertainty relation numerically for dimensions of each party's subsystem up to $d = 3$ and have found no violation. Moreover, we prove in Theorem 3 that the uncertainty relation (8) holds if the parties perform the measurements on the maximally entangled state.

We have also found an exotic form of an uncertainty relation, which is for a while numerically confirmed. Namely, we have tested numerically the inequality

$$I(A^{(1)} : B^{(1)}) + I(A^{(2)} : B^{(2)}) \leqslant \log c'' - 2 \log d, \tag{10}$$

TABLE I. Different coefficients appearing in uncertainty relations (4), (8), and (10). See the main text for more details.

Coefficient	Defined as . . .	Pertaining to the relation . . .
c	$\sum_d$ largest $\vert\langle b_i^{(1)}\vert b_j^{(2)}\rangle\vert^2$	$I(A:B^{(1)})+I(A:B^{(2)})\leqslant \log d+\log c$
c'	$\max_V \max_{i,j} \vert\langle b_i^{(1)}\vert VU^TV^\dagger\vert b_j^{(2)}\rangle\vert^2$	$I(A^{(1)}:B^{(1)})+I(A^{(2)}:B^{(2)})\leqslant 2\log d+\log c'$
c''	$\sum_{i,j,k,l}\frac{\vert\langle a_i^{(1)}\vert a_j^{(2)}\rangle\vert^{1/2}}{\vert\langle b_k^{(1)}\vert b_l^{(2)}\rangle\vert^{1/2}}$	$I(A^{(1)}:B^{(1)})+I(A^{(2)}:B^{(2)})\leqslant \log c''-2\log d$

with

$$c''=\sum_{ijkl} c_{ijkl}, \tag{11}$$

where

$$c_{ijkl}=\frac{\left|\langle a_i^{(1)}|a_j^{(2)}\rangle\right|^p}{\left|\langle b_k^{(1)}|b_l^{(2)}\rangle\right|^p} \tag{12}$$

for d up to 16 and $p=\frac{1}{2}$, and we have found no violation. Unfortunately, the minimal value of the RHS gets closer to $2\log d$ when the dimension d increases. Moreover, for some choices of observables, the RHS becomes singular.

For convenience in Table I we summarize all different coefficients for our uncertainty relations.

IV. ANALYTICAL RESULTS

Now, we prove the uncertainty relations for some states and observables. We begin with the uncertainty relations for one versus two observables. We shall consider an auxiliary uncertainty relation, which is not true in general. However, in the lemma below we will show that it holds for some states and observables, and then we will argue that this implies the validity of our relation (4). In contrast, the latter uncertainty relation is conjectured to hold in general.

Lemma 1. Suppose that Alice and Bob share a state ρ_{AB}. Alice performs a measurement in a basis $\mathcal{A}=\{|a_k\rangle\}$ corresponding to one-dimensional projectors $\{P_k=|a_k\rangle\langle a_k|\}$ and Bob performs a measurement in one of two bases $\mathcal{B}^{(s)}=\{|b_j^{(s)}\rangle\}$ corresponding to one-dimensional projectors $\{Q_j^{(s)}=|b_j^{(s)}\rangle\langle b_j^{(s)}|\}$, where the index $s=1,2$ corresponds to two bases. If after Alice's measurement the state is diagonal in a basis $|a_k\rangle\otimes|b_i^{(1)}\rangle$, i.e., it is of the form

$$\rho_{AB^{(1)}}=\sum_{ki} p_{ki}P_k\otimes Q_i^{(1)}, \tag{13}$$

then the following uncertainty relation holds:

$$I(A:B^{(1)})+I(A:B^{(2)})\leqslant \log d+\log\sum_{ij}c_{ij}^2, \tag{14}$$

where d is the dimension of each party's Hilbert space and

$$c_{ij}=\left|\langle b_i^{(1)}|b_j^{(2)}\rangle\right|^2. \tag{15}$$

Proof. If after Alice's measurement Bob performs a measurement in the basis $\mathcal{B}^{(1)}=|b_j^{(1)}\rangle$, then the state does not change. On the other hand, if Bob performs a measurement in the basis $\mathcal{B}^{(2)}=|b_j^{(2)}\rangle$, then the state takes the form

$$\begin{aligned}\rho_{AB}&=\sum_{ijk} p_{ki}P_k\otimes Q_j^{(2)}Q_i^{(1)}Q_j^{(2)}\\ &\equiv\sum_{ijk} p_{ki}c_{ij}P_k\otimes Q_j^{(2)}.\end{aligned} \tag{16}$$

