Bull. London Math. Soc. 53 (2021) 1263–1298 doi:10.1112/blms.12531

Dold sequences, periodic points, and dynamics

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Abstract

In this survey we describe how the so-called Dold congruence arises in topology, and how it relates to periodic point counting in dynamical systems.

1. *Introduction*

The arithmetic properties of integer sequences have pervasive connections to questions in number theory, topology, geometry, combinatorics, dynamical systems, and doubtless in many other places as well. These notes are a survey of a chain of ideas whose origin arguably lies in Fermat's little theorem, that a^p is congruent to a modulo p for any integer a and prime p. In one direction, it is natural to ask if there is a canonical way in which this fundamental congruence is the prime case of a more general type of statement about integer sequences. In another, it is natural to ask of any congruence if there is a counting argument that exhibits (in this case) $a^p - a$ as the cardinality of a set with a natural p-fold symmetry. For example, Petersen [**104**] gave a proof of Fermat's little theorem in 1872 along exactly these lines, by counting configurations of a colours in p boxes arranged in a circle (as a remark attached to a related argument for Wilson's theorem). Arguments of this sort have been, and continue to be, repeatedly reinvented or rediscovered — we refer to Dickson [**26**, pp. 75–86] for some of the early history. We will discuss questions that flow from these elementary considerations, in some cases starting from a quite different question in topology or dynamics.

A feature of this area is that many of the results not only have multiple independent (in some cases, independently repeated) proofs with different motivations, but have multiple histories. One consequence is that the same notion (or class of integer sequences) may have different names, and one of the things we try to do in these notes is to indicate some of these names and to be — at least internally — consistent in terminology. In order to explain some of the motivation behind the different threads here, we to some extent repeat some of these repetitions, favouring explanation of motivation over a strictly logical linear development.

Another feature of fixed-point theory is that the contexts and applications are widely varied. The preface to the handbook edited by Brown *et al.* [**20**] talks about 'the varied, and not easily classified, nature of the mathematics that makes up topological fixed point theory'. Given the existing literature on fixed-point theory, which sprawls in volume (MathSciNet indexes

Received 7 July 2020; revised 7 April 2021; published online 2 July 2021.

2020 *Mathematics Subject Classification* 11B50, 37C25, 55M20, 47H10 (primary).

The first author was supported by the National Science Centre, Poland, under grant no. 2016/23/D/ST1/01124. The second author was supported by the National Science Centre, Poland within the grant Sheng 1 UMO-2018/30/Q/ST1/00228. The third author was asked to give the 'Wandering Seminar' at Gdańsk University of Technology in February 2020, on the topic of 'Periodic points in dynamics'. The first and second authors had earlier been preparing a survey on Dold sequences, and — reflecting the wider history of these ideas — we discovered that we were in several cases talking about closely related things from different points of view.

more than ten thousand items under the primary subject classification 47H10, 'Fixed-point theorems', for example) and contains several weighty surveys and handbooks, this modest survey perhaps needs some justification. Our defence is merely that the circle of ideas discussed here has some particularly elementary entry points, and brings together facets of dynamical systems and topology in ways that seem interesting — to the authors at any rate.

One of the sources of integer sequences is counting periodic points in dynamics, which we will discuss from a particularly elementary point of view. There are many ways in which periodic points might be studied from a more sophisticated point of view, some of which are briefly outlined later.

Roughly speaking, in studying dynamical systems one often starts with a map — perhaps a smooth map on a differentiable manifold — and uses attributes of the map like hyperbolicity or local product structure to deduce specification or closing properties strong enough to construct periodic points for dynamical reasons (as opposed to the global topological production of periodic points mentioned at the end of Section 2). We will describe some of the consequences of starting at the other end, motivated by the following trivial observation (see Remark 3.6 for a specific example in the same spirit that originally triggered the interest of the third-named author in these kind of questions). Assume that a map has a single fixed point. Then, if the number of points fixed by the second iterate of the map is finite, it must be a non-negative *odd* number. This remark sets off a cascade of natural questions about the combinatorial and analytical properties of the sequences of periodic point counts of dynamical systems viewed — initially — simply as permutations of countable sets. We explore some of these questions here, with an emphasis on introducing a broad range of concepts using fairly simple examples and instances of results, rather than aiming for maximum generality.

Our focus will be on two closely related — but motivationally rather different — classes of integer sequences. Roughly speaking, the first class of sequences finds its natural home in topology, the second in dynamical systems and combinatorics. There is no particular reason to define one before the other: the class we describe first is fundamental for a journey into arithmetic starting with Fermat's little theorem, the class we describe second is fundamental for a journey into dynamics starting at the same place.

A sequence will be denoted $a = (a_n) = (a_n)_{n\in\mathbb{N}}$, and in particular a sequence is always indexed by the natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ unless explicitly indicated otherwise. With very few exceptions, the maps we are concerned with are self-maps, so we use phrases like 'a map of X' or 'a homeomorphism of X' to mean a self-map of X, a self-homeomorphism of X, and so on.

2. *Dold sequences*

The sequences which we will call *Dold sequences* as a result of the work of Dold [**27**] have (for example) also been called *sequences having divisibility* in the thesis of Moss [**92**], *pre-realisable sequences* by Arias de Reyna [**2**], *relatively realisable sequences* by Neumärker [**94**], *Gauss sequences* by Minton [**89**], and *generalised Fermat sequences* or *Fermat sequences* by Du, Huang, and Li [**28, 29**]. Doubtless there are other names, reflecting the long history and multiple settings in which they appear.

Definition 2.1 (Dold sequence). An integer sequence $a = (a_n)$ is called a *Dold sequence* if

$$\sum_{d|n} \mu\Big(\frac{n}{d}\Big) a_d \equiv 0 \tag{1}$$

modulo n for all $n \geqslant 1$.

Here μ denotes the classical Möbius function, defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \text{ has a squared factor, and} \\ (-1)^r & \text{if } n \text{ is a product of } r \text{ distinct primes.} \end{cases}$$

In particular, if $n = p$ is a prime, then (1) is the statement that $a_p \equiv a_1$ modulo p. Thus Fermat's little theorem says that the sequence (a^n) satisfies the Dold condition at every prime (and it is easy to show that it is in fact a Dold sequence, and in this case the congruence (1) is normally attributed to Gauss).

We start by making some remarks about Dold sequences.

(a) If $A \in \mathrm{Mat}_{m,m}(\mathbb{Z})$ is an integer matrix, then the sequence $(\mathrm{Tr}\, A^n)$ is a Dold sequence, generalising Fermat's little theorem. This observation in some form seems to have been known to Gauss, and has been rediscovered by many others including Browder [**17**], Peitgen [**102**], and Arnold [**4–7**]; we refer to notes of Smyth [**117**], Vinberg [**126**], and Deligne [**25**] for more on this. For prime n this was proved by Schönemann [**115**] in 1839. For $m = 1$ this has been proved many times; among these are work of Kantor [**65**], Weyr [**130**], Lucas [**80**], Pellet [**103**], Thue [**124**], Szele [**123**], and doubtless many others. We refer to a note of Steinlein [**120**] for an account and some of the history, and will discuss this again from a dynamical point of view in Section 4.3.

(b) The sequence of the number of fixed points of iterates of a map is a Dold sequence, but not all Dold sequences arise in this way (we will say more about this later).

(c) The original context considered by Dold [**27**] was to show that the sequence of *fixed-point indices* of iterations in a topological setting satisfies (1).

(d) Marzantowicz and Przygodzki [**82**] developed a theory of periodic expansions for integral arithmetic functions, giving a different characterisation of the sequence of fixed-point indices and Lefschetz numbers of iterations of a map. They also gave probably the first complete proof of the fact that sequences like $(\mathrm{Tr}\, A^n)$ are Dold sequences.

(e) Many sequences of combinatorial or arithmetic origin are Dold sequences, and in some cases it would be desirable to have a combinatorial or dynamical explanation. In simple cases this is clear, but (for example) it is not clear why the Bernoulli numerator, the Bernoulli denominator, and the Euler sequence (τ_n), (β_n), and $((-1)^n E_{2n})_{n\geqslant 1}$ respectively, are Dold sequences (here $\frac{\tau_n}{\beta_n} = |\frac{B_{2n}}{2n}|$ in lowest terms for all $n \geqslant 1$, and $\sum_{n\geqslant 0} E_n \frac{t^n}{n!} = \frac{2}{\mathrm{e}^t+\mathrm{e}^{-t}}$).

(f) If (b_n) is any integer sequence, then it follows from basic properties of the Möbius function (see Lemma 3.3(d)) that $(\sum_{d|n} d b_d)$ is a Dold sequence. Conversely, we prove in Lemma 2.6 that all Dold sequences can be obtained in this manner.

(g) Many multiplicative sequences are Dold sequences. One way to construct such sequences is to apply the above method and write such sequences in the form $(a_n) = (\sum_{d|n} d b_d)$. A simple argument (see Hardy and Wright [**56**, Theorem 265]) shows that (a_n) is multiplicative if and only if (b_n) is multiplicative. Choosing (b_n) to be the sequence $b_n = n^k$ of kth powers, $k \in \mathbb{N}_0$, shows that the sequence $a_n = \sigma_{k+1}(n)$ of sums of $(k+1)$st powers of divisors is a Dold sequence. On the other hand, the sequence $\sigma_0(n)$ of the number of divisors is not a Dold sequence.

(h) Beukers *et al.* [**13**] characterised the Dold congruence for the coefficients of the Laurent series associated to a multi-variable rational function.

(i) Samol and van Straten [**113**, Section 4] considered related congruences of arithmetic interest for sequences generated as the constant term of powers of Laurent polynomial; these results were later extended by Mellit and Vlasenko [**84**].

(j) Kenison *et al.* [**67**] have studied positivity questions for certain holonomic sequences (sequences satisfying a linear recurrence relation with polynomial coefficients), relating positivity to questions on vanishing of periods.

(k) While our emphasis is different, we also mention work of Marzantowicz and Wójcik [83] addressing related questions for periodic solutions of certain ordinary differential equations.

One of the questions we will be interested in is the characterisation of the intersection of classes of sequences defined in different ways. The next observation is a simple instance of this.

