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Conley type index applied to Hamiltonian inclusions

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ABSTRACT

In the paper we develop the theory of a cohomological index of the Conley type detecting invariant sets of a multivalued dynamical system generated by semilinear differential inclusion in an infinite dimensional Hilbert space. An application to the existence of periodic orbits to asymptotically linear Hamiltonian inclusions is presented.

1. Introduction

In [15], Conley introduced the theory of a homotopy index for invariant sets of a dynamical system in a locally compact metric space. This invariant, subsequently called the Conley index, proved to be a useful tool in studies concerning the behavior of various dynamical systems. In [31], Mrozek considered a cohomological version of the Conley index for multivalued flows defined on locally compact spaces. However, in many problems of nonlinear analysis one has to consider dynamical systems in infinite dimensional spaces. In [20], the authors constructed a version of the Conley index for flows determined by completely continuous perturbations of the special linear operators in infinite dimensional Hilbert spaces. Their method, resembling the construction of the Leray–Schauder degree, may be applied to existence and multiplicity results in problems having variational structure and involving strongly indefinite potentials.

In the present paper we introduce an infinite dimensional counterpart of the Mrozek index by the use of methods similar to those introduced in [20] (see [17] and also [22,23]) and apply it to the study of the existence of periodic orbits of Hamiltonian systems involving nonsmooth hamiltonians.

After this introduction the paper is organized as follows. First we introduce notation and establish some auxiliary results. In the second section we shall briefly discuss results concerning applications. In the third section we study multivalued flows and provide a construction of the cohomological Conley index for flows determined by the so-called L-vector fields. The final section is devoted to the proof of the main result from Section 2.

Given a metric space (X,d), a set $A \subset X$ and $\varepsilon > 0$, let $B_\varepsilon(A) := \{x \in X \mid d(x,A) := \inf_{a\in A} d(x,a) < \varepsilon\}$; $\operatorname{cl} A$ and $\operatorname{int} A$ stand for the *closure* and the *interior* of A, respectively. If $(\mathbb{E}, \|\cdot\|)$ is a (real) Banach space, then $\mathbb{E}^*$ stands for the (topological) dual of $\mathbb{E}$; by $\langle\cdot,\cdot\rangle$ we denote the duality pairing between $\mathbb{E}^*$ and $\mathbb{E}$, i.e., if $\xi \in \mathbb{E}^*$ and $u \in \mathbb{E}$, then $\langle \xi, u\rangle := \xi(u)$. If $x, y \in \mathbb{R}^n$, $n \in \mathbb{N}$, then $x \cdot y$ is the standard scalar product and $|x| := (x\cdot x)^{1/2}$ is the Euclidean norm in $\mathbb{R}^n$.

If $f : \mathbb{E} \to \mathbb{R}$ is a locally Lipschitz function, $x, u \in \mathbb{E}$, then $f^\circ(x;u)$ denotes the *Clarke generalized derivative of* f at x in the direction u and $\partial f(x) \subset \mathbb{E}^*$ is the *generalized gradient of* f at x (see e.g. [11, Chapter 2.1] or [4, Chapter 6]). For any $x \in \mathbb{E}$,

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the set $\partial f(x)$ is nonempty, weak*-compact and convex; for any $u \in E$, $f^\circ(x;u) = \max\{\langle \xi, u\rangle \mid \xi \in \partial f(x)\}$; the function $f^\circ(x;\cdot) : \mathbb{E} \to \mathbb{R}$ is Lipschitz continuous, positively homogeneous and subadditive and, for any $u \in E$, $f^\circ(\cdot;u) : \mathbb{E} \to \mathbb{R}$ is usc (upper semicontinuous); consequently $f^\circ : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ is usc. Hence the set-valued map $\mathbb{E} \ni x \mapsto \partial f(x) \subset \mathbb{E}^*$ is *upper hemicontinuous* (see [6, Chapter 3.2]) and *upper demicontinuous*, i.e., usc (in the sense of set-valued maps), provided $\mathbb{E}^*$ is endowed with the weak*-topology, in view of [6, Theorem 3.2.10] (see also [11, Proposition 2.1.5] and [21] for a general terminology concerning set-valued maps).

Proposition 1.1. *Given Banach spaces $\mathbb{E}_1 \subset \mathbb{E}_2$, suppose that the imbedding $j : \mathbb{E}_1 \to E_2$ is compact. If $f : \mathbb{E}_2 \to \mathbb{R}$ is locally Lipschitz, then $g := f|_{\mathbb{E}_1} : \mathbb{E}_1 \to \mathbb{R}$ is locally Lipschitz and the set-valued map $\mathbb{E}_1 \ni x \mapsto \partial g(x) \subset \mathbb{E}_1^*$ is completely continuous with compact convex values, i.e., it is usc and, for any bounded set $B \subset \mathbb{E}_1$, the set $\partial g(B)$ is relatively compact in $\mathbb{E}_1^*$ (in these statements $\mathbb{E}_1^*$ is considered with the norm-topology).*

Proof. It is clear that $g = f \circ j$ is locally Lipschitz and the values of ∂g are closed and convex. In order to prove the upper semicontinuity and the compactness of values of ∂g it is sufficient to show that given sequences (x_n) in $\mathbb{E}_1$ and (ξ_n) in $\mathbb{E}_1^*$ such that $\xi_n \in \partial g(x_n)$ for all $n \geqslant 1$, if $x_n \to x_0 \in \mathbb{E}_1$, then there is a subsequence (ξ_{n_k}) such that $\lim_{k\to\infty} \xi_{n_k} = \xi_0 \in \partial g(x_0)$.

To this end observe that the set $Y := \mathrm{cl}\{j(x_n)\}_{n=1}^\infty \subset \mathbb{E}_2$ is compact; hence the set $\partial f(Y)$ is bounded in $\mathbb{E}_2^*$ in view of [6, Proposition 3.2.4]. The adjoint $j^* : \mathbb{E}_2^* \to \mathbb{E}_1^*$ is compact. Therefore the set $j^*(\partial f(Y)) \subset \mathbb{E}_1^*$ is relatively compact. For each $n \geqslant 1$,

$$\xi_n \in \partial g(x_n) = \partial (f \circ j)(x_n) \subset j^*\bigl(\partial f\bigl(j(x_n)\bigr)\bigr) \subset j^*\bigl(\partial f(Y)\bigr)$$

in view of [11, Theorem 2.3.10, Remark 2.3.11]. Hence, passing to a subsequence if necessary, we may suppose that $\xi_n \to \xi_0 \in \mathbb{E}_1^*$. In particular $\xi_n \rightharpoonup \xi_0$ (weakly*). The closeness of the graph of ∂g (in $\mathbb{E}_1 \times \mathbb{E}_1^*$ with $\mathbb{E}_1^*$ having the weak*-topology—see [6, Proposition 3.2.5] or [11, Proposition 2.1.5]) implies that $\xi_0 \in \partial g(x_0)$.

To complete the proof we have to show that, given a bounded sequence (x_n) in $\mathbb{E}_1$, if $\xi_n \in \partial g(x_n)$ for all $n \geqslant 1$, then (ξ_n) has a convergent subsequence. To this end note that, by the compactness of j, $Y := \mathrm{cl}\{j(x_n)\}_{n=1}^\infty \subset \mathbb{E}_2$ is compact; hence $\partial f(Y)$ is bounded in $\mathbb{E}_2^*$. The same proof as above shows that, for each $n \geqslant 1$, $\xi_n \in j^*(\partial f(Y))$. The relative compactness of $j^*(\partial f(Y))$ ends the proof. □

2. Hamiltonian systems

Let $G : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$, $N \geqslant 1$, be 2π-periodic with respect to the first variable and locally Lipschitz with respect to the second one. We consider the Hamiltonian differential inclusion

$$\dot z \in J\partial G(t,z) \tag{1}$$

where

$$J = \begin{bmatrix} 0 & -I_N \\ I_N & 0 \end{bmatrix}$$

(I_N stands for the unit $(N \times N)$-matrix) is the standard *symplectic* matrix and, for any $t \in \mathbb{R}$, $\partial G(t,z)$ denotes the Clarke generalized gradient of $G(t,\cdot)$ at $z \in \mathbb{R}^{2N}$. We shall look for nontrivial 2π-periodic solutions to (1), i.e., 2π-periodic absolutely continuous functions $z : \mathbb{R} \to \mathbb{R}^{2N}$ such that, for a.a. (almost all) $t \in \mathbb{R}$, $\dot z(t) \in J\partial G(t,z(t))$ and $z \not\equiv 0$. This and similar problems has attracted a lot of attention, see the series of paper by Clarke, e.g. [12] and [18] where the principle of the least and dual action has been employed.

In order to apply an indirect variational attitude to (1), which is going to be considered here, it is customary to study the corresponding action functional on the fractional Sobolev space $H^{1/2}(S^1,\mathbb{R}^{2N})$ (here $S^1 := \mathbb{R}/2\pi\mathbb{Z}$ is the circle parameterized over $[0,2\pi]$) and to study its critical points. We briefly recall the setting (see [1, Chapter 3.2] or [33, Chapter 6]). For any $a \in \mathbb{R}$, let $e^{aJ} := \cos a \cdot I_{2N} + \sin a \cdot J : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$. We set

$$\mathbb{H} = H^{1/2}\bigl(S^1,\mathbb{R}^{2N}\bigr) := \Bigl\{ u \in L^2\bigl(S^1,\mathbb{R}^{2N}\bigr) \Bigm| \sum_{k\in\mathbb{Z}} |k||u_k|^2 < \infty \Bigr\}$$

where $(u_k) \subset \mathbb{R}^{2N}$ is the sequence of the Fourier coefficients of u:

$$u(t) = \sum_{k\in\mathbb{Z}} e^{ktJ} u_k.$$

It is clear that $\mathbb{H}$ is a real Hilbert space with the inner product

$$\langle u, v\rangle := 2\pi u_0 \cdot v_0 + 2\pi \sum_{k\in\mathbb{Z}^*} |k| u_k \cdot v_k, \quad u, v \in \mathbb{H}$$

(where $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$) and the norm $\|\cdot\|_{\mathbb{H}} = \sqrt{\langle\cdot,\cdot\rangle_{\mathbb{H}}}$.

Remark 2.1. It is well known (see e.g. [1, Theorem 3.1.1]) that smooth functions form a dense subset in $\mathbb{H}$ and the embedding $\mathbb{H} \hookrightarrow L^p(S^1, \mathbb{R}^{2N})$ is compact for any $p \geqslant 1$; in particular there is $C_p > 0$ such that, for $u \in \mathbb{H}$, $\|u\|_p \leqslant C_p\|u\|_{\mathbb{H}}$ where $\|\cdot\|_p$ stands for the standard norm in L^p. Moreover the embedding $H^1(S^1, \mathbb{R}^{2N}) \hookrightarrow \mathbb{H}$ is dense and compact.

Consider a map $L : \mathbb{H} \to \mathbb{H}$ given by

$$Lu(t) := \sum_{k\in\mathbb{Z}} (\operatorname{sgn} k) e^{ktJ} u_k, \quad u \in \mathbb{H}, \tag{2}$$

i.e., $(Lu)_k = (\operatorname{sgn} k)u_k$ for all $k \in \mathbb{Z}$ (where $\operatorname{sgn} 0 := 0$). For each $u \in \mathbb{H}$,

$$\|Lu\|^2_{\mathbb{H}} = 2\pi \sum_{k\in\mathbb{Z}} |k||u_k|^2 \leqslant \|u\|^2_{\mathbb{H}};$$

hence L is well defined, linear and bounded and $\langle Lu, v\rangle_{\mathbb{H}} = 2\pi \sum_{k\in\mathbb{Z}} ku_k \cdot v_k$ for $u, v \in \mathbb{H}$. Therefore L is self-adjoint. Observe that if $u, v \in H^1(S^1, \mathbb{R}^{2N}) \subset \mathbb{H}$, then

$$\langle -J\dot{u}, v\rangle_{L^2} = \int_0^{2\pi} -J\dot{u}(t) \cdot v(t)\, dt = 2\pi \sum_{k\in\mathbb{Z}} ku_k \cdot v_k = \langle Lu, v\rangle_{\mathbb{H}}$$

since $-J\dot{u}(t) = \sum_{k\in\mathbb{Z}} ke^{ktJ}u_k$ and, for any $a, b \in \mathbb{R}^{2N}$, $\langle e^{ktJ}a, e^{mtJ}b\rangle_{L^2} = 2\pi\,\delta_{km} a \cdot b$.

In what follows, for $k \in \mathbb{Z}$, let

$$\mathbb{H}(k) := \bigl\{u \in \mathbb{H} \bigm| u(t) = e^{ktJ}x,\ t \in S^1;\ x \in \mathbb{R}^{2N}\bigr\}.$$

The spaces $\mathbb{H}(k)$, $k \in \mathbb{Z}$, are mutually orthogonal; so are the spaces

$$\mathbb{H}_k := \mathbb{H}(-k) \oplus \mathbb{H}(k), \quad k = 0, 1, 2, \ldots. \tag{3}$$

Finally let

$$\mathbb{H}^- := \bigoplus_{k\geqslant 1} \mathbb{H}(-k), \quad \mathbb{H}^+ := \bigoplus_{k\geqslant 1} \mathbb{H}(k) \quad \text{and} \quad \mathbb{H}^n := \bigoplus_{k=0}^{n} \mathbb{H}_k, \quad n = 0, 1, 2, \ldots.$$

Then $\mathbb{H} = \mathbb{H}^- \oplus \mathbb{H}_0 \oplus \mathbb{H}^+$, $\mathbb{H} = \operatorname{cl} \bigcup_{n=1}^{\infty} \mathbb{H}^n$ and $\dim_{\mathbb{R}} \mathbb{H}^n = (2n + 1)2N$.