For simplicity, let us first assume that after Alice's measurement the state is of the form

$$\rho_{AB}=\sum_i p_iP_i\otimes Q_i^{(1)}, \tag{17}$$

which remains the same after Bob's measurement in the first basis. If Bob performs a measurement in the second basis, then the state becomes

$$\rho_{AB}=\sum_{ij} p_ic_{ij}P_i\otimes Q_j^{(2)}. \tag{18}$$

Let us calculate the sum of two mutual informations. We have

$$\begin{aligned}&I(A:B^{(1)})+I(A:B^{(2)})\\ &\quad=S(A)+I(A:B^{(2)})=S(A)+S(B^{(2)})-S(B^{(2)}|A)\\ &\quad=S(A)+S(B^{(2)})+\sum_i p_i\sum_j c_{ij}\log c_{ij}\\ &\quad\leqslant S(A)+S(B^{(2)})+\sum_i p_i\log\left(\sum_j c_{ij}^2\right)\\ &\quad=S(B^{(2)})+\sum_i p_i\log\left(\frac{\sum_j c_{ij}^2}{p_i}\right)\\ &\quad\leqslant \log d+\log\left(\sum_{ij}c_{ij}^2\right).\end{aligned} \tag{19}$$

In the fourth and sixth lines, we used the concavity of the logarithm. This concludes the proof for the state (17).

For the initial state after Alice's measurement of the more general form (16), it is enough to observe that this state can be obtained from the correlated one (17) by applying a local channel on Alice's side which (i) does not increase mutual informations $I(A:B^{(1)})$ and $I(A:B^{(2)})$; (ii) commutes with the second of Bob's measurement (in fact with both of them, but the first is irrelevant); and (iii) does not change the entropy $S(B^{(2)})$.

Because $\sum_i p_i\sum_j c_{ij}\log c_{ij}\leqslant\log\max_{i,j}c_{ij}$, we can replace $\sum_i p_i\sum_j c_{ij}\log c_{ij}$ in the third line of (19) by $\log\max_{i,j}c_{ij}$ and obtain the following inequality:

$$I(A:B^{(1)})+I(A:B^{(2)})\leqslant 2\log d+\log\max_{i,j}c_{ij}. \tag{20}$$

This is a special case of the Hall uncertainty relation. ■

For the above states and observables, we can immediately prove the uncertainty relation (4).

Theorem 1. Under the assumptions of Lemma 1, the uncertainty relation (4) holds.

Proof. Note that for a given i, $\{c_{ij}\}$ is a probability distribution. Hence, we have $\sum_j c_{ij}^2 \leqslant \max_j c_{ij}$. Taking the sum over i, we obtain $\sum_{ij} c_{ij}^2 \leqslant \sum_i \max_j c_{ij} \leqslant \sum c_{ij}$, where the last sum is over d largest coefficients c_{ij}. ■

In the following example, we show that the uncertainty relation (14) is not valid in general.

Example. Suppose that Alice's and Bob's subsystems are three-dimensional. Alice performs a measurement in the basis $\{|a_1\rangle = (1,0,0), |a_2\rangle = (0,1,0), |a_3\rangle = (0,0,1)\}$ and Bob performs a measurement either in the basis $|b_1^{(1)}\rangle = (1,0,0), |b_2^{(1)}\rangle = (0,1,0), |b_3^{(1)}\rangle = (0,0,1)$ or in the basis $\{|b_1^{(2)}\rangle = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), |b_2^{(2)}\rangle = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}), |b_3^{(2)}\rangle = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})\}$. Hence, the matrix of coefficients c_{ij} takes the following form:

$$c_{ij} = \left|\langle b_i^{(1)}|b_j^{(2)}\rangle\right|^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \tag{21}$$

Let us try to bound the sum of two mutual informations as in (14). We obtain

$$I(A:B^{(1)}) + I(A:B^{(2)}) = \log 3 + \log\frac{5}{4} = \log\frac{15}{4} < 2. \tag{22}$$

Now consider the following state:

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}\big(|a_1\rangle \otimes |b_3^{(1)}\rangle + |a_2\rangle \otimes |b_1^{(2)}\rangle\big). \tag{23}$$

The sum of two mutual informations is equal to 2 and hence it violates the bound (14). We stress, however, that the uncertainty relation (4) still holds in this case.

Now, we turn our attention to the uncertainty relations for two versus two observables. We assume that Alice and Bob share maximally entangled states and both Alice and Bob can choose one of two measurements. Our results are given in two theorems. In Theorem 2 we derive a state-dependent uncertainty relation (i.e., the RHS of the uncertainty relation depends on both the choice of observables and the choice of a maximally entangled state), and in Theorem 3 we derive a state-independent uncertainty relation (i.e., the RHS of the uncertainty relation depends only on the choice of observables and is valid for an arbitrary maximally entangled state).