Lemma 2.2 (Puri *et al.* [108, Lemma 2.4]). *If a completely multiplicative sequence is a Dold sequence, then it is the constant sequence with every term equal to* 1. *If a polynomial sequence is a Dold sequence, then it is a constant.*

Proof. If (a_n) is completely multiplicative, so $a_{mn} = a_m a_n$ for all $m, n \geqslant 1$, and satisfies (1) then $p^r \mid a_{p^{r-1}}(a_p - 1) = a_p^{r-1}(a_p - 1)$ for any prime p and all $r \geqslant 1$. Thus we can write $a_p = 1 + pk_p$ for some $k_p \in \mathbb{N}_0$, and deduce that $p^r \mid (1 + pk_p)^{r-1} pk_p$ for all $r \geqslant 1$. This gives $k_p \equiv 0$ modulo p^r for all $r \geqslant 1$, so $a_n = 1$ for all $n \geqslant 1$.

For the second assertion, assume that $h(n) = c_0 + c_1 n + \cdots + c_k n^k$ with $c_k \neq 0$ and $k \geqslant 1$ is a polynomial taking integer values on the integers with the property that $(h(n))$ is a Dold sequence. After clearing fractions in (1), we may assume without loss of generality that the coefficients $c_0, \ldots, c_k$ are integers. For any prime p we have

$$\tfrac{1}{p^2}\big(h(p^2) - h(p)\big) \in \Big(-\tfrac{c_1}{p} + \mathbb{Z}\Big) \cap \mathbb{Z}$$

by (1), so $p \mid c_1$. It follows (since p is any prime) that $c_1 = 0$. Thus we can write $h(n) = c_0 + n^2(c_2 + c_3 n + \cdots + c_k n^{k-2})$ for all $n \geqslant 1$. Now let p and q be different primes, so

$$h(p^2 q) - h(pq) - h(p^2) + h(p) \equiv -h(p^2) + h(p)$$

modulo $p^2 q$. Since the left-hand side is independent of q, this shows that

$$h(p^2) = h(p)$$

for all primes p. This contradicts the hypothesis that h is a non-constant polynomial. □

The motivation for some of these questions comes from the Lefschetz fixed-point theorem. For a suitable topological space X (for example, a finite CW-complex or, more generally, a space whose homology groups are finitely generated and trivial in all sufficiently high dimensions) and continuous map $f\colon X \to X$, the *Lefschetz number* $L(f)$ is defined by

$$L(f) = \sum_{k \geqslant 0} (-1)^k \operatorname{Tr}(f_*|_{H_k(X,\mathbb{Q})}),$$

the alternating sum of the traces of the linear maps induced by f on the singular homology groups $H_k(X, \mathbb{Q})$ of X with rational coefficients.

Theorem 2.3 (Lefschetz fixed-point theorem). *If $f\colon X \to X$ is a continuous map of a compact CW-complex or, more generally, a retract of a compact CW-complex, and $L(f) \neq 0$, then f has a fixed point.*

More generally, Dold [27] considered the sequence $(\operatorname{ind}(f^n, U))$ of indices under the hypothesis that $f\colon U \to X$ is a continuous map on an open subset U of an Euclidean neighbourhood retract X with compact set fixed by each iterate, and characterised the possible sequences arising as being those satisfying (1). For the case of n prime, the direct analogue of Fermat's little theorem, similar results were shown earlier by Steinlein [118] and by Zabreĭko and Krasnosel'skiĭ [134]. Using the Leray–Schauder degree to extend the concepts to infinite dimensions, the Dold relations have also been shown for suitable maps on Banach spaces by Steinlein [119].

As pointed out by Smale [**116**, p. 768] in his influential survey, 'the whole difficulty of the problem [...] is that it counts the periodic [points] geometrically, not algebraically'. That is, what is natural to count geometrically in the line of thought initiated by Lefschetz's theorem is not the same as simply counting the number of solutions to the equation $f(x) = x$.

2.1. *Linear recurrence Dold sequences*

We will see later that sequences of the form $(\mathrm{Tr}(A^n))$ for $A \in \mathrm{Mat}_{d,d}(\mathbb{Z})$ are automatically Dold sequences for dynamical reasons. Minton has shown that this is essentially the only way that a linear recurrence sequence can be a Dold sequence. Since Minton was working with sequences of rational numbers, he relaxed the conditions in the definition of a Dold sequence, insisting only that the divisibility holds except for powers of finitely many primes. For linear recurrence sequences this is equivalent, however, to considering rational multiples of Dold sequences.

THEOREM 2.4 (Minton [**89**, Theorem 2.15 and Remark 2.16]). *An integer linear recurrence sequence (a_n) is a rational multiple of a Dold sequence if and only if it is a* trace sequence, *meaning that there is an algebraic number field $\mathbb{K}$, rationals $b_1, \dots, b_r \in \mathbb{Q}$ and algebraic integers $\theta_1, \dots, \theta_r \in \mathbb{K}$ with*

$$a_n = \sum_{i=1}^{r} b_i \, \mathrm{Tr}_{\mathbb{K}|\mathbb{Q}}(\theta_i^n)$$

for all $n \geqslant 1$.

To see why the 'rational multiple' part is important, consider the characteristic sequence $(a_n) = (0, 1, 0, 1, \dots)$ of even numbers. This is a trace sequence since

$$a_n = \tfrac{1}{2}(1)^n + \tfrac{1}{2}(-1)^n$$

for all $n \geqslant 1$, but is not a Dold sequence since $a_2 - a_1 \not\equiv 0$ modulo 2. However, $(2a_n)$ is a Dold sequence because, for example, it counts the periodic points for a map on a set with two elements that swaps the elements.

We will see a simple instance of Theorem 2.4 in Lemma 4.9 for a specific historically important linear recurrence.

2.2. *Generating functions of Dold sequences*

Write $\mathscr{S}(R)$ for the set of all sequences (a_n) with values in a subring R of a field $\mathbb{K}$ of characteristic zero, and $\mathscr{S}$ if the field plays no role. We define bijections by

$$\mathsf{B}\colon \mathscr{S}(\mathbb{K}) \longrightarrow \mathscr{S}(\mathbb{K})$$

$$(a_n) \longmapsto (b_n) = \left(\frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) a_d \right)$$

and

$$\mathsf{C}\colon \mathscr{S}(\mathbb{K}) \longrightarrow \mathscr{S}(\mathbb{K})$$

$$(a_n) \longmapsto (c_n) = \left(\frac{1}{n}(a_n - c_1 a_{n-1} - \cdots - c_{n-1} a_1) \right).$$

That B and C are bijections is clear, and the inverse maps are given by the formulas

$$a_n = \sum_{d|n} d b_d \tag{2}$$

and

$$a_n = c_1 a_{n-1} + \cdots + c_{n-1} a_1 + n c_n$$

for all $n \geqslant 1$. To see that the formula (2) does indeed give the inverse of B, use Lemma 3.3(d). The sequence (c_n) is called the *generating sequence* of (a_n). As with much else in these notes, the various relationships between sequences described here have multiple names and appear in many different guises; in settings close to ours they may be found in work of Du *et al.* [**29**] and Arias de Reyna [**2**].

THEOREM 2.5. *For $(a_n) \in \mathscr{S}$ let $(b_n) = \mathsf{B}((a_n))$ and $(c_n) = \mathsf{C}((a_n))$. Then, in the ring of formal power series, we have*

$$\exp\left(\sum_{n\geqslant 1} \frac{a_n}{n} z^n\right) = \prod_{n\geqslant 1} (1-z^n)^{-b_n} = \left(1 - \sum_{n\geqslant 1} c_n z^n\right)^{-1}. \tag{3}$$

Proof. Using the Taylor expansion for the logarithm, we obtain

$$\begin{aligned}\log \prod_{n\geqslant 1} (1-z^n)^{-b_n} &= \sum_{n\geqslant 1} -b_n \log(1-z^n) = \sum_{n\geqslant 1}\sum_{k\geqslant 1} b_n \frac{z^{nk}}{k} \\ &= \sum_{m\geqslant 1}\sum_{n\mid m} n b_n \frac{z^m}{m} = \sum_{m\geqslant 1} \frac{a_m}{m} z^m,\end{aligned}$$

giving the first equality.

Write

$$f(z) = -\exp\left(-\sum_{n\geqslant 1} \frac{a_n}{n} z^n\right)$$

for the negative reciprocal of the left-hand expression in (3), and let $f(z) = \sum_{n\geqslant 0} c'_n z^n$ be its Taylor expansion. Comparing the values at zero gives $c'_0 = -1$, and the logarithmic derivative is given by

$$\frac{f'(z)}{f(z)} = -\sum_{n\geqslant 1} a_n z^{n-1}.$$

Hence

$$f'(z) = \sum_{n\geqslant 1} n c'_n z^{n-1} = \left(\sum_{n\geqslant 1} a_n z^{n-1}\right) \cdot \left(1 - \sum_{n\geqslant 1} c'_n z^n\right),$$

giving the equalities

$$n c'_n = a_n - c'_1 a_{n-1} - \cdots - c'_{n-1} a_1$$

for all $n \geqslant 1$. This proves that $c'_n = c_n$ for all $n \geqslant 1$, completing the proof. □

In the dynamical context relations like (3) may be used to prove integrality of certain sequences; we refer to Jaidee *et al.* [**58**], for example. By Möbius inversion we may deduce from the definition of B a classical relation due to Möbius [**91**],

$$\mathrm{e}^x = \prod_{n\geqslant 1} (1-x^n)^{-\mu(n)/n},$$

valid for $|x| < 1$. In a more arithmetic direction, applying this to the Artin–Hasse exponential series (defined in [8])

$$E_p(x) = \exp\left(x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + \frac{x^{p^3}}{p^3} + \cdots\right) = \exp\left(\sum_{n\geqslant 1} \frac{a_n}{n} x^n\right)$$

gives the corresponding (b_n) by the formula

$$b_n = \begin{cases} \frac{\mu(n)}{n} & \text{if } p \nmid n; \\ 0 & \text{otherwise.} \end{cases}$$

Thus Theorem 2.5 gives

$$E_p(x) = \prod_{p\nmid n} (1 - x^n)^{-\mu(n)/n},$$

from which the well-known fact that the coefficients of the Taylor expansion of E_p are p-adic integers may be deduced using the p-adic continuity of the binomial coefficient polynomials, or from 'Dwork's lemma' (see Koblitz [70, Chapter IV.2] for the latter argument).