Observe that $\mathbb{H}_0 = \ker L$ and $Lu = \pm u$ for $u \in \mathbb{H}(\pm k)$ for $k > 0$. Hence $L(\mathbb{H}_k) = \mathbb{H}_k$ if $k > 0$ and if $u = u^- + u_0 + u^+$ according to the above decomposition, then $Lu = u^+ - u^-$. Note that L is a Fredholm operator of index 0.

Remark 2.2. Observe that the quadratic forms $\pm\langle Lu, u\rangle_{\mathbb{H}}$, $u \in \mathbb{H}$, are strongly indefinite, i.e., unbounded from below and from above on any subspace of finite codimension; hence their Morse indices must be infinite.

Let us now specify the assumptions concerning $G : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$. Let $p \geqslant 1$ and let $q = \frac{p}{p-1}$ ($q = \infty$ if $p = 1$). We suppose that:

(G_1) for all $u \in \mathbb{R}^{2N}$, $G(\cdot, u) : \mathbb{R} \to \mathbb{R}$ is measurable and 2π-periodic; $G(\cdot, 0) \equiv 0$;
(G_2) there exists $\ell > 0$ such that, for a.a. $t \in [0, 2\pi]$ and all $x, x' \in \mathbb{R}^{2N}$,

$$\bigl|G(t, x) - G(t, x')\bigr| \leqslant \ell|x - x'|,$$

or

$(G_2)'$ for a.a. $t \in \mathbb{R}$, $G(t, \cdot) : \mathbb{R}^{2N} \to \mathbb{R}$ is locally Lipschitz and there is $\alpha > 0$ such that, for a.a. $t \in [0, 2\pi]$ and all $x \in \mathbb{R}^{2N}$,

$$\sup_{y\in\partial G(t,x)} |y| \leqslant \alpha\bigl(1 + |x|^{p-1}\bigr).$$

By (G_1), for any $x, v \in \mathbb{R}^{2N}$, $G^\circ(\cdot, x; v)$ is measurable (see [11, Lemma on p. 78]). Since, for any $t \in \mathbb{R}$ and $v \in \mathbb{R}^{2N}$, $G^\circ(t, \cdot; v)$ is usc, we see that if $u : \mathbb{R} \to \mathbb{R}^{2N}$ is measurable, then so is $G^\circ(\cdot, u(\cdot); v)$. Therefore, in view of [5, Theorem 8.2.14], the map $\partial G(\cdot, u(\cdot))$ is measurable and, by the Kuratowski, Ryll–Nardzewski theorem, the set $N(u)$ of all measurable selections of $\partial G(\cdot, u(\cdot))$ is nonempty. In view of (G_2) or $(G_2)'$, if $u \in L^p(S^1; \mathbb{R}^{2N})$ and $w \in N(u)$, then $w \in L^q(S^1; \mathbb{R}^{2N})$ and $\|w\|_q \leqslant c(1 + \|u\|_p^{p-1})$ for some constant $c > 0$.

Moreover, by [11, Theorem 2.7.5], the functional $\widetilde{\psi} : L^p(S^1, \mathbb{R}^{2N}) \to \mathbb{R}$ given by

$$\widetilde{\psi}(u) := -\int_0^{2\pi} G\bigl(t, u(t)\bigr)\, dt, \quad u \in L^p\bigl(S^1, \mathbb{R}^{2N}\bigr),$$

is well defined, locally Lipschitz and, for $u \in L^p(S^1, \mathbb{R}^{2N})$, $\partial\widetilde{\psi}(u) \subset N(u)$, i.e., if $\xi \in \partial\widetilde{\psi}(u) \subset [L^p]^*$, then there is $w \in L^q(S^1; \mathbb{R}^{2N})$ such that $w(t) \in \partial G(t, u(t))$ for a.a. $t \in S^1$ and, for $v \in L^p(S^1, \mathbb{R}^{2N})$,

$$\langle \xi, v\rangle = -\int_0^{2\pi} w(t)\cdot v(t)\,dt.$$

Therefore, in view of Proposition 1.1, (G_1) and (G_2) or $(G_2)'$ we have the following

Proposition 2.3. *Let $\psi := \widetilde{\psi}|_{\mathbb{H}}$. Then, for $u \in \mathbb{H}$,*

$$\psi(u) = -\int_0^{2\pi} G\bigl(t, u(t)\bigr)\,dt, \tag{4}$$

ψ is locally Lipschitz, the map $\mathbb{H} \ni u \mapsto \partial\psi(u)$ is completely continuous with compact convex values in $\mathbb{H}$ (we identify $\mathbb{H}^*$ with $\mathbb{H}$ via the Riesz theorem) and, given $u \in \mathbb{H}$ and $\xi \in \partial\psi(u)$, there is an L^q-selection $w : S^1 \to \mathbb{R}^{2N}$ of $\partial G(\cdot, u(\cdot))$ such that

$$\langle \xi, v\rangle_{\mathbb{H}} = -\int_0^{2\pi} w(t)\cdot v(t)\,dt$$

for any $v \in \mathbb{H}$. Moreover, there is a constant $c > 0$ such that, for any $u \in \mathbb{H}$ and $\xi \in \partial\psi(u)$,

$$\|\xi\|_{\mathbb{H}} \leqslant c\bigl(1 + \|u\|_{\mathbb{H}}^{p-1}\bigr). \tag{5}$$

In order to obtain the existence of solutions to (1) we shall study a functional $\Phi : \mathbb{H} \to \mathbb{R}$ given by

$$\Phi(u) := \frac{1}{2}\langle Lu, u\rangle_{\mathbb{H}} + \psi(u), \quad u \in \mathbb{H}. \tag{6}$$

It is evident that Φ is locally Lipschitz and the quadratic part of Φ is smooth. Hence, by [11, Theorem 2.3.3], for each $u \in \mathbb{H}$,

$$\partial\Phi(u) = Lu + \partial\psi(u). \tag{7}$$

We shall look for equilibria of $\partial\Phi$, i.e., the critical points of Φ. This is justified by the following result.

Proposition 2.4. *Suppose that $z \in \mathbb{H}$ is a critical point of Φ, i.e., $0 \in \partial\Phi(z)$, then $z \in H^1(S^1, \mathbb{R}^{2N})$ and z is a solution to (1).*

Proof. Clearly $z \in \mathbb{H} \hookrightarrow L^p(S^1, \mathbb{R}^{2N})$. By Proposition 2.3, there is $w \in L^q(S^1, \mathbb{R}^{2N})$ such that $w(t) \in \partial G(t, z(t))$ for a.a. $t \in [0, 2\pi]$ and, for each $h \in \mathbb{H}$,

$$\langle Lz, h\rangle_{\mathbb{H}} = \int_0^{2\pi} w(t)\cdot h(t)\,dt.$$

In particular this holds for any constant function $h \in \mathbb{H}$. Hence $\int_0^{2\pi} Jw(t)\,dt = 0$. For $t \in [0, 2\pi]$, let $u_1(t) := \int_0^t Jw(s)\,ds$ and $c := \frac{1}{2\pi}\int_0^{2\pi}(z(t) - u_1(t))\,dt$. Finally let

$$u(t) := c + u_1(t), \quad t \in [0, 2\pi].$$

Then $u(0) = u(2\pi)$ (hence we may treat u as being defined by periodicity on the whole $\mathbb{R}$ and on S^1), $\int_0^{2\pi} u(t)\,dt = \int_0^{2\pi} z(t)\,dt$ and $\dot{u}(t) = Jw(t)$ for a.a. $t \in [0, 2\pi]$. It is clear that $w \in L^2$ and, therefore, $u \in H^1(S^1, \mathbb{R}^{2N})$. For any $v \in H^1$,

$$\langle Lu, v\rangle_{\mathbb{H}} = \langle -J\dot{u}, v\rangle_{L^2} = \langle w, v\rangle_{L^2} = \langle Lz, v\rangle_{\mathbb{H}}.$$

Hence $Lu = Lz$ and $u - z \in \ker L = \mathbb{H}_0$, i.e., $u - z$ is a constant function. However $\int_0^{2\pi}(u - z)\,dt = 0$; thus $u = z$ and $\dot{z} = Jw$ a.e. □

In what follows we shall study a Hamiltonian system of the form (1) where G is assumed to have an asymptotically linear generalized gradient, that is—apart from (G_1), (G_2) or $(G_2)'$—we assume that there are symmetric $(2N \times 2N)$-matrices A_0 and A_∞ such that:

(G_3) $\sup_{y\in\partial G(t,u)} |y - A_0 u| = o(|u|)$ as $u \to 0$ uniformly with respect to $t \in [0, 2\pi]$;
(G_4) $\sup_{y\in\partial G(t,u)} |y - A_\infty u| = o(|u|)$ as $|u| \to \infty$ uniformly with respect to $t \in [0, 2\pi]$.[3]

In view of Remark 2.2, even if G is smooth and Φ is twice differentiable, the Morse indices of $\pm\Phi''(z)$ must be infinite. However, it is possible to define a certain relative index which will always be finite. In order to proceed we need some auxiliary concepts.

Given a symmetric $(2N \times 2N)$-matrix A with real (constant) coefficients, consider the following Hamiltonian system

$$\dot z = JAz. \tag{8}$$

The gradient of the energy functional Φ_A on $\mathbb{H}$, whose critical points correspond to solutions of (8), has the form

$$\nabla\Phi_A(u) = Lu - \mathcal{A}(u), \quad u \in \mathbb{H}, \tag{9}$$

where, for any $w \in \mathbb{H}$, $\langle \mathcal{A}(u), w\rangle_{\mathbb{H}} = \int_0^{2\pi} Au(t)\cdot w(t)\,dt$. In order to find an explicit formula for $\mathcal{A}$ on $\mathbb{H}_k$, $k \geqslant 0$, observe first that on $\mathbb{H}_0 \cong \mathbb{R}^{2N}$ the map $\mathcal{A}$ may be identified with A; for $k \geqslant 1$, let us identify $(u, v) \in \mathbb{H}_k$, where $u(t) = e^{-ktJ}x$, $v(t) = e^{ktJ}y$, $x, y \in \mathbb{R}^{2N}$, with $(x, y) \in \mathbb{R}^{4N}$ and consider the following change of variables $\mathbb{R}^{4N} \ni (x, y) \mapsto (a, b) \in \mathbb{R}^{4N}$ where $a = x + y$ and $b = J(y - x)$, i.e., $u + v$ corresponds to the function $\cos kt\cdot a + \sin kt \cdot b$. Under these identifications $L(x, u) = (-x, y)$ and $L(a, b) = (-Jb, Ja)$. Moreover $\mathcal{A}(a, b) = \frac{1}{k}(Aa, Ab)$. Therefore the restriction of $\nabla\Phi_A$ to $\mathbb{H}_k$, $k \geqslant 1$, may be identified with a linear map given by the $(4N \times 4N)$-matrix

$$T_k(A) = \begin{bmatrix} -\frac{1}{k}A & -J \\ J & -\frac{1}{k}A \end{bmatrix}.$$

Let $M^\pm(B)$ and $M^0(B)$ denote the number (with multiplicity) of positive (respectively negative) eigenvalues of a symmetric (real) matrix B and the dimension of its kernel, respectively. Observe that the *generalized Morse indices*

$$i^\pm(A) := M^\pm(-A) + \sum_{k=1}^{\infty}\big(M^\pm\big(T_k(A)\big) - 2N\big),$$

and the *generalized nullity*

$$i^0(A) := M^0(-A) + \sum_{k=1}^{\infty} M^0\big(T_k(A)\big),$$

introduced by Amann and Zehnder (see [2,3] and e.g. [36]), are well-defined and finite. Indeed, a simple computation shows that the matrix $\left[\begin{smallmatrix} 0 & -J \\ J & 0\end{smallmatrix}\right]$ has the eigenvalue ± 1 with multiplicity $2N$, so by a simple perturbation argument $M^\pm(T_k(A)) = 2N$ for all sufficiently large $k \geqslant 1$. Similarly $M^0(T_k(A)) = 0$ for all large k. Clearly, $i^-(A) + i^0(A) + i^+(A) = 2N$, $i^0(A) = \dim\operatorname{Ker}\nabla\Phi_A \leqslant 2N$ and it is easy to see that $i^0(A) = 0$ if and only if $\sigma(JA) \cap i\mathbb{R} = \emptyset$ (see [8, p. 105]).

We have the following main result of this section.

Theorem 2.5. *Consider the Hamiltonian system* (1) *with asymptotically linear G, i.e., satisfying assumptions* (G_1)–(G_4). *Assume that $i^0(A_0) = i^0(A_\infty) = 0$ and $i^+(A_0) \neq i^+(A_\infty)$ or $i^-(A_0) \neq i^-(A_\infty)$. Then Φ has a critical point $z \neq 0$, i.e., the system* (1) *has a nontrivial solution* (*in addition to the trivial one $z \equiv 0$*).

The proof, based on a variant of the Conley index theory presented below, will be given in the last section. The following example illustrates the easy use of Theorem 2.5.