Theorem 2. Suppose that Alice and Bob share a maximally entangled state $|\Phi\rangle_{AB}$ which is related to the maximally entangled state $|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{d}}\sum_i |i\rangle \otimes |i\rangle$ by the equation $|\Phi\rangle_{AB} = I \otimes V|\Phi^+\rangle_{AB}$, where V is a unitary operation acting on Bob's subsystem. Alice performs a measurement in one of two bases $\mathcal{A}^{(s)} = \{|a_k^{(s)}\rangle\}$ corresponding to one-dimensional projectors $\{P_k^{(s)} = |a_k^{(s)}\rangle\langle a_k^{(s)}|\}$ and Bob performs a measurement in one of two bases $\mathcal{B}^{(s)} = \{|b_j^{(s)}\rangle\}$ corresponding to one-dimensional projectors $\{Q_j^{(s)} = |b_j^{(s)}\rangle\langle b_j^{(s)}|\}$. The following uncertainty relation holds:

$$I(A^{(1)}:B^{(1)}) + I(A^{(2)}:B^{(2)}) \leqslant 2\log d + \log\tilde{c}', \tag{24}$$

where d is the dimension of each party's Hilbert space and $\tilde{c}' = \max_{i,j} |\langle b_i^{(1)}|VU^TV^\dagger|b_j^{(2)}\rangle|^2$, with U chosen in such a way that the relation

$$U^\dagger|a_k^{(2)}\rangle = |a_k^{(1)}\rangle \tag{25}$$

is satisfied for all k.

To prove Theorem 2, we will need two lemmas, which are given below.

Lemma 2. Mutual information between Alice and Bob calculated on a state

$$\sum_{k,j} \langle\Phi^+|_{AB} P_k^{(2)} \otimes Q_j^{(2)}|\Phi^+\rangle_{AB} P_k^{(2)} \otimes Q_j^{(2)} \tag{26}$$

is equal to mutual information between Alice and Bob calculated on a state

$$\sum_{k,j} \langle\Phi^+|_{AB} U^\dagger P_k^{(2)} U \otimes U^T Q_j^{(2)} U^* |\Phi^+\rangle_{AB} \times U^\dagger P_k^{(2)} U \otimes U^T Q_j^{(2)} U^*, \tag{27}$$

where $|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{d}}\sum_i |i\rangle \otimes |i\rangle$.

Proof. We prove it by showing that the former state can be transformed to the latter one by local unitary operations (which do not change mutual information). Indeed, we have

$$\begin{aligned}\sum_{k,j} &\langle\Phi^+|_{AB} P_k^{(2)} \otimes Q_j^{(2)}|\Phi^+\rangle_{AB} U^\dagger P_k^{(2)} U \otimes U^T Q_j^{(2)} U^* \\ &= \sum_{k,j} \langle\Phi^+|_{AB} U^\dagger P_k^{(2)} U \otimes U^T Q_j^{(2)} U^* |\Phi^+\rangle_{AB} \\ &\quad \times U^\dagger P_k^{(2)} U \otimes U^T Q_j^{(2)} U^*, \end{aligned} \tag{28}$$

where we used the identity $U \otimes U^*|\Phi^+\rangle_{AB} = |\Phi^+\rangle_{AB}$. ■

Lemma 3. Suppose that Alice and Bob share the maximally entangled state $|\Phi^+\rangle_{AB}$. Alice performs a measurement in one of two bases $\mathcal{A}^{(s)} = \{|a_k^{(s)}\rangle\}$ corresponding to one-dimensional projectors $\{P_k^{(s)} = |a_k^{(s)}\rangle\langle a_k^{(s)}|\}$ and Bob performs a measurement in one of two bases $\mathcal{B}^{(s)} = \{|b_j^{(s)}\rangle\}$ corresponding to one-dimensional projectors $\{Q_j^{(s)} = |b_j^{(s)}\rangle\langle b_j^{(s)}|\}$. The following uncertainty relation holds:

$$I(A^{(1)}:B^{(1)}) + I(A^{(2)}:B^{(2)}) \leqslant 2\log d + \log\tilde{c}, \tag{29}$$

where d is the dimension of each party's Hilbert space and $\tilde{c} = \max_{i,j} |\langle b_i^{(1)}|U^T|b_j^{(2)}\rangle|^2$ with U chosen in such a way that the relation

$$U^\dagger|a_k^{(2)}\rangle = |a_k^{(1)}\rangle \tag{30}$$

is satisfied for all k.