Theorem 2.5 gives a constructive alternative definition of Dold sequences. This has a parallel in Section 3, where it is expressed as the relation between counting closed orbits and counting periodic points.

LEMMA 2.6. *For $(a_n) \in \mathscr{S}$ let $(b_n) = \mathsf{B}((a_n))$ and $(c_n) = \mathsf{C}((a_n))$. Then the following are equivalent:*

(a) *(a_n) is a Dold sequence;*
(b) *$(b_n) \in \mathscr{S}(\mathbb{Z})$;*
(c) *$(c_n) \in \mathscr{S}(\mathbb{Z})$.*

Proof. The sequence (a_n) is a Dold sequence if and only if the values of (b_n) are integers by (2). Writing

$$g(z) = \prod_{n\geqslant 1} (1 - z^n)^{b_n} = \left(1 - \sum_{n\geqslant 1} c_n z^n\right),$$

it is clear that if all the powers b_n are integers, then so are all the coefficients c_n. For the reverse direction, assume that $c_n \in \mathbb{Z}$ for all $n \geqslant 1$ and assume that $b_1, \ldots, b_{m-1}$ are integers. The power series of g computed modulo z^{m+1} gives

$$g(z) \equiv \prod_{n=1}^{m} (1 - z^n)^{b_n} \equiv (1 - b_m z^m) \prod_{n=1}^{m-1} (1 - z^n)^{b_n} \equiv \prod_{n=1}^{m-1} (1 - z^n)^{b_n} - b_m z^m$$

modulo z^{m+1}. Since g has integer coefficients, we deduce that b_m is an integer, proving that all b_n are integers by induction. □

The different ways of expressing the property of being a Dold sequence facilitate a way to decompose a Dold sequence into elementary periodic sequences as follows. For $d \in \mathbb{N}$ we define

$$\mathsf{reg}_d(n) = \begin{cases} d & \text{if } d \mid n; \\ 0 & \text{if } d \nmid n. \end{cases}$$

The following decomposition — a different way of expressing the transformation B — is given by the following result of Jezierski and Marzantowicz [59, Proposition 3.2.7].

PROPOSITION 2.7. *Let $(a_n) \in \mathscr{S}$ and let $(b_n) = \mathsf{B}((a_n))$ as above. Then*

$$a_n = \sum_{k=1}^{\infty} b_k \mathsf{reg}_k(n). \tag{4}$$

Moreover, (a_n) is an integer Dold sequence if and only if $(b_n) \in \mathscr{S}(\mathbb{Z})$.

This is an immediate consequence of the formula for B and Lemma 2.6, and in the context of dynamics this may be thought of as building up a dynamical system as a union of closed orbits. The class of integer multiples of elementary periodic sequences is closed under pointwise multiplication, since we have

$$\mathsf{reg}_k \cdot \mathsf{reg}_\ell = \gcd(k,\ell)\mathsf{reg}_{\mathrm{lcm}(k,\ell)}. \tag{5}$$

When we study dynamical systems with finitely many points of each period, it will be clear that a bound on the number of periodic points is equivalent to a bound on the possible lengths of closed orbits. In the more general context of Dold sequences this is expressed as follows.

LEMMA 2.8 (Babenko and Bogatyĭ [10]). *A Dold sequence is bounded if and only if it is periodic. If this is the case, it can be written as a finite linear combination with integer coefficients of elementary periodic sequences reg_k.*

Proof. Let $(a_n)_n$ be a Dold sequence with $|a_n| < T$ for all $n \geqslant 1$. Then

$$|b_n| = \frac{1}{n}|\sum_{k|n} \mu(k) a_{k/d}| \leqslant \frac{T}{n}\sum_{k|n}|\mu(k)| \leqslant \frac{T}{n}\sigma_0(n),$$

where $\sigma_0(n)$ is the number of positive divisors of n. It is well known that for any $\beta > 0$ we have $\sigma_0(n) = \mathrm{o}(n^\beta)$ (see Hardy and Wright [56, Section 18.1]), so $|b_k| \to 0$ as $k \to \infty$. Since $b_n \in \mathbb{Z}$, we deduce that $b_n = 0$ for all but finitely many $n \in \mathbb{N}$, completing the proof. □

The representation of a Dold sequence in the form (4) is called its *periodic expansion*, and is useful in many problems in periodic point theory. For example, in the periodic expansion of the Lefschetz numbers of iterations of Morse–Smale diffeomorphisms, each (reg_k) for odd k represents a periodic orbit of minimal period k. This observation allows one to easily determine the so-called *minimal set of Lefschetz periods* $\mathrm{MPer}_L(f)$ for Morse–Smale diffeomorphisms (see Graff *et al.* [51]), studied by Guirao, Llibre, Sirvent, and other authors (we refer to [79] and the references therein for more on this).

The language of periodic expansions turns out to be useful in some dynamical problems in magnetohydrodynamics. This applies to the flow in magnetic flux tubes, which can be studied via the discrete field line mapping φ through the consecutive horizontal cross-sections of the tube. The bounds for coefficients of the periodic expansion of the fixed-point indices of iterations of φ at periodic points provide a class of topological constraints sought by the astrophysicists Yeates *et al.* [133] to explain unexpected behaviour in certain resistive-magnetohydrodynamic simulations of magnetic relaxation. We refer to work of Graff *et al.* [54] for more on this.

2.3. *Other characterisations*

András [1, Remark 2] noted that in the definition of the Dold property the Möbius function can be replaced by the Euler totient function ϕ defined as usual by $\phi(n) = |\{k \mid 1 \leqslant k \leqslant n;\ \gcd(k,n) = 1\}|$. That is, an integer sequence (a_n) is a Dold sequence if and only if

$$\sum_{d|n} \phi\Big(\frac{n}{d}\Big) a_d \equiv 0 \tag{6}$$

modulo n for all $n \geqslant 1$. Recently, Wójcik [132] proved that in fact the Dold property can be characterised with the Möbius function replaced by any other integer-valued function satisfying some natural constraints.

THEOREM 2.9 (Wójcik [132]). *Let $\psi\colon \mathbb{N} \to \mathbb{Z}$ be a function satisfying the two conditions*

$$\begin{cases} \psi(1) = \pm 1; \\ \displaystyle\sum_{k|n} \psi(k) \equiv 0 \pmod{n} \text{ for all } n \in \mathbb{N}. \end{cases} \tag{7}$$

Then (a_n) is a Dold sequence if and only if

$$\sum_{k|n} \psi\Big(\frac{n}{k}\Big) a_k \equiv 0 \tag{8}$$

modulo n for all $n \geqslant 1$.

Note that (7) is satisfied by the Möbius function, as the sum vanishes for $n > 1$ and is 1 for $n = 1$, and is satisfied by the Euler function as the sum in that case is equal to n.

REMARK 2.10. In the recent preprint [46] Graff *et al.* generalise the notion of Dold sequences to the setting of partially ordered sets. In this approach classical Dold sequences are the special case in which the partial order is given by the relation of divisibility. The Möbius function of a partially ordered set is an old concept, and became a central tool in combinatorics following Rota's seminal work [109]. We will mention later in Remark 3.7 how this plays a role in studying orbit growth for group actions.

2.4. *Polynomial Dold sequences*

The notion of Dold sequence has been generalised to sequences of polynomials by Gorodetsky [44]. For $n \in \mathbb{N}$, we define the polynomial $[n]_q$ in the variable q by

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1} \in \mathbb{Z}[q].$$

Clearly $\lim_{q\to 1}[n]_q = n$, so this may be thought of as the q-analogue of the positive integer n. Indeed, if $a > b$ with $a \equiv b$ modulo n, then

$$[a]_q = \frac{q^a - 1}{q - 1} = \frac{q^b(q^{a-b} - 1) + q^b - 1}{q - 1} \equiv \frac{q^b - 1}{q - 1} = [b]_q$$

modulo $[n]_q$ as a congruence in the polynomial ring $\mathbb{Z}[q]$.

DEFINITION 2.11. A sequence of polynomials $(a_n(q))$ with terms in $\mathbb{Z}[q]$ is said to satisfy the *q-Gauss congruences* (or *q-Dold congruences* in our terminology) if

$$\sum_{d|n} \mu(d) a_{n/d}(q^d) \equiv 0 \tag{9}$$

modulo $[n]_q$ in $\mathbb{Z}[q]$ for all $n \in \mathbb{N}$.

The fact that a given sequence $(a_n(q))$ satisfies the q-Dold congruences implies that the integer sequence $(a_n(1))$ satisfies the Dold congruences. As a consequence, some polynomial techniques are then able to be applied to find results about integer Dold sequences. In particular, tools like formal differentiation become available. We refer to work of Guo and Zudilin [55] and Straub [121] for more on this.

3. *Periodic points*

3.1. *Relating periodic points to closed orbits*

We will mostly be concerned with a 'dynamical system', which initially means simply a map $T\colon X \to X$ on a set X. The two fundamental notions we will deal with are periodic points and closed orbits.

Definition 3.1. Let $T\colon X \to X$ be a map.

(a) The *orbit* of a point $x \in X$ under T is the set

$$\mathscr{O}_T(x) = \{T^k x \mid k \in \mathbb{N}_0\}.$$

If $T^n x = x$ for some $n \geqslant 1$ then x is a *periodic point* and the set $\mathscr{O}_T(x)$ is a *closed orbit* of *length* $|\mathscr{O}_T(x)|$. We write $\mathsf{O}_T(n)$ for the number of closed orbits of length $n \geqslant 1$.

(b) If $T^k x = x$ for some $k \geqslant 1$ and $x \in X$, then x is a *periodic point*. The *least period* (or minimal period) of a periodic point x is $\min\{k \in \mathbb{N} \mid T^k x = x\}$. We write

$$\mathscr{F}_T(n) = \{x \in X \mid T^n x = x\}$$

for the set of points fixed by the nth iterate of T, and

$$\mathsf{F}_T(n) = |\mathscr{F}_T(n)|$$

for the number of points fixed by the nth iterate of T, or (equivalently) the number of points that are periodic with least period dividing n. We also write $\mathsf{L}_T(n)$ for the number of points in $\mathscr{F}_T(n)$ with least period n, so $n\mathsf{O}_T(n) = \mathsf{L}_T(n)$ for all $n \geqslant 1$.