Example 2.6. Suppose $\alpha : \mathbb{R}_+ \to [0, 1]$ is such that $\alpha|_{[0,1]} \equiv 0$, $\alpha|_{[2,+\infty)} \equiv 1$ and $\alpha(t) = t - 1$ for $1 \leqslant t \leqslant 2$. Let $\eta \in C^1(\mathbb{R}^2, \mathbb{R})$ be bounded with the bounded $\nabla\eta$, $\eta(0) = 0$ and $|\nabla\eta(x)| = o(|x|)$ as $|x| \to 0$. Suppose that $g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is bounded, measurable and 2π-periodic with respect to $t \in \mathbb{R}$ and Lipschitz with respect to $x \in \mathbb{R}^2$. Finally suppose that $g(\cdot, 0) \equiv 0$. Let $G : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be given by

$$G(t, x) := \frac{1 + \alpha(|x|)}{3}|x|^2 + \eta(x)g(t, x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^2. \tag{10}$$

Then, for $|x| \leqslant 1$, $G(t, x) = \frac{1}{2}A_0x\cdot x + \eta(x)g(t, x)$ and, for $|x| \geqslant 2$, $G(t, x) = \frac{1}{2}A_\infty x\cdot x + \eta(x)g(t, x)$ where $A_0 := \frac{2}{3}I_2$ and $A_\infty := \frac{4}{3}I_2$. It is clear that G satisfies (G_1) and (G_2) and, for any $t \in \mathbb{R}$,

$$\partial G(t, x) = \begin{cases} A_0x + g(t, x)\nabla\eta(x) + \eta(x)\partial g(t, x) & \text{for } |x| < 1; \\ A_\infty x + g(t, x)\nabla\eta(x) + \eta(x)\partial g(t, x) & \text{for } |x| > 2. \end{cases}$$

[3] It is clear that if, for any $u \in \mathbb{R}^{2N}$, $G(\cdot, u)$ is 2π-periodic and continuous, for any $t \in \mathbb{R}$, $G(t, \cdot)$ is locally Lipschitz and G satisfies (G_4), then $(G_2)'$ holds with $p = 2$.

Hence conditions (G_3) and (G_4) are satisfied, too. One easily verifies that $i^0(A_0) = i^0(A_\infty) = 0$, $i^-(A_0) = 2$ and $i^-(A_\infty) = 4$; therefore Theorem 2.5 provides a nontrivial periodic solution to (1) with G given by (10). Instead of factors $\frac{1}{3}$ and $\frac{2}{3}$ in definitions of A_0 and A_∞ one can take any positive numbers c_1, c_2 such that $c_i/2 \notin \mathbb{Z}$ and $c_1 - c_2$ is sufficiently large.

Theorem 2.5 was proved by Amann and Zehnder in [2, Theorem 12.11], see also [3, Theorem 2], under the assumption that G is C^2 with respect to the z-variable, by means of a saddle point reduction and the minimax arguments based on the generalized Morse theory (comp. [36, Theorem 7.2] and [13, Theorem IV.1.3]). For some other results in this direction (and the extensive bibliography on the subject)—see [8,14,38].

3. Multivalued flows and the Conley index

In this section we present the construction of the Conley-type index for multivalued flows defined on non-necessarily locally compact metric spaces.

Let X, Y be metric spaces. After [21] (see also [26]) we say that a set-valued map $\varphi : X \multimap Y$ is *admissible* if there exist a metric space Γ, a *proper surjection* $p : \Gamma \to X$ (i.e., p is continuous and, for any compact $K \subset X$, $p^{-1}(K)$ is compact) and a continuous map $q : \Gamma \to Y$ such that, for every $x \in X$, the fiber $p^{-1}(x)$ is *acyclic* (i.e., $H^*(p^{-1}(x)) = H^*(P)$ where P is a one-point space and H^* stands for the Alexander–Spanier cohomology with integer coefficients) and $\varphi(x) = q(p^{-1}(x))$.

It is clear that an admissible map is usc with nonempty compact values. The class of admissible maps is rich, e.g. any usc map $\varphi : X \multimap Y$ with compact acyclic (and, in particular, contractible or convex) values is admissible. Moreover the class of admissible maps is closed under superposition. For more details concerning admissible maps—see [21] or [26].

Definition 3.1. Let X be a metric space. By a *multivalued flow* on X we mean an usc mapping $\varphi : X \times \mathbb{R} \multimap X$ with nonempty and compact values such that, for every $s, t \in \mathbb{R}$ and $x, y \in X$,

(i) $\varphi(x, 0) = \{x\}$;
(ii) if $st \geqslant 0$, then $\varphi(x, t + s) = \varphi(\varphi(x, t) \times \{s\})$;
(iii) $y \in \varphi(x, t)$ if and only if $x \in \varphi(y, -t)$;
(iv) the map $\varphi(x, \cdot) : R \multimap X$ is continuous.

The flow φ is said to be *admissible*, if there exists $T > 0$ such that the restriction of φ to $X \times [0, T]$ is an admissible mapping.

Let Λ be a metric space. By a *parameterized family of* (*respectively admissible*) *multivalued flows* we mean an usc map $\eta : X \times \mathbb{R} \times \Lambda \multimap X$ such that, for any $\lambda \in \Lambda$, $\eta_\lambda := \eta(\cdot, \cdot, \lambda)$ is a multivalued (respectively admissible) flow.

Remark 3.2.

(a) Condition (iv) is formally absent in [31], but all relevant examples considered further on satisfy this hypothesis.
(b) Sometimes it is useful to admit the empty set as a value of φ; in this case we speak about a *partial* multivalued flow.
(c) Observe that, in view of (ii) if φ is admissible, then, for any $T \geqslant 0$, the restriction of φ to $X \times [0, T]$ is admissible; in fact, it is not difficult to prove that a flow φ is admissible if and only if the map φ is admissible.

Example and Definition 3.3. Let $L : \mathbb{E} \to \mathbb{E}$ be a bounded linear operator on a Banach space $\mathbb{E}$. A set-valued map $f : U \multimap \mathbb{E}$, $U \subset \mathbb{E}$, of the form $f(u) = Lu + F(u)$ for $u \in \mathbb{E}$, where $F : U \multimap \mathbb{E}$ is a completely continuous set-valued map with compact convex values having sublinear growth, i.e., there is a constant $C > 0$ such that, for each $u \in \mathbb{E}$ and $y \in F(u)$, $\|y\| \leqslant C(1 + \|u\|)$, is called an *L-vector field*.

Given an L-vector field $f := L + F : \mathbb{E} \to \mathbb{E}$, the standard fixed point argument (see e.g. [24, Theorem 5.2.2]) implies that, for each $x \in \mathbb{E}$, there is a *mild solution* to the Cauchy problem

$$\begin{cases} u' \in f(u) & \text{a.e. on } \mathbb{R}; \\ u(0) = x, \end{cases} \tag{11}$$

i.e., a continuous function $u : \mathbb{R} \to \mathbb{E}$ and a locally (Bochner) integrable function $w : \mathbb{R} \to \mathbb{E}$ such that $w(t) \in F(u(t))$ and $u(t) = e^{tL}x + \int_0^t e^{(t-s)L} w(s)\, ds$ for all $t \in \mathbb{R}$. Since L generates the uniformly continuous C_0-group of operators, u is a mild solution to (11) if and only if u is a *strong* solution, i.e., $u(t) = x + \int_0^t (Lu(s) + w(s))\, ds$ on $\mathbb{R}$ (see [24, Proposition 5.2.1] and [32]).

Let $S(x) \subset C(\mathbb{R}, \mathbb{E})$[4] be the set of all solutions to (11), $x \in \mathbb{E}$. In view of results from [27] (see also [24, Theorem 5.3.1] and [7]), for any $x \in \mathbb{E}$, $S(x)$ is an R_δ-set (i.e., the intersection of a decreasing sequence of compact contractible sets); in particular $S(x)$ is acyclic.

[4] Where $C(\mathbb{R}, \mathbb{E})$ stands for the Fréchet space (i.e., locally convex metrizable and complete) of all continuous maps $\mathbb{R} \to \mathbb{E}$ with the topology of the almost uniform convergence.

Consider a map $\varphi : \mathbb{E} \times \mathbb{R} \multimap \mathbb{E}$ given by the formula

$$\varphi(x,t) := \{u(t) \mid u \in S(x)\}, \quad x \in \mathbb{E},\ t \in \mathbb{R}. \tag{12}$$

We claim that φ is an admissible multivalued flow on $\mathbb{E}$ (we say that φ is *generated by* f).

It is easy to see that conditions (i)–(iv) from Definition 3.1 are satisfied; so it remains to show that φ is an admissible map. Let $\Gamma := \{(x,u,t) \in \mathbb{E} \times C(\mathbb{R},\mathbb{E}) \times \mathbb{R} \mid u \in S(x)\}$ and let $p : \Gamma \to \mathbb{E} \times \mathbb{R}$, $q : \Gamma \to \mathbb{E}$ be given by $p(x,u,t) := (x,t)$ and $q(x,u,t) = u(t)$ for $(x,u,t) \in \Gamma$. For any $(x,t) \in \mathbb{E} \times \mathbb{R}$, $p^{-1}(x,t) = S(x) \times \{t\}$; hence p has acyclic fibers. Clearly maps p, q are continuous and p is a surjection. Moreover, for any $(x,t) \in \mathbb{E} \times \mathbb{R}$, $q(p^{-1}(x,t)) = \varphi(x,t)$. In order to show that p is proper let (x_n, t_n) be a sequence in $\mathbb{E} \times \mathbb{R}$, $x_n \to x_0 \in \mathbb{E}$, $t_n \to t_0 \in \mathbb{R}$ and let $u_n \in S(x_n)$, i.e., $(u_n, t_n) \in p^{-1}(x_n, t_n)$. We are going to show that, up to a subsequence, $u_n \to u_0 \in S(x_0)$ almost uniformly (a.u). For any $n \in \mathbb{N}$, there is a locally (Bochner) integrable selection w_n of $F(u_n(\cdot))$ such that $u_n(t) = e^{tL}x_n + \int_0^t e^{(t-s)L} w_n(s)\,ds$, $t \in \mathbb{R}$. Since the sequence $z_n(t) := e^{tL}x_n$, $t \in \mathbb{R}$, a.u. converges to $z_0(t) := e^{tL}x_0$, $t \in \mathbb{R}$, we have to show that (y_n), where $y_n(t) = \int_0^t e^{(t-s)L} w_n(s)\,ds$, $t \in \mathbb{R}$, has a subsequence converging a.u. to some $y_0(t) := \int_0^t e^{(t-s)L} w_0(s)\,ds$, $t \in \mathbb{R}$, where $w_0(t) \in F(u_0(t))$ and $u_0(t) := z_0(t) + y_0(t)$ for all $t \in \mathbb{R}$.

The Gronwall inequality and the sublinear growth imply that (u_n) is bounded in $C(\mathbb{R},\mathbb{E})$; thus the sequence (w_n) is (locally) integrably bounded, i.e., there is a locally integrable $c \in L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$ such that $\|w_n(t)\| \leqslant c(t)$ for a.a. $t \in \mathbb{R}$ and all $n \in \mathbb{N}$. Hence the sequence (y_n) is equicontinuous. The complete continuity of F implies that fibers $\{y_n(t)\}$, $t \in \mathbb{R}$, are relatively compact. Hence, by the Ascoli–Arzela theorem, the sequence (y_n) has an a.u. convergent subsequence. On the other hand, again by the complete continuity of F, the sequence (w_n) has relatively compact fibers; thus—being (locally) integrably bounded—it has a subsequence weakly convergent in $L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{E})$ in view of the Diestel theorem (see [16, Corollary 3]). Passing to subsequences if necessary, we may assume that $y_n \to y_0$ in $C(\mathbb{R},\mathbb{E})$ and $w_n \rightharpoonup w_0$ (weakly) in $L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{E})$. Therefore, for all $t \in \mathbb{R}$, $y_0(t) = \int_0^t e^{(t-s)L} w_0(s)\,ds$. For each $n \in \mathbb{N}$, y_n is differentiable a.e. and, for a.a. $t \in \mathbb{R}$, $y_n'(t) = Ly_n(t) + w_n(t) \in Ly_n(t) + F(u_n(t))$. Hence y_n' converges weakly in L^1_{loc} to $Ly_0 + w_0$; moreover $u_n = z_n + y_n$ a.u. converges to $u_0 := z_0 + y_0$. By the so-called Convergence Theorem (see [6, Theorem 3.2.6]), we see that, for a.a. $t \in \mathbb{R}$, $Ly_0(t) + w_0(t) \in Ly_0(t) + F(z_0(t) + y_0(t))$, i.e., $w(t) \in F(z_0(t) + y_0(t))$ a.e. on $\mathbb{R}$; this completes the proof of the claim.[5]

Observe that the constructed above map φ is an *L-flow*, i.e., it is an admissible flow of the form:

$$\varphi(x,t) = e^{tL}x + U(x,t), \quad x \in \mathbb{E},\ t \in \mathbb{R}, \tag{13}$$

where $U : \mathbb{E} \times \mathbb{R} \multimap \mathbb{E}$ is completely continuous and admissible. Indeed, in our case

$$U(x,t) = \left\{ y = \int_0^t e^{(t-s)L} w(s)\,ds \in \mathbb{E} \;\middle|\; w \in L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{H}),\ w(\cdot) \in F\big(u(\cdot)\big) \right.$$
$$\left. \text{a.e. on } \mathbb{R} \text{ where } u(t) = e^{tL}x + \int_0^t e^{(t-s)L} w(s)\,ds \right\}. \tag{14}$$

By a *parameterized family of L-flows* we mean a family of flows $\eta : \mathbb{E} \times \mathbb{R} \times \Lambda \multimap \mathbb{E}$ (Λ is a metric space) of the form $\eta(x,t,\lambda) = e^{tL}x + U(x,t,\lambda)$, $x \in \mathbb{E}$, $t \in \mathbb{R}$ and $\lambda \in \Lambda$, where $U : \mathbb{E} \times \mathbb{R} \times \Lambda \multimap \mathbb{E}$ is an admissible completely continuous set-valued map.