Proof. We use Lemma 2 with U chosen as above and replace the second of Alice's measurement in the basis $\mathcal{A}^{(2)}$ given by the projectors $P_k^{(2)}$ by the first of Alice's measurement in the basis $\mathcal{A}^{(1)}$ given by the projectors $P_k^{(1)}$ and the second of Bob's measurement given in the basis $\mathcal{B}^{(2)}$ by the projectors $Q_j^{(2)}$ by the measurement in the basis $\mathcal{B}_U^{(2)}$ given by the projectors $U^T Q_j^{(2)} U^*$. We write the sum of two mutual informations in the following way:

$$\begin{aligned} &I(A^{(1)}:B^{(1)})+I(A^{(2)}:B^{(2)}) \\ &\quad= I(A^{(1)}:B^{(1)})+I(A^{(1)}:B_U^{(2)}) \\ &\quad= S(B^{(1)})-S(B^{(1)}|A^{(1)})+S(B_U^{(2)})-S(B_U^{(2)}|A^{(1)}) \\ &\quad\leqslant 2\log d-\left[S(B^{(1)}|A^{(1)})+S(B_U^{(2)}|A^{(1)})\right]. \end{aligned} \tag{31}$$

Let us now bound the terms in the square brackets. We have

$$\begin{aligned} &S(B^{(1)}|A^{(1)})+S(B_U^{(2)}|A^{(1)}) \\ &\quad=\sum_k \frac{1}{d}\left[S(B^{(1)})_{\rho_{Bk}}+S(B_U^{(2)})_{\rho_{Bk}}\right], \end{aligned} \tag{32}$$

where

$$\rho_{Bk}=d\,\mathrm{Tr}_A\big(P_k^{(1)}\otimes I|\Phi^+\rangle\langle\Phi^+|_{AB}\big). \tag{33}$$

The Maassen-Uffink uncertainty relation states that

$$S(B^{(1)})_{\rho_{Bk}}+S(B_U^{(2)})_{\rho_{Bk}}\geqslant-\log\tilde{c}. \tag{34}$$

Substituting (34) into (32) and then substituting the result into (31), we obtain

$$I(A^{(1)}:B^{(1)})+I(A^{(2)}:B^{(2)})\leqslant 2\log d+\log\tilde{c}. \tag{35}$$

■

We are now ready to prove Theorem 2.

Proof. We note that Alice's and Bob's measurements given by the projectors $P_k^{(s)}$ and $Q_j^{(s)}$ performed on the maximally entangled state $|\Phi\rangle_{AB}=I\otimes V|\Phi^+\rangle_{AB}$ are equivalent to measurements given by the projectors $P_k^{(s)}$ and $V^\dagger Q_j^{(s)}V$ performed on the maximally entangled state $|\Phi^+\rangle_{AB}$. Then, from Lemma 3 we immediately obtain our thesis. ■

Theorem 3. Suppose that Alice and Bob share an arbitrary maximally entangled state $|\Phi\rangle_{AB}$. Alice performs a measurement in one of two bases $\mathcal{A}^{(s)}=\{|a_k^{(s)}\rangle\}$ corresponding to one-dimensional projectors $\{P_k^{(s)}=|a_k^{(s)}\rangle\langle a_k^{(s)}|\}$ and Bob performs a measurement in one of two bases $\mathcal{B}^{(s)}=\{|b_j^{(s)}\rangle\}$ corresponding to one-dimensional projectors $\{Q_j^{(s)}=|b_j^{(s)}\rangle\langle b_j^{(s)}|\}$. The following uncertainty relation holds:

$$I(A^{(1)}:B^{(1)})+I(A^{(2)}:B^{(2)})\leqslant 2\log d+\log c', \tag{36}$$

where d is the dimension of each party's Hilbert space and $c'=\max_V\max_{i,j}|\langle b_i^{(1)}|VU^TV^\dagger|b_j^{(2)}\rangle|^2$, with U chosen in such a way that the relation

$$U^\dagger|a_k^{(2)}\rangle=|a_k^{(1)}\rangle \tag{37}$$

is satisfied for all k and maximum taken over all unitary operations V.