We will always assume that $\mathsf{O}_T(n) < \infty$ for all $n \geqslant 1$, so that all the quantities in Definition 3.1 are finite. When we refer to a 'map' or a 'dynamical system', we make this assumption, and (X, T) always means such a system.

Clearly the sequences $(\mathsf{O}_T(n))_{n\geqslant 1}$ and $(\mathsf{F}_T(n))_{n\geqslant 1}$ determine each other.

3.2. *Dirichlet series and arithmetic functions*

Definition 3.2. The *Dirichlet convolution* of functions $f, g\colon \mathbb{N} \to \mathbb{C}$ is the function $f{*}g\colon \mathbb{N} \to \mathbb{C}$ defined by

$$(f{*}g)(n) = \sum_{d|n} f(d) g\Big(\frac{n}{d}\Big)$$

Convolution is commutative and associative, and the following properties of the Möbius function are easily checked.

Lemma 3.3. *Define the function* $\mathrm{I}\colon \mathbb{N} \to \mathbb{C}$ *by* $\mathrm{I}(1) = 1$ *and* $\mathrm{I}(n) = 0$ *for all* $n > 1$.

(a) $f{*}\mathrm{I} = \mathrm{I}{*}f = f$ *for any function* $f\colon \mathbb{N} \to \mathbb{C}$.

(b) *If* $f\colon \mathbb{N} \to \mathbb{C}$ *has* $f(1) \neq 0$, *then there is a unique function* $g\colon \mathbb{N} \to \mathbb{C}$ *such that* $f{*}g = \mathrm{I}$. *This function is denoted* f^{-1}.

(c) *The Möbius function is multiplicative, meaning that* $\mu(mn) = \mu(m)\mu(n)$ *if* $\gcd(m, n) = 1$.

(d) *The Möbius function satisfies*

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1; \\ 0 & \text{if } n > 1. \end{cases}$$

(e) *If* $u(n) = 1$ *for all* $n \in \mathbb{N}$, *then* $u^{-1} = \mu$.

(f) *For functions* $f, g\colon \mathbb{N} \to \mathbb{C}$ *the following statements are equivalent:*
  (i) $f(n) = \sum_{d|n} g(d)$ *for all* $n \in \mathbb{N}$;
  (ii) $g(n) = \sum_{d|n} f(d)\mu(\frac{n}{d})$ *for all* $n \in \mathbb{N}$.

Proof. The first statement (a) is clear from the definition of convolution.

The equation $(f{*}g)(1) = f(1)g(1)$ determines $g(1)$. Assuming that $g(k)$ has been determined for $1 \leqslant k < n$, the equation

$$(f{*}g)(n) = f(1)g(n) + \sum_{1<d|n} f(d)g\Big(\frac{n}{d}\Big)$$

determines $g(n)$ uniquely, showing (b).

Let m and n be integers with $\gcd(m,n) = 1$, and factorise m and n as products of prime powers. Clearly no prime can appear as a factor of both m and n. If any prime appears with exponent 2 or more, then both sides of the equation $\mu(mn) = \mu(m)\mu(n)$ are zero. If m (n) is a product of k (respectively, ℓ) distinct primes, then mn is a product of $k+\ell$ primes, since m and n are coprime. Thus $\mu(m) = (-1)^k$ and $\mu(n) = (-1)^\ell$, so

$$\mu(mn) = (-1)^{k+\ell} = \mu(m)\mu(n),$$

showing (c).

The right-hand side of (d) is multiplicative. We claim that the left-hand side is also multiplicative. If m and n are coprime, then

$$\sum_{d|mn} \mu(d) = \sum_{d_1|m}\sum_{d_2|n} \mu(d_1 d_2) = \sum_{d_1|m} \mu(d_1) \sum_{d_2|n} \mu(d_2)$$

since $\gcd(d_1, d_2) = 1$ for $d_1 \mid m$ and $d_2 \mid n$, and μ is multiplicative. Since both sides of the claimed identity are multiplicative, it is enough to verify it when $n = p^r$, where we have

$$\sum_{d|p^r} \mu(d) = \mu(1) + \mu(p) = 0,$$

showing (d). The statement in (e) is simply the formula in (d) again.

The claim in (f) is usually called the *Möbius inversion formula*. We may write

$$f(n) = \sum_{d|n} g(d)$$

as the convolution identity $f = g{*}u$. By associativity of convolution and (e), we therefore have $f{*}\mu = g{*}u{*}\mu = g{*}\,\mathrm{I} = g$, so

$$g(n) = \sum_{d|n} f(d)\mu\Big(\frac{n}{d}\Big).$$

The converse direction is similar: convolve $g = f{*}\mu$ with u. □

The following relationship between periodic points and closed orbits is highlighted in Smale's influential survey [**116**, Section I.4].

Lemma 3.4. *Let* $T\colon X \to X$ *be a map. Then*

$$\mathsf{F}_T(n) = \sum_{d|n} d\mathsf{O}_T(d) \tag{10}$$

and so

$$\mathsf{O}_T(n)=\frac{1}{n}\sum_{d|n}\mu\Big(\frac{n}{d}\Big)\mathsf{F}_T(d). \tag{11}$$

Proof. Any point fixed by T^n must live on a closed orbit of length d for some d dividing n, so $\mathscr{F}_T(n)$ is the disjoint union of these closed orbits, and the number of points that live on closed orbits of length d is $d\mathsf{O}_T(d)$, showing (10). The equivalence of (10) and (11) follows at once from Lemma 3.3(f). □

Definition 3.5. The (Artin–Mazur or dynamical) *zeta function* ζ_T is the formal power series defined by

$$\zeta_T(z)=\exp\left(\sum_{n\geqslant 1}\frac{z^n}{n}\mathsf{F}_T(n)\right). \tag{12}$$

The *dynamical Dirichlet series* d_T is defined by

$$\mathsf{d}_T(s)=\sum_{n\geqslant 1}\frac{\mathsf{O}_T(n)}{n^s}. \tag{13}$$

Some remarks will help to familiarise these formal definitions.

(a) The basic relation (10) or, equivalently, (11), may be expressed just as in Theorem 2.5 in terms of these two generating functions by thinking of the collection of all closed orbits for a system (X,T) as a disjoint union of individual orbits, to give the identity

$$\zeta_T(z)=\prod_{n\geqslant 1}(1-z^n)^{-\mathsf{O}_T(n)}=\prod_{\tau}\left(1-z^{|\tau|}\right)^{-1}, \tag{14}$$

where the product is taken over all closed orbits τ of T, and the resulting identity

$$\mathsf{d}_T(s)\boldsymbol{\zeta}(s+1)=\sum_{n\geqslant 1}\frac{\mathsf{F}_T(n)}{n^{s+1}}, \tag{15}$$

where $\boldsymbol{\zeta}$ is the classical (Riemann) zeta function.

(b) If there is an exponential bound of the form

$$\limsup_{n\to\infty}\mathsf{F}_T(n)^{1/n}=R<\infty \tag{16}$$

then the dynamical zeta function is a convergent power series with radius of convergence $\rho(\zeta_T)=1/R$ by the Cauchy–Hadamard theorem.

(c) Write $s=\sigma+\mathrm{i}t$, and hence $|n^s|=n^\sigma$. If a Dirichlet series converges absolutely for $s_0=\sigma_0+\mathrm{i}t_0$, then it converges absolutely for all s with $\sigma>\sigma_0$. It follows that if $\mathsf{d}_T(s)$ does converge for some $s\in\mathbb{C}$ but does not converge for all $s\in\mathbb{C}$, then there is a real number $\sigma_{\text{abs}}=\sigma_{\text{abs}}(\mathsf{d}_T)$ (the 'abscissa of absolute convergence') with the property that $\mathsf{d}_T(s)$ converges absolutely if $\sigma>\sigma_{\text{abs}}$ but does not converge absolutely if $\sigma<\sigma_{\text{abs}}$. Thus we expect d_T to define a function on some right-half plane in $\mathbb{C}$ if there is a polynomial bound $\mathsf{O}_T(n)\leqslant An^B$ on the growth in closed orbits.

(d) A striking example to illustrate settings in which the dynamical Dirichlet series may be useful comes from the quadratic map $\alpha: x\mapsto 1-cx^2$ on the interval $[-1,1]$ at the Feigenbaum value $c=1.401155\cdots$. As pointed out by Ruelle [**110**], this map has dynamical zeta function

$$\zeta_\alpha(z)=\prod_{n=0}^{\infty}\left(1-z^{2^n}\right)^{-1}=\prod_{n=0}^{\infty}\left(1+z^{2^n}\right)^{n+1},$$

which satisfies the Mahler functional equation $\zeta(z^2) = (1-z)\zeta(z)$ and hence admits a natural boundary. In contrast, Everest *et al.* [**33**] point out that the dynamical Dirichlet series is given by $\mathsf{d}_\alpha(z) = 1/(1-2^{-z})$, with readily understood analytic behaviour.

REMARK 3.6. The third author's interest in the kind of questions discussed here to some extent began with an obscure error in a paper of England and Smith [**31**]. In order to show that an automorphism of a solenoid can have an irrational zeta function, they construct an explicit example and claim its zeta function is

$$\exp\left(4z + 4z^2 + \sum_{k=3}^{\infty}(7^k - 3^k)\frac{z^k}{k}\right). \tag{17}$$

Many years ago he was looking at the paper for other reasons, and noted that this could not be the zeta function of any map because $z \mapsto \frac{1-3z}{1-7z}$ is clearly a dynamical zeta function (corresponding to the sequence of coefficients $(7^n - 3^n)$, or counting the periodic points of the map dual to $x \mapsto \frac{7}{3}x$ on $\mathbb{Z}[\frac{1}{3}]$), and the basic Dold relation shows that it is not really possible to change finitely many terms of a sequence while preserving the property of counting periodic points for a map. Indeed, if $\zeta_T(z)$ is given by (17), then we would have $(\mathsf{F}_T(n)) = (4, 8, 316, 2320, 16\,564, 116\,920, \dots)$. Applying the Möbius transform (11) then gives

$$(\mathsf{O}_T(n)) = \left(4, 2, 104, 578, 3312, \tfrac{58\,300}{3}, \dots\right),$$

which is impossible. In fact irrational zeta functions are not only possible but common — indeed, in a certain sense, generic — for compact group automorphisms. We refer to work of Everest *et al.* [**34, 128, 129**] for more on this. Indeed, it seems that a typical compact group automorphism will admit a natural boundary for the zeta function; see Bell *et al.* [**11**] and the survey [**86**] for more on this. Bowen and Lanford [**15**] had earlier pointed out a large class of symbolic dynamical systems with irrational zeta functions by constructing uncountably many symbolic systems with distinct zeta functions, and showing that there are only countably many rational dynamical zeta functions.