Remark 3.4.

(i) It is easy to see that if $f : \mathbb{E} \times [0,1] \multimap \mathbb{E}$ is a *family of L-vector fields*, i.e., is given by $f(u,s) = Lu + F(u,s)$, $u \in \mathbb{E}$, $s \in [0,1]$, where $F : \mathbb{E} \times [0,1] \multimap \mathbb{E}$ is completely continuous with compact convex values and sublinear growth (independent of $s \in [0,1]$), then f generates the family η of L-flows defined, for $x \in \mathbb{H}$, $t \in \mathbb{R}$ and $s \in [0,1]$, by $y \in \eta(x,t,s)$ if and only if $y = u_s(t)$ where u_s is a solution to the problem $u' = f(u,s)$, $u(0) = x$.

(ii) If $f = L + F : \mathbb{E} \multimap \mathbb{E}$, where F is completely continuous with convex compact values but does not have the sublinear growth, then there is an open set $D \subset \mathbb{E} \times \mathbb{R}$ such that $\mathbb{E} \times \{0\} \subset D$ and, for any $x \in \mathbb{E}$, there is a solution $u : J_x \to \mathbb{E}$ to (11) where $0 \in J_x := \{t \in \mathbb{R} \mid (x,t) \in D\}$. The formula (12) defines thus a *local* L-flow $\varphi : D \multimap \mathbb{E}$.

Let $\varphi : X \times \mathbb{R} \multimap X$ be a multivalued flow.

Definition 3.5. Let $\Delta \subset \mathbb{R}$. A map $\sigma : \Delta \to X$ is a *Δ-trajectory of φ* if, for every $t, s \in \Delta$, $\sigma(t) \in \varphi(\sigma(s), t-s)$. If $x \in N \subset X$ and $0 \in \Delta$, then the set of all Δ-trajectories *in N originating in x* (i.e., such that $\sigma(0) = x$ and $\sigma(t) \in N$ for $t \in \Delta$) is denoted by $\mathrm{Tr}_N(\varphi; \Delta, x)$.

It is an easy exercise to show the following (see [9,34]).

[5] The same argument works also in case L is assumed merely to be the generator of a strongly continuous C_0-semigroup of operators; in this case $S(x)$ stands for the set of *mild* solutions to (11).

Lemma 3.6.

(i) *Every trajectory of* φ *is a continuous mapping.*
(ii) *Let* $\sigma_1 : \Delta_1 \to X, \sigma_2 : \Delta_2 \to X$ *be trajectories of* φ *which coincide on* $\Delta_1 \cap \Delta_2$. *Then the mapping* $\sigma : \Delta_1 \cup \Delta_2 \to X$ *defined by*

$$\sigma(t) = \begin{cases} \sigma_1(t) & \text{for } t \in \Delta_1, \\ \sigma_2(t) & \text{for } t \in \Delta_2, \end{cases}$$

is a trajectory of φ.

Remark 3.7. Recall Example 3.3 and assume that $\varphi : \mathbb{E} \times \mathbb{R} \multimap \mathbb{E}$ is an L-flow defined by (12) or (13) and (14). It is easy to see that if $x \in \mathbb{E}$, $u \in S(x)$, $0 \in \Delta \subset \mathbb{R}$ and $u(t) \in N \subset \mathbb{E}$ for $t \in \Delta$, then $u \in \mathrm{Tr}_N(\varphi; \Delta, x)$.

We shall show that the converse statement is also true: if N is closed and $\sigma \in \mathrm{Tr}_N(\varphi; \mathbb{R}, x)$, then $\sigma \in S(x)$. To this end it is enough to show that the restriction $\sigma|_I : I \to \mathbb{E}$ of σ to any compact interval $I \subset \mathbb{R}$, $0 \in I$, is a solution to (11) on I, i.e., there is a (Bochner) integrable function $w : I \to \mathbb{E}$ such that $w(\cdot) \in F(\sigma(t))$ and $\sigma(t) = x + \int_0^t (Lu(\xi) + w(\xi))\,d\xi$ for $t \in I$.

For simplicity, let $I := [0,1]$. Fix $n \in \mathbb{N}$ and, for $0 \leqslant k \leqslant 2^n$, let $t_k := k2^{-n}$. Since $\sigma(t_1) \in \varphi(x, t_1)$, there is $v_1 \in S(x)$ such that $\sigma(t_1) = v_1(t_1)$. Let $u_n(t) := v_1(t)$ for $t \in [0, t_1]$. Suppose that, for some $0 < k < 2^n$, the function u_n is defined on $[0, t_k]$ and $u_n(t_k) = \sigma(t_k)$. Since $\sigma(t_{k+1}) \in \varphi(\sigma(t_k), t_{k+1} - t_k)$, there is $v_{k+1} \in S(\sigma(t_k))$ such that $\sigma(t_{k+1}) = v_{k+1}(t_{k+1})$. Let $u_n(t) := v_{k+1}(t - t_k)$ for $t \in [t_k, t_{k+1}]$. In this way we have defined inductively a function u_n on I for any positive integer n. It is easy to see that, for all $n \in \mathbb{N}$, u_n is a solution to (11) on I, i.e., belongs to the compact subset in $C(I, \mathbb{H})$ of all solution to (11) on I. Therefore the sequence (u_n) has a convergent subsequence; without loss of generality we may suppose that $u_n \to u$ where u is a solution to (11) on I. Since, for any $n \in \mathbb{N}$ and $0 \leqslant k \leqslant 2^n$, $u_n(k2^{-n}) = \sigma(k2^{-n})$, we gather that $\sigma \equiv u$ on I.

Definition 3.8. Let $N \subset X$. We set

$$\mathrm{Inv}(N, \varphi) := \{x \in N \mid \mathrm{Tr}_N(\varphi; \mathbb{R}, x) \neq \emptyset\}, \qquad \mathrm{Inv}^{\pm}(N, \varphi) := \{x \in N \mid \mathrm{Tr}_N(\varphi; \mathbb{R}_{\pm}, x) \neq \emptyset\}.$$

A set $K \subset X$ is *invariant* (respectively *positively* (*negatively*) *invariant*) with respect to φ if

$$\mathrm{Inv}(K, \varphi) = K \quad \big(\text{respectively } \mathrm{Inv}^{\pm}(K, \varphi) = K\big).$$

There is also a stronger version of invariance (being equivalent to the above one in the singlevalued case). Namely we say that $K \subset X$ is *strongly* (*positively, negatively*) *invariant* if, for every $x \in K$, $\varphi(x, \mathbb{R}) \subset K$ (respectively $\varphi(x, \mathbb{R}^+) \subset K$, $\varphi(x, \mathbb{R}^-) \subset K$).

Note that given $N \subset X$, the set $K := \mathrm{Inv}(N, \varphi)$ is the maximal invariant with respect to φ subset of N.

The following result is a version of the generalized Barbashin theorem (comp. [10, Proposition 16]).

Proposition 3.9. *Let* $N \subset X$ *be closed and* $\eta : X \times \mathbb{R} \times \Lambda \multimap X$ *be a parameterized family of multivalued flows. Then the graph of the set-valued map* $\Lambda \ni \lambda \mapsto \mathrm{Inv}(N, \eta_\lambda) \subset N$ *is closed.*

Proof. Take a sequence $(\lambda_n, x_n) \in \Lambda \times N$ such that $x_n \in \mathrm{Inv}(N, \eta_{\lambda_n})$ and $(\lambda_n, x_n) \to (\lambda_0, x_0)$. We shall show that $x_0 \in \mathrm{Inv}(N, \eta_{\lambda_0})$.

In view of Lemma 3.6(ii) it is enough to construct a trajectory $\sigma_0 : [0,1] \to N$ of η_{λ_0} such that $\sigma_0(0) = x_0$. For each $n \in \mathbb{N}$, choose a trajectory $\sigma_n : \mathbb{R} \to N$ of η_{λ_n} such that $\sigma_n(0) = x_n$ and consider a sequence $y_n := \sigma_n(1) \in \eta_{\lambda_n}(x_n, 1)$. Since η is usc and has compact values, there exists a subsequence y_{n_k} convergent to a point $y \in \eta_{\lambda_0}(x_0, 1)$. Set $\sigma_0(1) := y$.

Repeating this procedure for the interval $[0, \frac{1}{2}]$ we obtain a subsequence of the sequence $(\sigma_{n_k}(\frac{1}{2}))$ converging to a point $\sigma_0(\frac{1}{2}) \in \eta_{\lambda_0}(x_0, \frac{1}{2})$. The upper semicontinuity of η implies that $\sigma_0(1) \in \eta_{\lambda_0}(\sigma_0(\frac{1}{2}), \frac{1}{2})$. Therefore we can define $\sigma_0(t)$ for all *dyadic numbers* $t \in \bigcup_{q=0}^{\infty} \{\frac{p}{2^q} \mid 0 \leqslant p \leqslant 2^q\}$ using the above procedure as an inductive step. Moreover, for all dyadic numbers $s \geqslant t$, $\sigma_0(s) \in \eta_{\lambda_0}(\sigma_0(t), s - t)$ and $\sigma_0(s) \in N$ as the limit of points from N.

Now let $t \in [0,1]$ be arbitrary. Define a set

$$K(t) = \bigcap_{t' < t < t''} \eta\big(\sigma_0(t'), t - t', \lambda_0\big) \cap \eta\big(\sigma_0(t''), t - t'', \lambda_0\big),$$

where t', t'' are dyadic numbers. The set $K(t)$ is nonempty being an intersection of a family of closed sets with finite intersection property in a compact set $\eta_{\lambda_0}(\{x\} \times [0,1])$. Actually, in view of conditions (i) and (iv) from Definition 3.1, this set is a singleton. Let $\sigma_0(t)$ be the unique member of $K(t)$. One easily verifies that $[0,1] \ni t \mapsto \sigma_0(t)$ is a $[0,1]$-trajectory of η_{λ_0} (comp. [34]). $\square$

We are going now to describe briefly the Conley index due to Mrozek [31]. Suppose that $\varphi : X \times \mathbb{R} \multimap X$ is a multivalued flow on a *locally compact* space X.

Definition 3.10. (See [31].) A *compact* set $N \subset X$ is an *isolating neighborhood* for φ if $\mathrm{Inv}(N,\varphi) \subset \mathrm{int}\, N$. We say that a set K invariant with respect to φ is *isolated* if there is an isolating neighborhood N such that $K = \mathrm{Inv}(N,\varphi)$.

Observe that, in view of Proposition 3.9, isolated invariant sets are compact.

Let N be an isolating neighborhood for φ and let $x, y \in N$. For any $t \in \mathbb{R}$, by a *t-connection from x to y in N* we mean a $[0,t]$-trajectory (or a $[t,0]$-trajectory in case $t \leqslant 0$) σ of φ in N such that $\sigma(0) = x$ and $\sigma(t) = y$. The set of all t-connections from x to y, $t \in \mathbb{R}$, is denoted by $\mathrm{Conn}_N(\varphi; t, x, y)$. In view of [34, Theorem 6.1], if $y \in \varphi(x,t)$, then $\mathrm{Conn}_X(\varphi; t, x, y) \neq \emptyset$.

We define a map $\varphi_N : N \times \mathbb{R} \multimap N$ by the formula

$$\varphi_N(x,t) = \bigl\{ y \in X \bigm| \mathrm{Conn}_N(\varphi; t, x, y) \neq \emptyset \bigr\}, \quad x \in X,\ t \in \mathbb{R}.$$

It is easy to see that, for all $x \in X$ and $t \in \mathbb{R}$, $\varphi_N(x,t) \subset N \cap \varphi(x,t)$ and φ_N is a *partial* multivalued flow on N (see [31, Proposition 4.7]).

Definition 3.11. A pair (P_1, P_2) of subsets of $N \subset X$ is an *index pair* for φ in N provided:

(i) P_1, P_2 are compact and strongly positively invariant with respect to φ_N;
(ii) $\mathrm{Inv}^-(N,\varphi) \subset \mathrm{int}_N P_1$, $\mathrm{Inv}^+(N,\varphi) \subset N \setminus P_2$;
(iii) $\mathrm{cl}(P_1 \setminus P_2) \subset \mathrm{int}\, N$.

The following two results are crucial for the construction of the Conley index (see [31, Theorems 4.1, 5.2]).

Theorem 3.12.

(i) *If K is an isolated invariant set for the flow φ with an isolating neighborhood N, then for every neighborhood W of K, there exists an index pair for φ in N such that* $\mathrm{cl}(P_1 \setminus P_2) \subset W$.
(ii) *If the flow φ is admissible and K is an isolated invariant set, then the Alexander–Spanier cohomology* (*graded*) *group $H^*(P_1, P_2)$ does not depend on the choice of an isolating neighborhood N and an index pair (P_1, P_2) of φ in N.*

Theorem 3.12(ii) justifies the following concept (see [31, Definition 5.1]).

Definition 3.13. By the *cohomological Conley index of an isolated invariant set $K \subset X$ of an admissible flow φ* we mean the (graded) group

$$CH^*(K,\varphi) := H^*(P_1, P_2),$$

where (P_1, P_2) is an index pair for φ in an isolating neighborhood N of K.