Proof. Proof immediately follows from Theorem 2, as maximization over V gives the upper bound in the worst case (i.e., for a maximally entangled state for which the sum of two mutual informations is maximal). ■

There is still an open question of whether the uncertainty relation (36) holds for nonmaximally entangled states. Let us suppose that mutual information between the results of measurements performed on the nonmaximally entangled state of the form $|\Phi\rangle_{AB}=\sum_i\sqrt{\lambda_i}|\lambda_{\mathrm{A}i}\rangle\otimes|\lambda_{\mathrm{B}i}\rangle$ ($|\lambda_{\mathrm{A}i}\rangle\otimes|\lambda_{\mathrm{B}i}\rangle$ is the Schmidt basis) is smaller than mutual information between the results of measurements performed on the maximally entangled state $|\Phi\rangle_{AB}=\frac{1}{\sqrt{d}}\sum_i|\lambda_{\mathrm{A}i}\rangle\otimes|\lambda_{\mathrm{B}i}\rangle$. In such a case, we can replace the former by the latter and prove the analog of Theorems 2 and 3 for the results of measurements performed on a nonmaximally entangled state. However, for a measurement of the observables $A=X+Z$ on Alice's side and $B=X-Z$ on Bob's side, mutual information is $I(A:B)=0.049$ when the parties perform the measurement on the nonmaximally entangled state $|\Phi\rangle_{AB}=\sqrt{0.0332}|0\rangle\otimes|0\rangle+\sqrt{0.9668}|1\rangle\otimes|1\rangle$, and it is $I(A:B)=0$ when the parties perform the measurement on the maximally entangled state $|\Phi^+\rangle_{AB}$. Hence, the straightforward generalization of the proof (but not the uncertainty relation) fails.

V. CONCLUSIONS

We have proposed mutual uncertainty relations within a distant labs paradigm which bound the sum of mutual informations between Alice's and Bob's results of measurements for different observables. We have proved these uncertainty relations for some states and observables. We have also tested numerically the inequalities and found no numerical violations. Remarkably, the mutual uncertainty relation (4) for one versus two observables (one on Alice's side and two on Bob's side) is much stronger than the Hall uncertainty relation (3) derived from the Maassen-Uffink uncertainty relation. On the other hand, the uncertainty relation for two versus two observables has the coefficient c' on the RHS analogous to the coefficient a on the RHS of the Maassen-Uffink uncertainty relation. It would be interesting to check if the following uncertainty relation holds:

$$I(A^{(1)}:B^{(1)})+I(A^{(2)}:B^{(2)})\leqslant 2\log d+\log c''', \tag{38}$$

where $c'''=\max_V\sum|\langle b_i^{(1)}|VU^TV^\dagger|b_j^{(2)}\rangle|^2$. Here U is chosen in such a way that the relation $U^\dagger|a_k^{(2)}\rangle=|a_k^{(1)}\rangle$ is satisfied for all k, and V is a unitary operator. The sum is taken over d largest coefficients $|\langle b_i^{(1)}|VU^TV^\dagger|b_j^{(2)}\rangle|^2$.

Let us also remark on the generalization of our uncertainty relations to continuous variables when both parties measure operators such as position x and momentum p. In such a case, there always exists a state for which at least one mutual piece of information can be arbitrarily large (this is related to the fact that the Hilbert space is infinite-dimensional) and hence one cannot bound the sum of both mutual informations. To obtain nontrivial uncertainty relations for continuous variables, one should encompass finite resources such as the bounded average energy, which would make the relations quite different from the proposed ones.

Note added. Recently, our conjectured uncertainty relation for one versus two observables was proved in a general case by P. J. Coles and M. Piani [15].

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APPENDIX

Here we present details of numerical calculations. The numerical evidence was obtained using the genetic algorithm. Genetic organisms were represented as vectors of real numbers in the range 0 to 1 which were subjected to mutation, crossover, and selection. The crossover was done by the random selection of elements which are being swapped (i.e., individual elements were selected for swapping). We used a population of 25 organisms, and the elite of three best organisms were always taken to the next generation unchanged.

We used a technique which we call *mutation scaling*, which is supposed to allow us to approach the (maybe local) maximum. The mutation was done by adding $s(1-2r)$ to the elements selected for mutation, where r is a random number in the range 0 to 1 while s is a scaling factor. The scaling factor starts with 1 and is increased to $s' = \min(1, 1.1s)$ if the current generation brings improvement (i.e., if in the current generation there is an organism that is better than the best organism in the previous generation). The scaling factor is decreased to $s' = s/1.05$ if the current generation does not bring improvement (i.e., the best organism in the current generation is the same as the best organism in the previous generation). If s decreases to 10^{-9}, the scaling factor is set back to 1.

To optimize the unitary operators of a given dimension, we used a function which returns a unitary matrix given a vector of random numbers (implementation of an algorithm proposed in [16]) suitable in optimizations with the genetic algorithm.

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