REMARK 3.7. As mentioned in Remark 2.10, the fundamental congruence (1) can be interpreted in any partially ordered set. Miles and Ward [**87, 88**] used a similar extension of Lemma 3.4 for closed orbits of group actions and bounds on the values of the Möbius function on the lattice of finite-index subgroups of nilpotent groups to find asymptotics for orbit growth properties of some simple algebraic actions of abelian and nilpotent groups.

4. *Universal dynamical systems*

Periodic points in dynamics arise in several settings:

- In various classical mechanical settings, periodic orbits represent possible repetitive motions. For example, in the three body problem great efforts have gone into establishing the existence of infinitely many periodic orbits.
- More generally, the Arnold conjecture in symplectic geometry concerns how the additional rigidity of symplectic manifolds impacts on the number of closed orbits; we refer to Arnold's famous book of problems [**3**, 1972-33] for more on this important problem.
- As invariants of suitable notions of equivalence. For example, if $T\colon X \to X$ and $S\colon Y \to Y$ are homeomorphisms of compact metric spaces (topological dynamics) then a *topological conjugacy* between the systems (X, T) and (Y, S), meaning a homeomorphism $\phi\colon X \to Y$ with the property that $\phi \circ T = S \circ \phi$, implies that $\mathsf{F}_T(n) = \mathsf{F}_S(n)$ for all $n \in \mathbb{N}$.

• As indicators of complexity. For example, under certain natural conditions the logarithmic growth rate of $\mathsf{F}_T(n)$ is related to the topological entropy of T. We refer to the work of Katok [**66**] for foundational results of this sort.

• In a different direction, work of Markus and Meyer [**81**] shows that certain 'exotic' orbit growth patterns that arise in one-dimensional solenoids are a generic feature of high-dimensional Hamiltonian dynamical systems.

• As a product of the presence of other phenomena via closing lemmas, specification properties, and so on.

• As an insight into typical behaviour in various contexts. For example, Artin and Mazur [**9**] show there is a C^k-dense set of C^k-maps on a compact smooth manifold without boundary with an exponential bound on the growth in $\mathsf{F}(n)$ (a result later extended and strengthened by Kaloshin [**62**]).

• In contrast, Kaloshin [**63**] showed that arbitrarily fast growth in $\mathsf{F}(n)$ is Baire generic in the space of C^2 or smoother diffeomorphisms.

• Kaloshin and Hunt [**64**] showed that for a measure-theoretic notion of generic (prevalent) the growth is typically only a little faster than exponential, with a growth bound of the shape $\exp(Cn^{1+\delta})$.

• Counting periodic points in algebraic systems can lead to subtle Diophantine problems (we refer to work of Lind [**77**] on quasihyperbolic toral automorphisms and work of Everest *et al.* [**23, 32**], for example; very similar problems occur also for endomorphisms of algebraic groups (and related maps) in positive characteristic [**21, 22**]).

A less familiar question is to ask not for what is *typical* in various contexts but to ask for what is *possible*.

4.1. *Combinatorial maps*

DEFINITION 4.1. A sequence (a_n) with terms in $\mathbb{N}_0$ is called *realisable* if there is some map $T\colon X \to X$ with $a_n = \mathsf{F}_T(n)$ for all $n \geqslant 1$.

It is clear that (11) gives the only constraint on the periodic points of a bijection. That is, if (o_n) is any sequence with values in $\mathbb{N}_0$ then there is a bijection $T\colon X \to X$ with $\mathsf{O}_T(n) = o_n$ for all $n \in \mathbb{N}$, simply by defining X to be the disjoint union of o_n many closed orbits of length n for each $n \in \mathbb{N}$. Thus a sequence (a_n) is realisable if and only if

$$\frac{1}{n}\sum_{d|n}\mu\Big(\frac{n}{d}\Big)a_d \in \mathbb{N}_0 \tag{18}$$

for all $n \geqslant 1$. Note that this requires two conditions:

- (Non-negative) The sum over divisors must be non-negative.
- (Congruence) The sum over divisors must be divisible by n for each n.

Thus a realisable sequence is a Dold sequence in the sense of Definition 2.1, together with a sign condition on the signed linear combinations in (18).

Figure 1 illustrates this with (the start of) a construction of a map with the property that the number of closed orbits of length n is n for each $n \in \mathbb{N}$. The set X consists of the black dots and the map T is given by the cyclic motions illustrated by the arrows (and of course the figure is to be extended to the right in a similar way).

4.2. *Relating Dold and realisable sequences*

For specific classes of dynamical systems the relation between the Dold condition and realisability may be quite involved, but at the level of full generality the relationship is

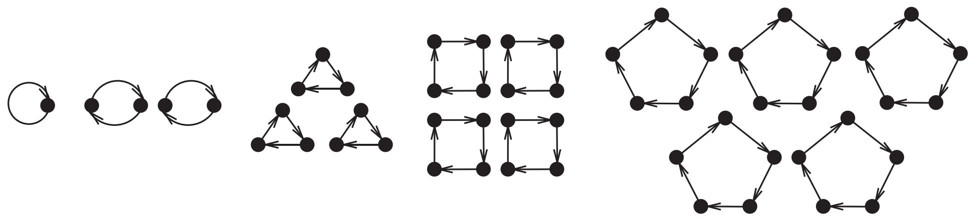

FIGURE 1. *Building a map with n closed orbits of length n for each n.*

straightforward — and explains the sense in which the realisable sequences may be viewed as a positive cone in the space of Dold sequences. The next result is shown in many places, including work of Du, Huang, and Li [28] and Arias de Reyna [2].

PROPOSITION 4.2. *The following conditions are equivalent for an integer sequence:*

(a) (a_n) *is a Dold sequence;*
(b) (a_n) *can be written as a difference of two realisable sequences; and*
(c) *for every prime p and $n, \ell \in \mathbb{N}$ we have $a_{np^\ell} \equiv a_{np^{\ell-1}}$ modulo p^ℓ.*

Proof. The defining relations (18) and (1) make it clear that (b) implies (a). Assume now that (a) holds, and let $b_n = \sum_{d|n} \mu(\frac{n}{d}) a_d$, so we know that $n \mid b_n$ for all $n \geqslant 1$. Let $b_n^\pm = \max(0, \pm b_n)$, so that b_n^+ and b_n^- are non-negative integers with

$$b_n = b_n^+ - b_n^-$$

for all $n \geqslant 1$. Correspondingly, define $a_n^\pm = \sum_{d|n} d b_d^\pm$. Then (a_n^+) and (a_n^-) are realisable by (18) and $a_n = a_n^+ - a_n^-$ for all $n \geqslant 1$.

To see that (b) implies (c), it is enough to show (c) for realisable sequences because the congruence condition is closed under subtraction of sequences. So assume that (a_n) is realisable by some system (X, T). To show (c) it is enough to prove that $a_{np^\ell} \equiv a_{np^{\ell-1}}$ modulo p^ℓ under the additional assumption that $p \nmid n$. By (10) we have

$$\mathsf{F}_T(p^\ell n) - \mathsf{F}_T(p^{\ell-1} n) = \sum_{d|p^\ell n} d\mathsf{O}_T(d) - \sum_{d|p^{\ell-1} n} d\mathsf{O}_T(d)$$

which must be divisible by p^ℓ as any d with $d \mid p^\ell n$ but $d \nmid p^{\ell-1} n$ has $p^\ell \mid d$, showing (c).

Finally, assume that (a_n) satisfies (c). Note that in order to show that (a_n) is a Dold sequence it is enough to prove that

$$\sum_{d|n} \mu\Big(\frac{n}{d}\Big) a_d \equiv 0 \tag{19}$$

modulo p^ℓ for any prime p with $p^\ell \parallel n$ (meaning that $p^\ell \mid n$ but $p^{\ell+1} \nmid n$). So assume that $p^\ell \parallel n$ and $n = p^\ell m$. Then in the sum on the left-hand side of (19) only the terms with $p^{\ell-1} \mid d$ do not vanish. Thus, by multiplicativity of the Möbius function, we get

$$\begin{aligned} \sum_{d|n} \mu\Big(\frac{n}{d}\Big) a_d &\equiv \sum_{d|m} \mu\Big(\frac{m}{d}\Big) a_{p^\ell d} - \sum_{d|m} \mu\Big(\frac{m}{d}\Big) a_{p^{\ell-1} d} \\ &\equiv \sum_{d|m} \mu\Big(\frac{m}{d}\Big) \big(a_{p^\ell d} - a_{p^{\ell-1} d}\big) \equiv 0 \end{aligned}$$

modulo p^ℓ, showing (a). □

Hence, for example, we can easily see from property (c) in Proposition 4.2 that the class of Dold sequences is closed under taking sums, differences, and products.

4.3. *Linear recurrence realisable sequences*

Linear recurrence sequences form a natural class of well-studied sequences (we refer to [**36**] for an overview of some of their properties and an extensive bibliography). The first natural direction in which to extend Fermat's little theorem is to integer matrices.