This index has the following properties:

Theorem 3.14.

(i) *If $CH^*(K,\varphi)$ is nontrivial, then $K \neq \emptyset$.*
(ii) (*Continuation*). *Assume that $\eta : X \times \mathbb{R} \times [0,1] \multimap X$ is a family of admissible flows and let $N \subset X$ be an isolating neighborhood for all flows η_t, $t \in [0,1]$.*[6] *Then*

$$CH^*\bigl(\mathrm{Inv}(N,\eta_0), \eta_0\bigr) = CH^*\bigl(\mathrm{Inv}(N,\eta_1), \eta_1\bigr).$$

(iii) (*Additivity*). *Let K_1, K_2 be disjoint isolated invariant sets for an admissible flow φ. Then $CH^*(K_1 \cup K_2, \varphi) = CH^*(K_1,\varphi) \oplus CH^*(K_2,\varphi)$.*

Proof. Part (i) is obvious. The property (ii) is exactly Corollary 6.2 in [31]. Part (iii) follows from the fact that K_1, K_2 are compact, thus we can find disjoint isolating neighborhoods N_1, N_2 and $N = N_1 \cup N_2$ is an isolating neighborhood of $K_1 \cup K_2$. □

Let us finally mention that multivalued flows generated by differential inclusions in *finite-dimensional* spaces were also studied from a viewpoint of the Conley index theory in [19,29,30] by approximation techniques.

In what follows we shall construct a variant of the Conley index for L-flows defined on a (real) Hilbert space $(\mathbb{H}, \langle\cdot,\cdot\rangle)$. Let $L : \mathbb{H} \to \mathbb{H}$ be a linear bounded operator with the spectrum denoted by $\sigma(L)$. Assume that:

(L_1) $\mathbb{H} = \bigoplus_{k=0}^{\infty} \mathbb{H}_k$ with all subspaces $\mathbb{H}_k$ being mutually orthogonal and of finite dimension;

[6] One easily sees that the parameter space can be any compact path connected space instead of [0,1].

(L_2) $L(\mathbb{H}_0) \subset \mathbb{H}_0$ where $\mathbb{H}_0$ is the invariant subspace of L corresponding to the part of spectrum $\sigma_0(L) := i\mathbb{R} \cap \sigma(L)$ lying on the imaginary axis;
(L_3) $L(\mathbb{H}_k) = \mathbb{H}_k$ for all $k > 0$;
(L_4) $\sigma_0(L)$ is isolated in $\sigma(L)$, i.e., $\sigma_0(L) \cap \mathrm{cl}(\sigma(L) \setminus \sigma_0(L)) = \emptyset$.

Let $\eta : \mathbb{H} \times \mathbb{R} \times \Lambda \multimap \mathbb{H}$ be a parameterized family of L-flows, $\eta(x,t,\lambda) = e^{tL}x + U(x,t,\lambda)$ for $x \in \mathbb{H}$, $t \in \mathbb{R}$ and $\lambda \in \Lambda$, and let $X \subset \mathbb{H}$.

Proposition 3.15. *If $X \subset \mathbb{H}$ is closed and bounded, then the set-valued map $\Lambda \ni \lambda \mapsto \mathrm{Inv}(X, \eta_\lambda) \subset X$ is usc and has compact (possibly empty) values.*

Proof. In view of Proposition 3.9, it is sufficient to show that, given a sequence (λ_n, x_n) in $\Lambda \times X$ such that $x_n \in \mathrm{Inv}(X, \eta_{\lambda_n})$ and $\lambda_n \to \lambda_0 \in \Lambda$, (x_n) has a convergent subsequence, i.e., the set $S := \{x_n\}_{n=1}^\infty$ is relatively compact. Suppose it is not so.

Denote by $\mathbb{H}_-$ (respectively $\mathbb{H}_+$) the closed L-invariant subspace corresponding to the part of the spectrum $\sigma(L)$ of L with negative (respectively positive) real part. In view of the above assumptions, $\mathbb{H}$ splits into the direct sum $\mathbb{H} = \mathbb{H}_- \oplus \mathbb{H}_0 \oplus \mathbb{H}_+$ (see [25, Theorem 6.17]). Let $P_\pm : \mathbb{H} \to \mathbb{H}_\pm$ and $P_0 : \mathbb{H} \to \mathbb{H}_0$ be the orthogonal projections. Since $\sigma_0(L)$ is isolated in $\sigma(L)$, for each $\varrho > 0$, there is $t_0 > 0$ such that, for all $x \in \mathbb{H}_+$ and $t \geqslant t_0$

$$\|e^{tL}x\| \geqslant \varrho \|x\| \tag{15}$$

and, for $x \in \mathbb{H}_-$ and $t \leqslant -t_0$,

$$\|e^{tL}x\| \geqslant \varrho \|x\|. \tag{16}$$

Clearly $S \subset \mathrm{cl}\, P_-(S) \times \mathrm{cl}\, P_0(S) \times \mathrm{cl}\, P_+(S)$. The set $\mathrm{cl}\, P_0(S)$ is compact as a closed bounded subset of a finite-dimensional space $\mathbb{H}_0$. Therefore either $\mathrm{cl}\, P_-(S)$ or $\mathrm{cl}\, P_+(S)$ is noncompact. Assume that $P_+(S)$ is not relatively compact. Hence there exists an $\varepsilon > 0$ such that $P_+(S)$ does not admit a finite ε-net and we can choose a sequence $(x_{n_i}) \subset S$ such that $z_i := P_+(x_{n_i})$, $i \geqslant 1$, satisfy $\|z_i - z_j\| \geqslant \varepsilon$ whenever $i \neq j$. Choose $\delta > 0$ and $t_0 > 0$ such that $X \subset B_\delta(0)$ and the inequality (15) holds for $\varrho = \frac{3\delta}{\varepsilon}$. For $i \geqslant 1$, set $u_i := e^{t_0 L} x_{n_i}$ and take an arbitrary $v_i \in U(x_{n_i}, t_0, \lambda_{n_i})$; then

$$u_i + v_i \in e^{t_0 L} x_{n_i} + U(x_{n_i}, t_0, \lambda_{n_i}) = \eta(x_{n_i}, t_0, \lambda_{n_i}) \subset X \subset B_\delta(0).$$

Thus, for $i \neq j$,

$$3\delta \leqslant \|u_i - u_j\| \leqslant \|u_i + v_i\| + \|v_i - v_j\| + \|u_j + v_j\| < 2\delta + \|v_i - v_j\|$$

and, consequently,

$$\|v_i - v_j\| > \delta.$$

But, for each $i \geqslant 1$, v_i belongs to the set $\bigcup_{j=1}^\infty U(x_j, t_0, \lambda_j)$ being relatively compact in view of the complete continuity of U. Thus (v_i) has a convergent subsequence: a contradiction. □

Definition 3.16. By an *isolating neighborhood* for an L-flow $\varphi : \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$, we mean a *bounded and closed* set $X \subset \mathbb{H}$ such that $\mathrm{Inv}(X, \varphi) \subset \mathrm{int}\, X$.[7]

Corollary 3.17. *Let $\eta : \mathbb{H} \times \mathbb{R} \times \Lambda \multimap \mathbb{H}$ be a family of L-flows. If, for some $\lambda_0 \in \Lambda$, X is an isolating neighborhood for η_{λ_0}, then it is an isolating neighborhood for all λ from an open neighborhood V of λ_0 in Λ.*

Proof. By definition, $\mathrm{Inv}(X, \eta_{\lambda_0}) \subset \mathrm{int}\, X$. The upper semicontinuity of the map from Proposition 3.15 implies the existence of a neighborhood V of λ_0 in Λ such that, for $\lambda \in V$,

$$\mathrm{Inv}(X, \eta_\lambda) \subset \mathrm{int}\, X. \quad \square$$

For the rest of this section recall Example 3.3 and suppose that $f := L + F$ is an L-vector field that generates an L-flow $\varphi : \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$, i.e., for $x \in E$, $t \in \mathbb{R}$,

$$y \in \varphi(x,t) \iff y = u(t)$$

where $u \in S(x)$, i.e., u is a solution to the problem

$$u' = Lu + F(u), \quad u(0) = x. \tag{17}$$

In particular we have

[7] Observe that the ambient space $\mathbb{H}$ is *no longer* locally compact; hence Definition 3.10 does not apply.

Lemma 3.18. *Let $X \subset \mathbb{H}$ be an isolating neighborhood for φ. There is $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \leqslant \varepsilon_0$, X is an isolating neighborhood for the flow generated by the L-vector field $f_\varepsilon(x) := Lx + \operatorname{cl}\operatorname{conv} F(D_\varepsilon(x))$, $x \in \mathbb{H}$, where $D_\varepsilon(x) := \{y \in \mathbb{H} \mid \|y - x\| \leqslant \varepsilon\}$.*

Proof. It is obvious that, for each $\varepsilon > 0$, f_ε is an L-vector field. Consider a map $h : \mathbb{H} \times [0,1] \multimap \mathbb{H}$ defined by the formula $h(x,s) = f_s(x)$, $x \in \mathbb{H}$, $s \in [0,1]$. It is easy to see that h is a family of L-flows. Moreover, X is an isolating neighborhood for the flow generated by $f = f_0$. In view of Corollary 3.17, we gather that X is an isolating neighborhood for the flow generated by f_t whenever t is small enough. $\square$

Denote by $P_n : \mathbb{H} \to \mathbb{H}$ the orthogonal projection of $\mathbb{H}$ onto $\mathbb{H}^n := \bigoplus_{k=0}^n \mathbb{H}_k$ and consider two sequences of L-vector fields $f_n : \mathbb{H}^n \multimap \mathbb{H}^n$ and $h_n : \mathbb{H}^{n+1} \times [0,1] \multimap \mathbb{H}^{n+1}$, $n \in \mathbb{N}$, given by the formulae

$$f_n(x) = Lx + P_n\big(F(x)\big), \quad x \in \mathbb{H}^n,$$
$$h_n(x,s) = Lx + (1-s)P_n\big(F(x)\big) + sP_{n+1}\big(F(x)\big), \quad x \in \mathbb{H}^{n+1},\ s \in [0,1].$$

For each n, set-valued maps f_n and h_n are usc with compact convex values and have sublinear growth. Hence, for every n, f_n generates a multivalued flow $\varphi_n : \mathbb{H}^n \times \mathbb{R} \multimap \mathbb{H}^n$, and h_n generates a family $\eta_n : \mathbb{H}^{n+1} \times \mathbb{R} \times [0,1] \multimap \mathbb{H}^{n+1}$ of flows.

Let X be an isolating neighborhood for φ.

Lemma 3.19. *There exists $n_0 \in \mathbb{N}$ such that, for $n \geqslant n_0$, $X^n := X \cap \mathbb{H}^n$ is an isolating neighborhood for φ_n and $\eta_{n-1}(\cdot,\cdot,s)$, $s \in [0,1]$, in the sense of Definition 3.10.*

Proof. Define a family of L-vector fields $h : \mathbb{H} \times [0,1] \multimap \mathbb{H}$ by

$$h(x,s) = Lx + (1+n)(1-ns)P_{n+1}\big(F(x)\big) + n\big[(n+1)s - 1\big]P_n\big(F(x)\big)$$

for $\frac{1}{n+1} < s \leqslant \frac{1}{n}$ and $h(x,0) = f(x)$. In view of Remark 3.4(i), h generates a family $\eta : \mathbb{H} \times \mathbb{R} \times [0,1] \multimap \mathbb{H}$ of L-flows. By Proposition 3.15, the graph S of the map $[0,1] \ni s \mapsto \operatorname{Inv}(X, \eta_s)$ is compact in $[0,1] \times X$ and $S \cap (\{0\} \times X) \subset \{0\} \times \operatorname{int} X$. Therefore, for some $s_0 > 0$, we have $S \cap ([0,s_0] \times X) \subset [0,s_0] \times \operatorname{int} X$; in other words, for $0 \leqslant s \leqslant s_0$, $\operatorname{Inv}(X, \eta_s) \subset \operatorname{int} X$. One takes $n_0 > 1/s_0$. $\square$

For any $n \geqslant n_0$, where n_0 is given in Lemma 3.19, let $K^n := \operatorname{Inv}(X^n, \varphi_n)$; then K^n is a compact invariant set with X^n as an isolating neighborhood; by Theorem 3.12(i), X^n admits an index pair (Y^n, Z^n) and the Conley index $CH^*(K^n, \varphi_n) = H^*(Y^n, Z^n)$ is well defined. Moreover, by Lemma 3.19, X^{n+1} is an isolating neighborhood for the flow $\eta_n(\cdot,\cdot,s)$, $s \in [0,1]$. Clearly, for each $n \geqslant 1$, $f_{n+1} = h_n(\cdot,1)$ and, thus, $\varphi_{n+1} = \eta_n(\cdot,\cdot,1)$.

We shall show that the Conley indices $CH^*(K^n, \varphi_n)$ and $CH^*(K^{n+1}, \varphi_{n+1})$, $n \geqslant n_0$, are closely related. To this end, for each $n \geqslant n_0$, consider a family flows $\theta_n : \mathbb{H}^{n+1} \times \mathbb{R} \times [0,1] \multimap \mathbb{H}^{n+1}$ generated by the family of vector fields $g_n(x,s) := Lx + P_n(F(P_n x + s(x - P_n x))$, $x \in \mathbb{H}^{n+1}$, $s \in [0,1]$.