LEMMA 4.3. *If $A \in \mathrm{Mat}_{d,d}(\mathbb{N}_0)$ is a matrix with non-negative integer entries, then the sequence $(\mathrm{Tr}(A^n))$ is realisable.*

Proof. Consider a directed graph $\mathsf{G} = (V, E)$ comprising a vertex set

$$V = \{v_1, v_2, \ldots, v_d\}$$

and a set E of directed edges between elements of V with $a_{i,j} \in \mathbb{N}_0$ directed edges from the vertex v_i to the vertex v_j for i and j in $\{1, \ldots, d\}$. Writing $A = (a_{i,j})$ for the resulting adjacency matrix, the matrix $A^n = (a_{i,j}^{(n)})$ has (i, j)th entry $a_{i,j}^{(n)}$ equal to the number of walks in the graph G of length n starting at vertex v_i and ending at vertex v_j. In particular, $\mathrm{Tr}\, A^n$ is thus equal to the number of walks of length n that begin and end at the same vertex. If we let $X \subseteq E^{\mathbb{Z}}$ be the set of bi-infinite walks (e_i) in the graph G, then X is a compact set in the topology inherited from the Tychonoff topology on $E^{\mathbb{Z}}$, and the left shift map $\sigma \colon X \to X$ defined by $\sigma((e_i)) = (e_{i+1})$ is a homeomorphism. Moreover, $\mathsf{F}_\sigma(n) = \mathrm{Tr}\, A^n$ for all $n \geqslant 1$ since there is a one-to-one correspondence between sequences fixed by σ^n and walks of length n that start and end at the same vertex. □

COROLLARY 4.4. *Let B be a square matrix with integer entries. Then the sequence $(\mathrm{Tr}(B^n))$ is a Dold sequence.*

Proof. Fix n and choose a non-negative matrix A with $A \equiv B$ modulo n. Then the condition (1) for A holds by Lemma 4.3, and implies the condition (1) for B, since all the expressions involved only involve operations that commute with reduction modulo n. □

We can further specialise this result to deduce congruences of Euler–Fermat type.

COROLLARY 4.5. *Let A denote a square matrix with integer entries. Then*

$$\mathrm{Tr}(A^{p^r}) \equiv \mathrm{Tr}(A^{p^{r-1}})$$

modulo p^r for any prime p and $r \in \mathbb{N}$.

Proof. This is the Dold congruence condition (1) of the sequence $(\mathrm{Tr}(A^n))$ applied to n being a prime power. □

This kind of reasoning may be used to find non-trivial congruences; we refer to work of Arias de Reyna [**2**] and Sun [**122**], for example.

Minton's result in Theorem 2.4 gives a complete description of linear recurrence Dold sequences. This gives partial information about realisable linear recurrence sequences, but not complete information because of the sign conditions and possible rational multiplier. To see how this looks in a simple setting, we describe in Example 4.8 one of the early results in this direction, where a complete picture is readily found.

Sign conditions for linear recurrence sequences present difficulties in several directions. There are many ways to see that the absolute value of a linear recurrence sequence is not in general a linear recurrence; the next example was shown to us by Jason Bell.

EXAMPLE 4.6. Let $a_n = 1 + n\mathrm{i}$ for all $n \geqslant 1$, so that the sequence (a_n) satisfies the linear recurrence relation $a_{n+2} = 2a_{n+1} - a_n$ for all $n \geqslant 1$. Then we have $|a_n| = \sqrt{1+n^2}$, so the sequence $(|a_n|)$ cannot satisfy a linear recurrence relation because the terms of a linear recurrence sequence must lie in a finitely generated field extension of the rationals and the extension $\mathbb{Q}(\sqrt{1+n^2} \mid n \in \mathbb{N})/\mathbb{Q}$ is not finitely generated.

Clearly, if a sequence has constant sign, then taking its absolute value preserves the property of being a linear recurrence sequence. The same holds under the weaker assumption that the signs change periodically. As it turns out, for real-valued sequences this condition is also necessary.

LEMMA 4.7. *Let (a_n) be a linear recurrence sequence with real values. Then $(|a_n|)$ is also a linear recurrence sequence if and only if the sequence of signs $(\operatorname{sgn}(a_n))$ with values in $\{-1, 0, 1\}$ is ultimately periodic.*

Proof. Assume that $(|a_n|)$ is a linear recurrence, and consider the sets

$$J_- = \{n \in \mathbb{N} \mid a_n < 0\}, \qquad J = \{n \in \mathbb{N} \mid a_n = 0\}, \qquad J_+ = \{n \in \mathbb{N} \mid a_n > 0\}.$$

For $n \notin J$, the sequence $\operatorname{sgn}(a_n) = |a_n|/a_n$ can be written as a quotient of two linear recurrence sequences. Since this sequence takes values ± 1, we may apply the Hadamard quotient theorem [**36**, Theorem 4.4] to see that there is a linear recurrence sequence (b_n) with $\operatorname{sgn}(a_n) = b_n$ for all $n \notin J$. By the Skolem–Mahler–Lech theorem [**36**, Theorem 2.1], the sets J_-, J, and J_+ are unions of finite sets and finitely many arithmetic progressions. It follows that $(\operatorname{sgn}(a_n))$ is ultimately periodic □

The problem of determining if a given linear recurrence sequence of algebraic numbers has positive terms matters in multiple fields, and the decidability and complexity (in the sense of computer science) of this question has a long history. Mignotte *et al.* [**85**] and Vereshchagin [**125**] independently used Diophantine analysis to show the problem is decidable for linear recurrence sequences of degree no more than 4, and Ouaknine and Worrell [**99**] brought in additional tools from number theory and real algebraic geometry to prove decidability and establish the complexity of the question for linear recurrence sequences over the integers of degree up to 9, under the assumption that there are no repeated characteristic roots. We refer to their paper [**99**] for more of the history, and for additional references.

EXAMPLE 4.8 (Fibonacci recurrences). Consider the sequence

$$(1, a, 1+a, 1+2a, 2+3a, 3+5a, \dots)$$

for $a \in \mathbb{N}_0$. For which values of a is this realisable?

- If $a = 0$ the sequence begins $(1, 0, 1, 1, 2, 3, 5, \dots)$. If this is realised by some (X, T) then $\mathsf{O}_T(1) = 1$ (because there is one fixed point), which means that $\mathsf{F}_T(2) = \mathsf{O}_T(1) + 2\mathsf{O}_T(2) = 0$ is impossible.
- If $a = 1$ the sequence begins $(1, 1, 2, \dots)$ which is impossible as $\mathsf{F}_T(3) - \mathsf{F}_T(1)$ must be divisible by 3.

Theorem 2.4 shows that the space of all Dold solutions of this recurrence has rank one, which means the space of realisable solutions must have rank zero or one. In fact it has rank one, and there is a complete description. Of course this result is an easy consequence of Minton's Theorem 2.4, because in this case the sign issues are easy, but we include a proof to illuminate the sort of issues that come into play in a simple example.

Lemma 4.9 (Puri and Ward [**107**]). *The sequence*

$$u = (u_n) = (1, a, 1 + a, 1 + 2a, 2 + 3a, 3 + 5a, \dots) \tag{20}$$

is realisable if and only if $a = 3$.

Proof. Lemma 4.3 applied to the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

gives a system (X, T) with $\mathsf{F}_T(n) = L_n$ for $n \geqslant 1$, where L_n denotes the nth term of the Lucas sequence $(1, 3, 4, 7, \dots)$. Thus the sequence (20) is indeed realisable if $a = 3$.

For the converse direction, write $F = (F_n) = (1, 1, 2, 3, 5, 8, \dots)$ for the Fibonnaci numbers. Since (L_n) is realisable, we have

$$\sum_{d|n} \mu\Big(\frac{n}{d}\Big) L_d \equiv 0$$

modulo n for all $n \geqslant 1$. For $n = p$ a prime, it follows that

$$L_p = F_{p-2} + 3F_{p-1} \equiv L_1 = 1 \tag{21}$$

modulo p (the first equality is an easy induction). It follows from (21) that

$$F_{p-1} \equiv 1 \iff F_{p-2} \equiv -2 \tag{22}$$

modulo p for any prime p. The Fibonacci numbers satisfy $F_{p+1} \equiv 0$ modulo p if p is a prime congruent to ± 2 modulo 5 (see [**56**, Theorem 180]). Let p be any such prime. The resulting identities $F_{p+1} = F_p + F_{p-1} = 2F_{p-1} + F_{p-2} \equiv 0$ modulo p and (21) together imply that $F_{p-1} \equiv 1$ modulo p if $p \equiv \pm 2$ modulo 5.

Assume now that (20) is realisable. Then $u_p \equiv u_1 = 1$ modulo p for any prime p. Now $u_n = F_{n-2} + aF_{n-1}$ for $n \geqslant 1$, so $F_{p-2} + aF_{p-1} \equiv 1$ modulo p. If $p \equiv \pm 2$ modulo 5 we thus have

$$(F_{p-2} - 1) + a \equiv 0 \tag{23}$$

modulo p, since $F_{p-1} \equiv 1$. For such p, (22) gives $F_{p-2} \equiv -2$ modulo p, so (23) gives $a \equiv 3$ modulo p. By Dirichlet's theorem there are infinitely many primes p congruent to ± 2 modulo 5, so $a = 3$ as claimed. □

Extending this line of reasoning gives the following picture. For a given (homogeneous) linear recurrence relation there will be some initial conditions that give realisable sequences. There is always one such sequence because $(0, 0, 0, \dots)$ is realisable. Since any natural number multiple of a realisable sequence is also realisable (by taking disjoint unions of a given system with itself), we can ask for the 'rank' or 'dimension' of the realisable subspace, meaning the number of linearly independent realisable solutions. Lemma 4.9 shows that there is a one-dimensional subset of realisable sequences satisfying the Fibonacci recurrence relation.

A more involved argument along the same lines using special primes gives the following result, originally shown by Puri [**106**] using other methods. Once again Theorem 2.4 gives an upper bound on the dimensions, but other arguments are needed to give exact values or lower bounds.

THEOREM 4.10 (Everest *et al.* [**35**, Theorem 2.1]). *Let Δ denote the discriminant of the characteristic polynomial of a non-degenerate binary recurrence relation. Then the realisable subspace has*

(a) *dimension* 0 *if* $\Delta < 0$*;*
(b) *dimension* 1 *if* $\Delta = 0$ *or* $\Delta > 0$ *and non-square;*
(c) *dimension* 2 *if* $\Delta > 0$ *is a square.*

The fact that realisable sequences can be multiplied by natural numbers and added while remaining realisable gives the following illustration of case (c): any sequence $u_n = a2^n + b$ with $a, b \in \mathbb{N}$ is realisable and satisfies the recurrence relation $u_{n+2} = 3u_{n+1} - 2u_n$ for all $n \in \mathbb{N}$.