Lemma 3.20. *For any $n \geqslant n_0$, and $s \in [0,1]$, X^{n+1} is an isolating neighborhood for the flow $\theta_n(\cdot,\cdot,s)$ and $\operatorname{Inv}(X^{n+1}, \theta_n(\cdot,\cdot,s)) = K^n \times \{0\}$.*

Proof. Let $n \geqslant n_0$ and $s \in [0,1]$. Recall that $\mathbb{H}^{n+1} = \mathbb{H}^n \oplus \mathbb{H}_{n+1}$ and that both spaces $\mathbb{H}^n$, $\mathbb{H}_{n+1}$ are L-invariant; moreover $L : \mathbb{H}_{n+1} \to \mathbb{H}_{n+1}$ is an isomorphism. The nonlinear part of the field $g_n(\cdot,s)$ takes values in $\mathbb{H}^n$. If $x = (x_n, y) \in \operatorname{Inv}(X^{n+1}, \theta_n(\cdot,\cdot,s))$, where $y \in \mathbb{H}_{n+1}$ and $y \neq 0$, then trajectory $\sigma \in \operatorname{Tr}_{X^{n+1}}(\theta_n(\cdot,\cdot,s); \mathbb{R}, x)$ such that $\sigma(0) = x$ is unbounded in the direction of $\mathbb{H}_{n+1}$. However X^{n+1} is bounded: a contradiction. On the other hand, for $x \in \mathbb{H}^n$, $g_n(x,s) = Lx + P_n(F(P_n(x)) = f_n(x)$ and the second assertion follows. $\square$

For $n \in \mathbb{N}$, let $\mathbb{H}_n^\pm := \mathbb{H}_n \cap \mathbb{H}^\pm$. One can easily check that the pair $(\widetilde{Y}^n, \widetilde{Z}^n)$, where

$$\widetilde{Y}^n := Y^n \times D_{n+1}^+ \times D_{n+1}^-, \qquad \widetilde{Z}^n := Z^n \times D_{n+1}^+ \times D_{n+1}^- \cup Y^n \times \partial D_{n+1}^+ \times D_{n+1}^-$$

and

$$D_n^\pm := \big\{x \in \mathbb{H}_n^\pm \bigm| \|x\| \leqslant r\big\}, \qquad \partial D_n^\pm := \big\{x \in \mathbb{H}_n^\pm \bigm| \|x\| = r\big\},$$

is an index pair for $\theta_n(\cdot,\cdot,0)$ in the isolating neighborhood X^{n+1} provided $r > 0$ is small enough.

Let us identify the circle with the quotient $S^1 := [0,1]/\{0,1\}$ and recall that the *suspension* SX of a (pointed) space X is defined by the smash product $SX := S^1 \wedge X$; for $m \in \mathbb{N}$, we define $S^m X := S(S^{m-1}X)$ (comp. [37]).

Let $\nu(n) := \dim \mathbb{H}_n^+$ and note that the quotient $\widetilde{Y}^n/\widetilde{Z}^n$ has the homotopy type of the suspension $S^{\nu(n)}(Y^n/Z^n)$. Thus there is a natural (graded) isomorphism

$$H^*\big(\widetilde{Y}^n/\widetilde{Z}^n, *\big) \cong H^*\big(S^{\nu(n)}\big(Y^n/Z^n\big), *\big). \tag{18}$$

Proposition 3.21. *There is an isomorphism*

$$c^* : H^*\big(Y^{n+1}/Z^{n+1}, *\big) \to H^*\big(S^{\nu(n)}\big(Y^n/Z^n\big), *\big).$$

Proof. In view of Lemmas 3.19 and 3.20, for $n \geqslant n_0$, the set X^{n+1} is an isolating neighborhood for admissible flows $\theta_n(\cdot,\cdot,s)$ and $\eta_n(\cdot,\cdot,s)$, $s \in [0,1]$. Moreover $\theta_n(\cdot,\cdot,1) = \eta_n(\cdot,\cdot,0)$. Therefore, by the continuation property of the cohomological Conley index (see Theorem 3.14(ii)), we obtain an isomorphism $\mu : H^*(Y^{n+1}, Z^{n+1}) \cong H^*(\widetilde{Y}^n, \widetilde{Z}^n)$.

It is well known (see e.g. [35]) that, for any metric pair (Y, Z), there is an isomorphism $\kappa : H^*((Y/Z), *) \cong H^*(Y, Z)$. Composing κ for (Y^{n+1}, Z^{n+1}) with μ, κ^{-1} for $(\widetilde{Y}^n, \widetilde{Z}^n)$ and (18) we get the required isomorphism c^*. □

There is also a natural (desuspension) isomorphism $H^*(S^{\nu(n)}(Y^n/Z^n), *) \cong H^{*-\nu(n)}(Y^n/Z^n, *)$ (see [37]). Hence, for each $n \geqslant n_0$ and $q \in \mathbb{Z}$, there is an isomorphism

$$\gamma_n : H^{q+\nu(n)}\big(Y^{n+1}, Z^{n+1}\big) \to H^q\big(Y^n, Z^n\big).$$

Define $\rho : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ by $\rho(0) = 0$ and $\rho(n) = \sum_{i=0}^{n-1} \nu(i)$ for $n \geqslant 1$. For a fixed $q \in \mathbb{Z}$ and $n \geqslant n_0$, consider the sequences of isomorphisms

$$\gamma_n : H^{q+\rho(n+1)}\big(Y^{n+1}, Z^{n+1}\big) \to H^{q+\rho(n)}\big(Y^n, Z^n\big)$$

as described above. It is clear that this sequence forms an inverse system of (graded) groups. Therefore the following definition is justified.

Definition 3.22. By the *cohomological L-index of the isolating neighborhood X of the L-flow* φ we understand the graded group $CH^*(X,\varphi) = \{CH^q(X,\varphi)\}_{q \in \mathbb{Z}}$ where, for $q \in \mathbb{Z}$,

$$CH^q(X,\varphi) := \varprojlim_{n \geqslant n_0} \big\{H^{q+\rho(n)}\big(Y^n, Z^n\big), \gamma_n\big\}.$$

This definition is correct since it clearly does not depend on the choice of the sequence (Y^n, Z^n) and, as it was pointed out in [22] (in a single-valued case), the group $CH^q(X,\varphi)$ may be nontrivial both for negative and positive $q \in \mathbb{Z}$.

Example 3.23. Recall the linear systems (8) and (9). If $i^0(A) = 0$, then $\nabla\Phi_A$ is a linear isomorphism and $S := \{0\}$ is an isolated invariant set for the flow φ generated by $\nabla\Phi_A$. Since all other orbits of (8) are unbounded, it follows that, for any $r > 0$, the closed ball $D_r(0) := \{z \in \mathbb{H} \mid \|z\| \leqslant r\}$ is an isolating neighborhood for φ. We shall show that $CH^*(D_r(0),\varphi) = H^*(S^{i^+(A)}, *)$, i. e. the only nontrivial group $CH^q(D_r(0),\varphi) = \mathbb{Z}$ for $q = i^+(A)$. Let k_0 be such that $M^\pm(T_k(A)) = 2N$ for all $k > k_0$. Then $\nabla\Phi_A : \mathbb{H}^{k_0} \to \mathbb{H}^{k_0}$ is a linear selfadjoint isomorphism and the classical Conley index (see [15]) for the flow generated in this subspace is the homotopy type of a sphere S^t, where $t = M^+(-A) + \sum_{k=1}^{k_0} M^+(T_k(A))$. Now go back to Definition 3.22. In our case, for any $i \geqslant 1$, $\nu(i) = 2N$ and $\rho(k_0) = 2Nk_0$. Therefore for any $q \in \mathbb{Z}$ the qth cohomology group is $CH^q(D_r(0),\varphi) = H^{q+2Nk_0}(S^t, *)$. It is equal to $\mathbb{Z}$ if and only if $q + 2Nk_0 = t$. Hence $q = t - 2Nk_0 = i^+(A)$.

We now formulate basic properties for $CH^*(X,\varphi)$. They are obvious consequences of respective properties provided in Theorem 3.14 for a finite-dimensional situation.

Proposition 3.24. *Let X be an isolating neighborhood for an L-flow φ generated by an L-vector field f. If $CH^*(X,\varphi) \neq \{0\}$, then $\mathrm{Inv}(X,\varphi) \neq \emptyset$. In particular there is a bounded solution* (*lying in X*) *to problem* (17).

Proof. Let h be a family of L-vector fields defined in the proof of Lemma 3.19 and let η be the family of L-flows generated by h. Then $f = h(\cdot,0)$ and $\varphi = \eta_0$. Suppose that $\mathrm{Inv}(X,\varphi) = \emptyset$. Proposition 3.15 implies that there is $s_0 > 0$ such that, for $s \in [0, s_0]$, $\mathrm{Inv}(X,\eta_s) = \emptyset$. Thus, for $n > 1/s_0$ (keeping the above notation), $\mathrm{Inv}(X^n,\varphi_n) = \emptyset$; by Theorem 3.14(i), $H^*(Y^n, Z^n) = 0$ for an arbitrary index pair in X^n. The inverse limit of trivial groups is also trivial: a contradiction. The last statement is a consequence of Remark 3.7. □

Proposition 3.25. *Let Λ be a compact, connected and locally contractible metric space. Assume that $\eta : \mathbb{H} \times \mathbb{R} \times \Lambda \multimap \mathbb{H}$ is a family of L-flows generated by a family of L-vector fields $f : \mathbb{H} \times \Lambda \to \mathbb{H}$. Let X be an isolating neighborhood for the flow η_{λ_0} for some $\lambda_0 \in \Lambda$. Then there is a compact neighborhood $C \subset \Lambda$ of λ_0 such that $CH^*(X,\eta_\mu) = CH^*(X,\eta_\lambda)$ for all $\mu, \lambda \in C$.*

Proof. Corollary 3.17 implies the existence of a neighborhood C of λ_0 such that X is an isolating neighborhood for all η_λ for $\lambda \in C$. We can assume that C is compact and contractible, thus path connected. Repeating the argument from the proof of Lemma 3.19 for the family $f(\cdot,\mu)$, $\mu \in C$, of L-vector fields we get $n_0 \in \mathbb{N}$ such that, for $n > n_0$, the set X^n is an isolating neighborhood for the flows generated by the vector fields $f_n(\mu) : \mathbb{H}^n \multimap \mathbb{H}^n$ where $f_n(\mu)(x) = P_n(f(x,\mu))$ for all $\mu \in C$ and $x \in \mathbb{H}_n$. Then we apply Theorem 3.14(ii). □

Proposition 3.26. *Let X, X' be two isolating neighborhoods for an L-flow φ generated by an L-vector field f. Assume that $X \subset X'$ and $\operatorname{Inv}(X',\varphi) \subset \operatorname{int} X$. Then $CH^*(X,\varphi) = CH^*(X',\varphi)$.*

Proof. We keep the notation introduced directly before and in the proof of Lemma 3.19. There is n_0 such that, for $n \geqslant n_0$, X^n and $X'_n := X' \cap \mathbb{H}^n$ are isolating neighborhoods for φ_n on $\mathbb{H}^n$. Suppose that $CH^*(X,\varphi) \neq CH^*(X',\varphi)$. Then, in view of Theorem 3.12(ii), $\operatorname{Inv}(X'_n,\varphi_n) \not\subset \operatorname{int} X^n$. It is easy to see that if $x_n \in \operatorname{Inv}(X'_n,\varphi_n) \cap (X'_n \setminus \operatorname{int} X^n)$, then $x_n \in \operatorname{Inv}(X', \eta(\cdot,\cdot,\frac{1}{n})) \cap (X' \setminus \operatorname{int} X)$. By Theorem 3.15, $(x_n, 1/n)$ has a subsequence convergent to $(x,0)$. But then Proposition 3.9 implies that $x \in \operatorname{Inv}(X',\varphi) \setminus \operatorname{int} X$: a contradiction. □

4. Periodic solutions to Hamiltonian systems

In this section we are going to prove Theorem 2.5. Recall Section 2 and observe that the operator $L : \mathbb{H} \to \mathbb{H}$ given by (2), where $\mathbb{H} := H^{1/2}(S^1, \mathbb{R}^{2N})$, together with the family $\{\mathbb{H}_k\}_{k \geqslant 0}$ given by (3), satisfy assumptions (L_1)–(L_4) from Section 3. In what follows we shall study the flow generated by $\partial\Phi$ (see (6) and (7)), i.e., by the vector field of the form $L + F$ where $F := \partial\psi$.

According to Corollary 2.3, estimate (5) holds for some $p \geqslant 1$. In view of Remark 3.4, if $p > 2$, then the flow generated by $\partial\Phi$, i.e., given by (12) (or by (13) and (14)) with $F = \partial\psi$, is defined on some open set $D \subset \mathbb{H} \times \mathbb{R}$ such that $\mathbb{H} \times \{0\} \subset D$ and satisfies axioms from Definition 3.1. In other words $\varphi : D \multimap \mathbb{H}$ is a *local L-flow*. All notions introduced in Section 3 concerning global L-flows may be defined for local ones and the L-index may be defined by the following simple trick. If X is an isolated neighborhood for the local L-flow generated by $\partial\Phi$ and $X \subset B_\rho(0)$, where $\rho > 0$, then let us define $d : \mathbb{H} \to \mathbb{R}$ by

$$d(x) = \begin{cases} 1 & \text{for } \|x\| \leqslant \rho; \\ 1 + \rho - \|x\| & \text{for } rho < \|x\| \leqslant \rho + 1; \\ 0 & \text{for } \|x\| \geqslant \rho + 1. \end{cases}$$

Clearly the vector field $\widetilde{F} : \mathbb{H} \to \mathbb{H}$, given by $\widetilde{F}(x) := d(x)\partial\psi(x)$, $x \in \mathbb{H}$, is completely continuous with convex compact values and sublinear growth. Therefore the L-vector field $L + \widetilde{F}$ generates a global L-flow $\widetilde{\varphi}$ which coincides with φ on X. Hence X is an isolating neighborhood for $\widetilde{\varphi}$ and we can define $CH^*(X,\varphi) := CH^*(X,\widetilde{\varphi})$. From this viewpoint the assumption concerning p plays only a technical role and, without loss of generality, in what follows we may assume that $1 \leqslant p \leqslant 2$. Therefore we have:

Proposition 4.1. *The set-valued map $\partial\Phi = L + \partial\psi$ is an L-vector field in $\mathbb{H}$ and it generates an L-flow $\varphi : \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$.*

If $\Phi : \mathbb{H} \to \mathbb{R}$ were a smooth functional, then Φ would be a Lyapunov function for the flow generated by $\partial\Phi = \nabla\Phi$. However when Φ is only locally Lipschitz, it is rarely the case (except for special cases). Therefore we are going to introduce an associated *single-valued* pseudo-gradient L-field of the form $L + V$, where V is locally Lipschitz and completely continuous, such that Φ is a Lyapunov function for the flow generated by $L + V$. For this reason we need some more preparation.

For any compact convex set $A \subset \mathbb{H}$, define

$$|||A||| := \sup_{a \in A} \inf_{b \in A} \langle a, b\rangle_{\mathbb{H}}.$$

By the von Neumann mini-max equality, $|||A||| = \inf_{b \in A} \sup_{a \in A} \langle a, b\rangle_{\mathbb{H}} \geqslant \inf_{b \in A} \|b\|^2 \geqslant 0$; hence $|||A||| = 0$ if and only if $0 \in A$. Let us define $\delta : \mathbb{H} \to \mathbb{R}$ by

$$\delta(x) := \inf_{u \in \mathbb{H}} \big(\big|\big|\big| \partial\Phi(u) \big|\big|\big| + \|u - x\|_{\mathbb{H}} \big), \quad x \in \mathbb{H}.$$

For each $u \in \mathbb{H}$, the function $\mathbb{H} \ni x \mapsto \delta_u(x) := |||\partial\Phi(u)||| + \|u - x\|_{\mathbb{H}}$ is Lipschitz with constant 1; hence the lower envelope $\delta = \inf_{u \in \mathbb{H}} \delta_u$ is continuous. It is clear that $\delta(x) \leqslant \delta_x(x) := |||\partial\Phi(x)|||$ for any $x \in \mathbb{H}$.

Now let $K(\Phi) := \{z \in \mathbb{H} \mid 0 \in \partial\Phi(z)\}$ be the set of critical points of Φ.

Proposition 4.2. *If $X \subset \mathbb{H}$ is bounded and* $\operatorname{cl} X \cap K(\Phi) = \emptyset$, *then* $\inf_{x \in X} \delta(x) > 0$.

Proof. Suppose that $\inf_X \delta = 0$, i.e., there are sequences $x_n \in X$ and $u_n \in \mathbb{H}$ such that

$$\big|\big|\big| \partial\Phi(u_n) \big|\big|\big| + \|u_n - x_n\|_{\mathbb{H}} \to 0 \quad \text{as } n \to \infty.$$

Hence the sequence (u_n) is bounded and $|||\partial\Phi(u_n)||| \to 0$, i.e., for each $n \geqslant 1$, there is $y_n \in \partial\psi(u_n)$ such that $Lu_n + y_n \to 0$. The complete continuity of $\partial\psi$ implies that (up to a subsequence) $y_n \to y \in \mathbb{H}$. Thus $Lu_n \to -y$. Since (u_n) is bounded and L is a Fredholm operator, we infer that (up to a subsequence) $u_n \to u$ and $x_n \to u$. The upper semicontinuity of $\partial\Phi$ implies that $u \in K(\Phi)$: contradiction because $u \in \operatorname{cl} X$. □

Theorem 4.3. *For any $\varepsilon > 0$, there exists a locally Lipschitz L-field $W : \mathbb{H} \setminus K(\Phi) \to \mathbb{H}$ of the form $W(y) = Ly + V(y)$ where $V : \mathbb{H} \setminus K(\Phi) \to \mathbb{H}$ is completely continuous with sublinear growth such that, for each $y \in \mathbb{H}$ and $q \in \partial\Phi(y)$,*

$$\langle W(y), q\rangle_{\mathbb{H}} \geqslant \frac{1}{2}\delta(y).$$

Moreover $V(y) \in \operatorname{cl}\operatorname{conv}\partial\psi(B_\varepsilon(y))$ for every $y \in \mathbb{H} \setminus K(\Phi)$.

Proof. For any $u \in \mathbb{H} \setminus K(\Phi)$, $|||\partial\Phi(u)||| \geqslant \delta(u) > 0$. Hence there is $v_u \in \partial\psi(u)$ such that

$$\inf_{q\in\partial\Phi(u)} \langle q, Lu + v_u\rangle_{\mathbb{H}} > \frac{1}{2}\delta(u).$$

Observe now that

$$\inf_{q\in\partial\Phi(u)} \langle q, Lu + v_u\rangle_{\mathbb{H}} = -\Phi^\circ(u; -Lu - v_u).$$

The function

$$\mathbb{H} \ni y \mapsto \inf_{q\in\partial\Phi(y)} \langle q, Ly + v_u\rangle_{\mathbb{H}} - \frac{1}{2}\delta(y)$$

is lower semicontinuous and takes a positive value for $y = u$. Hence, for $\varepsilon > 0$, there is an open neighborhood $N_u \subset B_\varepsilon(u)$ of u such that, for all $y \in N_u$,

$$\inf_{q\in\partial\Phi(y)} \langle q, Ly + v_u\rangle_{\mathbb{H}} > \frac{1}{2}\delta(y).$$

Consider a locally finite partition of unity $\{\lambda_s\}_{s\in S}$ consisting of locally Lipschitz functions with supports $\{\operatorname{supp}\lambda_s\}$ refining the cover $\{N_u\}_{u\in\mathbb{H}\setminus K(\Phi)}$ of $\mathbb{H} \setminus K(\Phi)$, i.e., for any $s \in S$, there is $u_s \in \mathbb{H}$ such that $\operatorname{supp}\lambda_s \subset N_{u_s}$. For each $s \in S$, let $v_s := v_{u_s}$ and define

$$V(y) := \sum_{s\in S} \lambda_s(y)v_s, \quad y \in \mathbb{H} \setminus K(\Phi).$$

It is clear that V is well defined, locally Lipschitz and maps bounded sets in $\mathbb{H} \setminus K(\Phi)$ into compact ones. Moreover, V has sublinear growth since so does $\partial\psi$.

Let $y \in \mathbb{H} \setminus K(\Phi)$ and let $S_y := \{s \in S \mid \lambda_s(y) \neq 0\}$. If $s \in S_y$, then $y \in N_{u_s} \subset B_\varepsilon(u_s)$; hence $u_s \in B_\varepsilon(y)$ and $v_s \in \partial\psi(B_\varepsilon(y))$. Therefore $V(y) \in \operatorname{cl}\operatorname{conv}\partial\psi(B_\varepsilon(y))$ and

$$\inf_{q\in\partial\Phi(y)} \langle q, Ly + v_s\rangle_{\mathbb{H}} > \frac{1}{2}\delta(y).$$

Hence, for any $q \in \partial\Phi(y)$

$$\langle q, Ly + V(y)\rangle_{\mathbb{H}} = \sum_{s\in S_y} \lambda_s(y)\langle q, Ly + v_s\rangle_{\mathbb{H}} \geqslant \frac{1}{2}\delta(y).$$

Putting $W(y) := Ly + V(y)$, $y \in \mathbb{H} \setminus K(\Phi)$ we complete the proof. □

Suppose that $x \in \mathbb{H} \setminus K(\Phi)$ and let $\eta(x,\cdot) : J_x \to \mathbb{H} \setminus K(\Phi)$ be the unique (local, i.e., defined on some open interval $J_x := (t^-(x), t^+(x))$, $0 \in J_x$) solution of the Cauchy problem

$$\dot\eta(x,t) := \frac{\partial}{\partial t}\eta(x,t) = W\big(\eta(x,t)\big), \quad \eta(x,0) = x. \tag{19}$$

The function $\Phi \circ \eta(x,\cdot)$ is absolutely continuous and, for almost all $t \in I_x$,

$$\frac{\partial}{\partial t}\Phi \circ \eta(x,t) \geqslant \inf_{q\in\partial\Phi(\eta(x,t))} \langle q, \dot\eta(x,t)\rangle_{\mathbb{H}} = \inf_{q\in\partial\Phi(\eta(x,t))} \langle q, W\big(\eta(x,t)\big)\rangle_{\mathbb{H}} \geqslant \frac{1}{2}\delta\big(\eta(x,t)\big) > 0;$$

in other words Φ strictly increases along the trajectory $\eta(x,\cdot)$.

Theorem 4.4. *Suppose that y belongs to the ω-limit set $\omega(x)$ of the point $x \in \mathbb{H} \setminus K(\Phi)$ with respect to the local dynamical system η generated by W. Then $y \in K(\Phi)$.*

Proof is standard. Recall that, by definition $y \in \omega(x)$ if and only if $y = \lim_{t\to t^+(x)} \eta(x,t)$. We show that Φ is constant on $\omega(x)$. Indeed, let $z \in \omega(x)$. Hence there are sequences $t_n \nearrow t^+(x)$, $s_n \nearrow t^+(x)$ such that $\eta(x,t_n) \to y$ and $\eta(x,s_n) \to z$. We may assume that $\cdots < t_n < s_n < t_{n+1} < s_{n+1} < \cdots t^+(x)$. Hence $\cdots < \Phi(\eta(x,t_n)) < \Phi(\eta(x,s_n)) < \Phi(\eta(x,t_{n+1})) < \Phi(\eta(x,s_{n+1})) < \cdots$. By continuity, $\Phi(y) = \Phi(z)$.

Suppose to the contrary that $y \notin K(\Phi)$, i.e., $\delta(y) > 0$, and consider the trajectory $\eta(y,\cdot) : J_y \to \mathbb{H}$. It is easy to see that $\{\eta(y,t) \mid t \in J_y\} \subset \omega(x)$. But then (at least locally) Φ increases along $\eta(y,\cdot)$: a contradiction. $\square$

In the rest of this section we shall establish the existence of critical points of Φ via the study of invariant sets for η.

Theorem 4.5. *Let φ be the L-flow generated by the L-vector field $\partial\Phi$. Assume that X is an isolating neighborhood for φ and $CH^*(X,\varphi) \neq 0$. Then $K(\Phi) \cap X \neq \emptyset$.*

Proof. Suppose to the contrary that $K(\Phi) \cap X = \emptyset$ and let $\varepsilon_0 > 0$ be as in Lemma 3.18 stated for our flow φ. By Propositions 3.18 and 3.25, $CH^*(X,\varphi_\varepsilon) = CH^*(X,\varphi) \neq 0$ for any $0 < \varepsilon \leqslant \varepsilon_0$, where φ_ε is a flow generated by the field $f_\varepsilon := L + \operatorname{cl}\operatorname{conv}\partial\psi(D_\varepsilon(\cdot))$.

Fix $0 < \varepsilon \leqslant \varepsilon_0$ and let $\lambda : \mathbb{H} \to [0,1]$ be given by

$$\lambda(x) = \frac{d(x, K(\Phi))}{d(x,X) + d(x,K(\Phi))}, \quad x \in \mathbb{H}.$$

Then $\lambda|_X \equiv 1$ and $\lambda|_{K(\Phi)} \equiv 0$. Take a map $V : \mathbb{H} \setminus K(\Phi) \to \mathbb{H}$ given by Proposition 4.3 and let $\widetilde{V}(x) = \lambda(x)V(x)$ for $x \in \mathbb{H} \setminus K(\Phi)$ and $\widetilde{V}(x) = 0$ for $x \in K(\Phi)$. Then $\widetilde{V} : \mathbb{H} \to \mathbb{H}$ is locally Lipschitz, completely continuous and has sublinear growth. Let $\widetilde{\eta}$ be the L-flow generated by $\widetilde{W} := L + \widetilde{V}$.

Consider a family $h : \mathbb{H} \times [0,1] \multimap \mathbb{H}$ of L-vector fields given by $h(x,s) := sf_\varepsilon(x) + (1-s)\widetilde{W}(x)$, $s \in [0,1]$, $x \in \mathbb{H}$; it generates the family of L-flows $\theta : \mathbb{H} \times \mathbb{R} \times [0,1] \multimap \mathbb{H}$. It is clear that, for each $s \in [0,1]$ and $x \in X$, $h(x,s) \subset f_\varepsilon(x)$. Thus, for all $s \in [0,1]$, $\operatorname{Inv}(X,\theta_s) \subset \operatorname{Inv}(X,\varphi_\varepsilon)$; thus X is an isolating neighborhood for θ_s. Therefore

$$CH^*(X,\widetilde{\eta}) = CH^*(N,\theta_0)) = CH^*(N,\theta_1) = CH^*(N,\varphi_\varepsilon) \neq 0.$$

By Proposition 3.24, the invariant part $\operatorname{Inv}(N,\widetilde{\eta})$ is nonempty, i.e., there is $x \in \operatorname{Inv}(X,\widetilde{\eta})$ such that $\widetilde{\eta}(x,t) \in X$ for all $t \in \mathbb{R}$. Thus, for each $t \in J_x$, $\eta(x,t) \subset X$ where η is given by (19). Since $\operatorname{Inv}(X,\widetilde{\eta})$ is compact, the ω-limit set $\omega(x)$ of x with respect to η is nonempty and contained in X. By Theorem 4.4, $y \in K(\Phi) \cap N$: a contradiction. $\square$

Now recall (6), (4) and assumptions (G_3) and (G_4).