The proofs of the definitive Lemma 4.9 and Theorem 4.10 rely on special properties of quadratics. The general picture in higher degree is much less clear, and new phenomena arise. A partial result with a slightly more permissive definition is found by Everest *et al.*

THEOREM 4.11 (Everest *et al.* [**35**, Theorem 2.3]). *Let f be the characteristic polynomial of a non-degenerate linear recurrence sequence with integer coefficients. If f is separable with ℓ irreducible factors and a dominant root, then the dimension of the subspace of solutions of the recurrence with the property that their absolute value is realisable cannot be more than ℓ. If $f(0) \neq 0$ then equality holds if either the dominant root is not less than the sum of the absolute values of the other roots or the dominant root is strictly greater than the sum of the absolute values of its conjugates.*

Extending this to all linear recurrences, and dealing with the sign conditions, remains open apart from the general result of Minton in Theorem 2.4, which gives an upper bound of the possible dimensions for the original problem of the realisability of the terms of a linear recurrence as opposed to the absolute value of the terms as considered here. The Fibonacci case above only really used the congruence condition in (18), but the general case also engages the non-negativity condition, which can be difficult to work with. A simple example of a dominant root cubic is given by the so-called 'Tribonacci' relation.

EXAMPLE 4.12. The sequence $(3, 1, 3, 7, 11, 21, 39, \dots)$ satisfying

$$u_{n+3} = u_{n+2} + u_{n+1} + u_n$$

is realisable, since its nth term is $(\mathrm{Tr}(A^n))$, where A is the companion matrix to the polynomial $x^3 - x^2 - x - 1$. Theorem 4.11 says that any realisable sequence satisfying the same cubic recurrence must be a multiple of this one.

In a rather different direction, work of Kim, Ormes, and Roush [**69**] on the Spectral Conjecture of Boyle and Handelman [**16**] gives a checkable criterion for a given linear recurrence sequence to be realised by an irreducible subshift of finite type. Bertrand-Mathis [**12**] discussed related questions from a language-theoretic point of view.

The original proofs of Theorems 4.10 and 4.11 again ultimately involve knowing there are infinitely many primes satisfying certain conditions. For the quadratic case the argument rests on inert primes. The higher degree argument in [**35**] also uses the result of Vinogradov [**127**] (which says that the sequence of fractional parts of $p\beta$ as p runs through the primes is dense in (0,1) for irrational β) to ensure there are infinitely many primes satisfying a sign condition needed for the argument.

4.4. *Time-changes preserving realisability*

The two viewpoints on realisable sequences — on the one hand, being defined by a dynamical system (X, T) and, on the other, being a sequence of non-negative integers satisfying the combinatorial condition (4.15) — raises questions about 'functorial' properties. That is, how to interpret operations that are natural in one setting in the other setting. This idea is explored further in work of Pakapongpun *et al.* [**100, 101, 108**]. The very simplest of these, for example, is to note that changing T to T^2 changes the sequence $(\mathsf{F}_T(n))$ to the sequence $(\mathsf{F}_T(2n))$. We can interpret this observation by saying that the map $n \mapsto 2n$ for $n \in \mathbb{N}$ is a realisability-preserving time-change in the following sense.

DEFINITION 4.13 (Jaidee *et al.* [**58**]). For a system (X, T) define $\mathscr{P}_T$, the set of *realisability-preserving time-changes for* (X, T), to be the set of maps $h\colon \mathbb{N} \to \mathbb{N}$ with the property that $(\mathsf{F}_T(h(n)))$ is a realisable sequence. Also define the monoid of *universally realisability-preserving time-changes*, $\mathscr{P} = \bigcap_{\{(X,T)\}} \mathscr{P}_T$, where the intersection is taken over all systems (X, T).

Not only is $\mathscr{P}$ a monoid — closed under composition — it is in fact closed under infinite composition in the following sense. If $(h_1, h_2, \dots)$ is a sequence of functions in $\mathscr{P}$ with the property that the sequence

$$(h_1(n), h_2(h_1(n)), h_3(h_2(h_1(n))), \dots) \tag{24}$$

stabilises for every $n \in \mathbb{N}$, then the infinite composition $h = \cdots \circ h_3 \circ h_2 \circ h_1$, defined by setting $h(n)$ to be the stabilised value or limit of (24), is also in $\mathscr{P}$.

THEOREM 4.14 (Jaidee *et al.* [**58**]). *The monoid $\mathscr{P}$ has the following properties.*

(a) *A polynomial lies in $\mathscr{P}$ if and only if it is a monomial.*
(b) *The monoid $\mathscr{P}$ is uncountable.*

The fact that monomials lie in $\mathscr{P}$ was shown first in Moss' thesis [**92**], and of course means that if a sequence (a_n) satisfies the Dold congruence, then for any $k, \ell \in \mathbb{N}$ the sequence $(a_{\ell n^k})$ also does. The proof that $\mathscr{P}$ is uncountable relies on a simpler remark also due to Moss: For a given prime p, the map $g_p\colon \mathbb{N} \to \mathbb{N}$ defined by

$$g_p(n) = \begin{cases} n & \text{if } p \nmid n; \\ pn & \text{if } p \mid n \end{cases}$$

lies in $\mathscr{P}$. Using the closure under infinite composition property mentioned above, this allows us to embed the power set of the set of primes into $\mathscr{P}$. One of the questions raised in [**58**] asked if a non-trivial permutation could lie in $\mathscr{P}$, and we answer this here, in a stronger form.

THEOREM 4.15. *The only surjective map $\mathbb{N} \to \mathbb{N}$ in $\mathscr{P}$ is the identity.*

Proof. Let $\sigma\colon \mathbb{N} \to \mathbb{N}$ be a surjective map, assume that $\sigma \in \mathscr{P}$, and fix $k \in \mathbb{N}$. Applying σ to reg_k gives the sequence $(\mathsf{reg}_k(\sigma(n)))$, which by definition of reg_k can only take on the values 0 and k, and by surjectivity of σ must take on the value k. By thinking about the possible closed orbits (or from the Dold congruence), we know that a realisable sequence only taking on a single non-zero value k is necessarily an integer multiple of the sequence reg_d for some d dividing k. If $d(k)$ is the smallest integer with the property that $k \mid \sigma(d(k))$, it follows that $d(k) \mid k$ and $k \mid \sigma(n)$ if and only if $d(k) \mid n$ for any $n \in \mathbb{N}$. That is,

$$\sigma^{-1}(k\mathbb{N}) = d(k)\mathbb{N}. \tag{25}$$

Let $s = \Omega(k)$ be the number of prime divisors of k counted with multiplicity, and define the divisors $m_0, m_1, \dots, m_s$ of k so that $m_0 = 1$ and $\frac{m_{j+1}}{m_j}$ is a prime for $j = 1, \dots, s-1$. Then there are strict inclusions

$$m_0\mathbb{N} \supsetneq m_1\mathbb{N} \supsetneq m_2\mathbb{N} \supsetneq \cdots \supsetneq m_s\mathbb{N} = k\mathbb{N}.$$

Since σ is surjective, applying σ^{-1} gives strict inclusions

$$d(m_0)\mathbb{N} \supsetneq d(m_1)\mathbb{N} \supseteq d(m_2)\mathbb{N} \supsetneq \cdots \supsetneq d(m_s)\mathbb{N} = d(k)\mathbb{N} \tag{26}$$

by (25). If $d(k)$ is a proper divisor of k then $\Omega(d(k)) < \Omega(k)$, contradicting (26). It follows that $d(k) = k$ for all $k \in \mathbb{N}$. Now the definition of $d(k)$ shows that $k \mid \sigma(n)$ if and only if $k \mid n$ for any $k, n \in \mathbb{N}$. Applying this with $k = n$ and $k = \sigma(n)$ shows that $\sigma(n) = n$ for all $n \in \mathbb{N}$. □

4.5. *Repair phenomenon*

As remarked earlier, the Fibonacci sequence (F_n) is not realisable. Moss and Ward [**93**] strengthened this observation, and showed that the Fibonacci sequence sampled along the squares can be 'repaired' to be realisable in the following sense.

THEOREM 4.16 (Moss and Ward [**93**]). *If j is odd, then the set of primes dividing denominators of $\frac{1}{n}\sum_{d|n}\mu(\frac{n}{d})F_{d^j}$ for $n \in \mathbb{N}$ is infinite. If j is even, then the sequence (F_{n^j}) is not realisable, but the sequence $(5F_{n^j})$ is.*

This motivated the following definition.

DEFINITION 4.17 (Miska and Ward [**90**]). For a sequence A of non-negative integers,

$$\operatorname{Fail}(A) = \begin{cases} \operatorname{lcm}\left(\{\operatorname{Denom}\big((\frac{1}{n}\mu * A)(n)\big) \mid n \geqslant 1\}\right) & \text{if this is finite;} \\ \infty & \text{if not.} \end{cases}$$

The sequence A is said to be *almost realisable* if $\operatorname{Fail}(A) < \infty$ and it satisfies the sign condition.

Thus the result of [**93**] states in particular that (F_n) is not almost realisable, but (F_{n^2}) is almost realisable with $\operatorname{Fail}((F_{n^2})) = 5$. This is part of a wider phenomenon in which a Lucas sequence sampled along the squares becomes realisable if it is multiplied by its discriminant.

There is another combinatorial setting where a similar almost realisability occurs. Write $S^{(1)}(n, k)$ for the (signless) Stirling numbers of the first kind, defined for any $n \geqslant 1$ and $0 \leqslant k \leqslant n$ to be the number of permutations of $\{1, \dots, n\}$ with exactly k cycles, and write $S^{(2)}(n, k)$ for $n \geqslant 1$ and $1 \leqslant k \leqslant n$ for the Stirling numbers of the second kind, so $S^{(2)}(n, k)$ counts the number of ways to partition a set comprising n elements into k non-empty subsets.