Proposition 4.6. *If G satisfies conditions (G_1)–(G_4), then*

(i) $|\Phi(u) - \Phi_{A_0}(u)| = o(\|u\|^2_{\mathbb{H}})$ *as* $\|u\|_{\mathbb{H}} \to 0$, $|\Phi(u) - \Phi_{A_\infty}(u)| = o(\|u\|^2_{\mathbb{H}})$ *as* $\|u\|_{\mathbb{H}} \to \infty$;
(ii) $\sup_{y\in\partial\Phi(u)} \|y - \nabla\Phi_{A_0}(u)\|_H = o(\|u\|_{\mathbb{H}})$ *as* $\|u\|_{\mathbb{H}} \to 0$ *and* $\sup_{y\in\partial\Phi(u)} \|y - \nabla\Phi_{A_\infty}(u)\|_{\mathbb{H}} = o(\|u\|_{\mathbb{H}})$ *as* $\|u\|_{\mathbb{H}} \to \infty$.

Proof. Since $G(\cdot,0) \equiv 0$, we see that $\Phi(0) = \Phi_{A_0}(0) = \Phi_{A_\infty}(0)$. Moreover $\partial\Phi - \nabla\Phi_{A_0} = \partial\psi + \mathcal{A}_0$ and $\partial\Phi - \nabla\Phi_{A_\infty} = \partial\psi + \mathcal{A}_\infty$. We shall proceed for an arbitrary $p \geqslant 1$.

For any $t \in [0,2\pi]$, $u \in \mathbb{R}^{2N}$ and $y \in \partial G(t,u)$, in view of (G_2) or $(G_2)'$,

$$|y - A_0u| \leqslant \ell + \|A_0\||u| \quad \bigl(\text{or} \quad |y - A_0u| \leqslant \alpha\bigl(1 + |u|^{p-1}\bigr) + \|A_0\||u|\bigr). \tag{20}$$

Take any $\varepsilon > 0$ and $r > \max\{1, p-1\}$. By (G_3) and (20), there is $c(\varepsilon) > 0$ such that, for any $t \in [0,2\pi]$, $u \in \mathbb{R}^{2N}$ and $y \in \partial G(t,u)$,

$$|y - A_0u| \leqslant \varepsilon|u| + c(\varepsilon)|u|^r. \tag{21}$$

Let $u \in \mathbb{H}$ and $y \in \partial\psi(u)$; then, by Corollary 2.3, there is $w \in L^q$ such that $w(t) \in \partial G(t,u(t))$ on $[0,2\pi]$ and, for all $v \in \mathbb{H}$, $\langle y, v\rangle_{\mathbb{H}} = -\int_0^{2\pi} w(t)\cdot v(t)\,dt$. By (21), for $t \in [0,2\pi]$, $|w(t) - A_0u(t)| \leqslant \varepsilon|u(t)| + c(\varepsilon)|u(t)|^r$. Hence, for any $v \in \mathbb{H}$, by the Hölder inequality and Remark 2.1,

$$\begin{aligned}\bigl|\langle y + \mathcal{A}_0(u), v\rangle_{\mathbb{H}}\bigr| &\leqslant \int_0^{2\pi} \bigl|w(t) - A_0u(t)\bigr|\bigl|v(t)\bigr|\,dt \leqslant \int_0^{2\pi} \bigl(\varepsilon\bigl|u(t)\bigr| + c(\varepsilon)\bigl|u(t)\bigr|^r\bigr)\bigl|v(t)\bigr|\,dt \\ &\leqslant \bigl(\varepsilon\|u\|_2 + c(\varepsilon)\|u\|^r_{2r}\bigr)\|v\|_2 \leqslant C_2\bigl(C_2\varepsilon\|u\|_{\mathbb{H}} + C_{2r}c(\varepsilon)\|u\|^r_{\mathbb{H}}\bigr)\|v\|_{\mathbb{H}}.\end{aligned}$$

Taking the supremum over $\|v\|_{\mathbb{H}} \leqslant 1$, we see that $\sup_{y\in\partial\Phi(u)} \|y - \nabla\Phi_{A_0}(u)\|_H = o(\|u\|_{\mathbb{H}})$ as $\|u\|_{\mathbb{H}} \to 0$.

For any $u \in \mathbb{H}$, the function $[0,1] \ni s \mapsto \xi(s) := \Phi(su) - \Phi_{A_0}(su) = \psi(su) + \frac{1}{2}\int_0^{2\pi} s^2 A_0u(t)\cdot u(t)\,dt$ is absolutely continuous and, for almost all $s \in [0,1]$,

$$\frac{d}{ds}\xi(s) \leqslant \sup_{y\in\partial\psi(su)} \bigl\langle y + s\mathcal{A}_0(u), u\bigr\rangle_{\mathbb{H}}.$$

Hence

$$\left|\Phi(u)-\Phi_{A_0}(u)\right| = \left|\xi(1)\right| \leqslant \int_0^1 \left|\frac{d}{ds}\xi(s)\right| \leqslant \sup_{s\in[0,1]} \sup_{y\in\partial\Phi(su)} \left\|y-\nabla\Phi_{A_0}(su)\right\|_{\mathbb{H}} \|u\|_{\mathbb{H}} = o\big(\|u\|^2_{\mathbb{H}}\big)$$

as $\|u\| \to 0$.

Analogously, for any $\varepsilon > 0$, there is $c(\varepsilon) > 0$ such that

$$|y - A_\infty u| \leqslant \varepsilon|u| + c(\varepsilon)$$

for any $t \in [0, 2\pi]$, $u \in \mathbb{R}^{2N}$ and $y \in \partial G(t,u)$. Thus, for any $u \in H$, $y \in \partial\psi(u)$ and $v \in \mathbb{H}$, by Remark 2.1,

$$\left|\langle y + \mathcal{A}_\infty(u), v\rangle_{\mathbb{H}}\right| \leqslant C_s\big(C_r\varepsilon\|u\|_{\mathbb{H}} + c(\varepsilon)\big)\|v\|_{\mathbb{H}};$$

this shows that $\sup_{y\in\partial\Phi(u)} \|y - \nabla\Phi_{A_\infty}(u)\|_{\mathbb{H}} = o(\|u\|_{\mathbb{H}})$ as $\|u\|_{\mathbb{H}} \to \infty$. Similarly as above we show the last part of the assertion. □

Now we are ready for the

Proof of Theorem 2.5. Assume that $i^+(A_0) \neq i^+(A_\infty)$ and observe that $\nabla\Phi_{A_0}$ and $\nabla\Phi_{A_\infty}$ are selfadjoint and have trivial null-spaces since $i^0(A_0) = i^0(A_\infty) = 0$. Thus there is $c > 0$ such that $\|\nabla\Phi_{A_0}u\| \geqslant c\|u\|$ and $\|\nabla\Phi_{A_\infty}u\| \geqslant c\|u\|$ for all $u \in \mathbb{H}$.

By Proposition 4.6, there are $0 < r < R$ such that, for $u \in \mathbb{H}$ and $y \in \partial\Phi(u)$: if $\|u\| \leqslant r$, then $\|y - \nabla\Phi_{A_0}(u)\|_{\mathbb{H}} < \frac{c^2}{2\|\nabla\Phi_{A_0}\|}\|u\|$ and if $\|u\| \geqslant R$, then $\|y - \nabla\Phi_{A_\infty}(u)\|_{\mathbb{H}} < \frac{c^2}{2\|\nabla\Phi_{A_\infty}\|}\|u\|$.

Now suppose that the only critical point of Φ is in $x = 0$. Applying Theorem 4.3 we define a single-valued L-vector field $W = L + V : \mathbb{H} \setminus \{0\} \to \mathbb{H}$ such that for $u \in \mathbb{H}$, $u \neq 0$, and $\xi \in \partial\Phi(u)$ we have $\langle W(u), \xi\rangle_H \geqslant \frac{1}{2}\delta(u)$. Define a new L-vector field $\widetilde{W} : \mathbb{H} \to \mathbb{H}$ as follows:

$$\widetilde{W}(u) := \begin{cases} \nabla\Phi_{A_0}(u) & \text{if } \|u\| < \frac{r}{2};\\ (1 - \frac{2\|u\|}{r} + 1)\nabla\Phi_{A_0}(u) + (\frac{2\|u\|}{r} - 1)W(u) & \text{if } \frac{r}{2} \leqslant \|u\| \leqslant r;\\ W(u) & \text{if } r < \|u\| < R;\\ (\|u\| - R)\nabla\Phi_{A_\infty}(u) + (1 - \|u\| + R)W(u) & \text{if } R \leqslant \|u\| \leqslant R+1;\\ \nabla\Phi_{A_\infty}(u) & \text{if } \|u\| > R+1. \end{cases}$$

Let $u \in \mathbb{H}$ and $y \in \partial\Phi(u)$; if $0 < \|u\| \leqslant r$, then

$$\langle\nabla\Phi_{A_0}(u), y\rangle = \langle\nabla\Phi_{A_0}(u), \nabla\Phi_{A_0}(u)\rangle + \langle\nabla\Phi_{A_0}(u), y - \nabla\Phi_{A_0}(u)\rangle \geqslant c^2\|u\|^2 - \|\nabla\Phi_{A_0}\| \cdot \|u\| \cdot \|y - \nabla\Phi_{A_0}(u)\| \geqslant \frac{1}{2}c^2\|u\|$$

and, analogously, if $\|u\| \geqslant R$, then

$$\langle\nabla\Phi_{A_\infty}(u), y\rangle \geqslant \frac{1}{2}c^2\|u\|.$$

Therefore, for all $u \in \mathbb{H} \setminus \{0\}$ and $q \in \partial\Phi(u)$, $\langle\widetilde{W}(u), q\rangle \geqslant \min\{\frac{1}{2}\delta(u), \frac{1}{2}c^2\|u\|\} > 0$. Hence, by remarks after Theorem 4.3, Φ is a Lyapunov function for the flow η generated by $\widetilde{W}$. Thus the conclusion of Theorem 4.5 holds. On the other hand the closed balls $D_{\frac{r}{2}}(0), D_{R+2}(0))$ are isolating neighborhoods for η. Clearly by Example 3.23, $CH^*(D_{\frac{r}{2}}(0), \eta)$ has the only nontrivial group in dimension $i^+(A_0)$. Similarly $CH^*(D_{R+2}(0), \eta)$ has the only nontrivial group in dimension $i^+(A_\infty)$, since by a homotopy argument it is equal to the index of the flow generated by a linear L-vector field $\nabla\Phi_{A_\infty}$.

In view of Proposition 3.26, we have $\mathrm{Inv}(D_{R+2}(0), \eta) \not\subset B_{\frac{r}{2}}(0)$. Take $x \in \mathrm{Inv}(D_{R+2}(0), \eta) \setminus \{0\}$. We have shown in Theorem 4.4 that $y \in \omega(x)$ is a critical point of Φ. The same is clearly true for $y \in \alpha(x)$. But since Φ increases along the orbits of η, either $\omega(x) \neq \{0\}$ or $\alpha(x) \neq \{0\}$. This contradiction completes the proof.

In the case when $i^-(A_0) \neq i^-(A_\infty)$, a similar argument for $-\Phi$ works. □

Example 4.7. Consider $G : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ constructed similarly as in Example 2.6 but such that $G(t,x,y) = \frac{1}{2}(x^2 + y^2) + (x^3 - 3xy^2)\cos(3t)$ if $x^2 + y^2 < 1$ and $t \in \mathbb{R}$; hence G is smooth in a neighborhood of 0. Here $A_0 := I_2$ and $i^0(A_0) = 2$. In [20, Example 5.1], it is shown that $\{0\}$ is an isolated invariant set and $CH^3(X, \varphi) = \mathbb{Z} \oplus \mathbb{Z}$ for a small isolating neighborhood X of $\{0\}$. On the other hand, the continuation argument 3.25 implies that, for sufficiently large R, the ball $D_R(0)$ is an isolating neighborhood for φ and the only nontrivial group in $CH^*(D_R(0))$ is in dimension 4. Thus, in spite of the fact that the assumptions of Theorem 2.5 are not satisfied, the same argument as in the proof of Theorem 2.5 shows the existence of a nontrivial periodic solution to the problem (1).

In the forthcoming paper we shall study more carefully the situation present in Example 4.7, i.e., asymptotically linear locally Lipschitz hamiltonians under the resonance, i.e., without nondegeneracy assumption $i^0(A_0) = 0$, $i^0(A_\infty) = 0$, as well as non-autonomous asymptotically linear hamiltonians, i.e., such that (G_3) and (G_4) hold for some *time-dependent* symmetric matrices $A_0(t)$, $A_\infty(t)$, $t \in \mathbb{R}$, with 2π-periodic entries. Problems of this type for smooth hamiltonians have been studied in e.g. [14,28,38] by means of the generalized Morse theory.

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