THEOREM 4.18 (Miska and Ward [**90**]). *For each $k \geqslant 1$ define sequences*

$$S_k^{(1)} = \left(S^{(1)}(n + k - 1, k)\right)_{n \geqslant 1}$$

and

$$S_k^{(2)} = \left(S^{(2)}(n + k - 1, k)\right)_{n \geqslant 1}.$$

Then:

(a) *For $k \geqslant 1$ the sequence $S_k^{(1)}$ is not almost realisable.*
(b) *For $k \leqslant 2$ the sequence $S_k^{(2)}$ is realisable.*

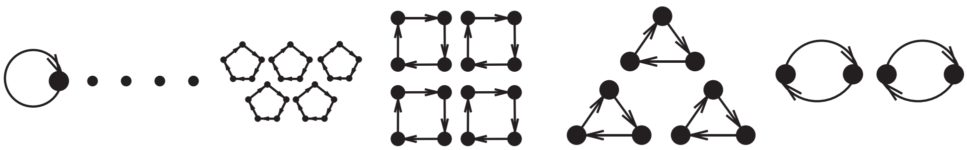

FIGURE 2. *Building a continuous map with n closed orbits of length n for each n.*

(c) *For $k \geqslant 3$ the sequence $S_k^{(2)}$ is not realisable, but is almost realisable with* $\mathrm{Fail}(S_k^{(2)}) \mid (k-1)!$ *for all $k \geqslant 1$.*

Computing the repair factor $\mathrm{Fail}(S_k^{(2)})$ is problematic, because *a priori* it involves an unbounded calculation. Calculations suggest that

$$\mathrm{Fail}\Big(S_k^{(2)}\Big) = \mathrm{lcm}\left(\{\mathrm{Denom}\big((\tfrac{1}{n}\mu * A)(n)\big) \mid 1 \leqslant n \leqslant k^\sharp\}\right)$$

where $k^\sharp$ is the largest prime power less than k, but this is not proved. What little is known about the sequence

$$\left(\mathrm{Fail}\Big(S_k^{(2)}\Big)\right)_{k\geqslant 1} = (1, 1, 2, 6, 12, 60, 30, 210, 840, 2520, 1260, \dots)$$

is recorded in the entries A341617 and A341991 of the Online Encyclopedia of Integer Sequences [**98**].

4.6. *Continuous maps*

A simple observation from [**108**] is that an arbitrary sequence of closed orbit counts (o_n), when viewed as a set, can be given a compact topology for which the map is a homeomorphism. This is easiest to see if there is a fixed point as this can be placed 'at infinity'. Returning to Figure 1, we can simply locate the fixed point at the origin, squeeze the closed orbits into some bounded subset of $\mathbb{R}^2$, and define the compact space to be the union of the orbits as illustrated in Figure 2, to produce a continuous map with (in this case) n closed orbits of length n for each $n \in \mathbb{N}$.

LEMMA 4.19. *A sequence (a_n) with $a_n \in \mathbb{N}_0$ is realised by a homeomorphism of a compact metric space if and only if it satisfies (18).*

Even at this level of generality one has to respect topological constraints. The spaces constructed above are zero-dimensional, which permits the extreme flexibility in the number of orbits. On a specific topological space there are additional conditions forced onto the sequence by the global topology, which will be discussed further in Section 6. Two examples of this will illustrate some of what may arise.

EXAMPLE 4.20. A continuous map on a disk must have a fixed point by Brouwer's theorem.

EXAMPLE 4.21. A continuous map $f\colon [0,1] \to [0,1]$ must obey Šarkovs'kiĭ's theorem [**114**]. This shows there is a total order $\prec$ on $\mathbb{N}$ with the property that if $m \prec n$ then the existence of a point of minimal period m for f implies the existence of a point of minimal period n for f, and that this order has the following shape:

$$3 \prec 5 \prec 7 \prec \cdots \prec 3\cdot 2 \prec 5\cdot 2 \prec \cdots \prec 3\cdot 2^2 \prec 5\cdot 2^2 \prec \cdots \prec 2^3 \prec 2^2 \prec 2 \prec 1.$$

4.7. *Smooth maps*

It is less clear how to create smooth models. The following result requires a delicate construction, and this was done by Windsor.

THEOREM 4.22 (Windsor [**131**]). *A sequence* (a_n) *with* $a_n \in \mathbb{N}_0$ *is realised by a* C^∞ *map of the* 2*-torus if and only if it satisfies* (18).

The start of this construction is straighforward — to build closed orbits of given length using rotations. The challenge is to ensure that no new orbits are created in the limiting processes involved.

OPEN PROBLEM 4.23 (Smooth models in general). Is it true that for every smooth manifold M of dimension at least 2 a sequence (a_n) with $a_n \in \mathbb{N}_0$ is realised by a C^∞ map of M if and only if it satisfies (18) and the consequences of the Lefschetz fixed-point theorem?

5. *Lefschetz sequences*

DEFINITION 5.1. A sequence of integers (a_n) is a *Lefschetz sequence* if there exist square integer matrices A and B such that

$$a_n = \operatorname{Tr} A^n - \operatorname{Tr} B^n$$

for all $n \in \mathbb{N}$.

Expressing the trace in terms of the eigenvalues of the matrices gives the following characterisation.

PROPOSITION 5.2. *A sequence of integers* (a_n) *is a Lefschetz sequence if and only if there exist algebraic numbers* $\lambda_1, \dots, \lambda_s$ *and integers* $m_1, \dots, m_s$ *such that*

$$a_n = \sum_{i=1}^{s} m_i \lambda_i^n$$

for all $n \in \mathbb{N}$.

It is clear from Corollary 4.4 that a Lefschetz sequence is a Dold sequence.

We recall that a sequence (c_n) (taking values in a field) is a *linear recurrence sequence* if there exists an integer p (the *order* of the recurrence) and constants

$$\alpha_0, \dots, \alpha_{p-1}$$

with $\alpha_0 \neq 0$ such that

$$c_{n+p} = \alpha_{p-1} c_{n+p-1} + \cdots + \alpha_0 c_n$$

for all $n \geqslant 1$. In applications, it may happen that the recurrence is only satisfied for $n \geqslant n_0$, in which case the sequence nevertheless satisfies a linear recurrence with $p' = p + n_0 - 1$, $\alpha'_i = \alpha_{i-n_0}$ for $i \geqslant n_0$ and $\alpha'_i = 0$ otherwise. The following lemma is well known (it may be found, for example, in the monograph of Salem [**112**]; we also refer to [**36**] for results of this sort and an extensive guide to the literature on recurrence sequences).

Lemma 5.3. *A power series*

$$f(z)=\sum_{n=0}^{\infty} c_n z^n \tag{27}$$

with coefficients in a field $\mathbb{K}$ *represents a rational function* $\frac{P(z)}{Q(z)}$ *(where* P *and* Q *are polynomials) if and only if its coefficients satisfy a linear recurrence relation. Furthermore, the order of the recurrence is at most* $\max(\deg P, \deg Q)$.

It is clear that Lefschetz sequences constitute a proper subset of the set of all Dold sequences. There are many ways to see this, including the following.

- **Construction:** There are dynamical systems (X,T) with the property that $\sum_{n=1}^{\infty} \mathsf{F}_T(n)z^n$ is far from rational, each of which gives a Dold sequence that is not Lefschetz.
- **Symmetry:** As discussed in Section 4.4, there is a notion of 'symmetry' in the space of all zeta functions that shows, for example, if (a_n) is a realisable sequence then (a_{n^2}) is also a realisable sequence. These are 'symmetries' of the space of realisable (and hence of Dold) sequences that clearly do not preserve the Lefschetz property (see Jaidee *et al.* [**58**]).
- **Cardinality:** There are only countably many sequences that arise as the difference of the traces of powers of integer matrices, but there are uncountably many Dold sequences.

Theorem 5.4. *A sequence of integers* (a_n) *is a Lefschetz sequence if and only if* (a_n) *is a Dold sequence with the property that its generating sequence* (c_n) *is a linear recurrence sequence.*

Proof. Assume first that (a_n) is a Lefschetz sequence (and hence a Dold sequence), so that we may write $a_n=\sum_{i=1}^{s} m_i\lambda_i^n$ for all $n\geqslant 1$. From Theorem 2.5 we can calculate

$$\begin{aligned}\left(1-\sum_{n\geqslant 1} c_n z^n\right) &= \exp\left(-\sum_{n\geqslant 1}\frac{a_n}{n}z^n\right)=\prod_{i=1}^{s}\exp\left(-m_i\sum_{n\geqslant 1}\frac{\lambda_i^n}{n}z^n\right)\\ &=\prod_{i=1}^{s}\exp\left(m_i\log(1-\lambda_i z)\right)=\prod_{i=1}^{s}(1-\lambda_i z)^{m_i},\end{aligned}$$

where the last three equalities are formal. We deduce that the generating function $\sum_{n\geqslant 1} c_n z^n$ is rational, so by Lemma 5.3 the sequence (c_n) satisfies a linear recurrence relation.

This reasoning can be inverted: if the generating sequence (c_n) is linear recurrent, then by Theorem 5.3 the series $1-\sum_{n\geqslant 1} c_n z^n$ represents a rational function $\frac{P(z)}{Q(z)}$ (where P and Q are polynomials with rational coefficients), and since its constant term is 1, we may factorise it over the complex numbers as

$$1-\sum_{n\geqslant 1} c_n z^n=\prod_{i=1}^{s}(1-\lambda_i z)^{m_i}$$

for complex numbers λ_i and integers m_i. Repeating the previous formal computations gives $a_n=\sum_{i=1}^{s} m_i\lambda_i^n$. Since the numbers λ_i arose as roots of P and Q, they are algebraic numbers, and hence (a_n) is a Lefschetz sequence by Proposition 5.2. □

5.1. *Generating sequences for Lefschetz sequences*

Generating sequences of Lefschetz numbers were considered by Graff *et al.* [**52**], and a certain characterisation of generating sequences of Lefschetz numbers of iterations was given. Here we will describe a simple computational criterion for verifying that a given sequence (c_n) is not a generating sequence of a Lefschetz sequence (provided the dimensions of respective matrices are bounded from above). We start by recalling a well-known test for rationality in terms of the *Kronecker–Hankel determinants*. We refer to Salem [**112**] or Koblitz [**70**] for convenient proofs.

LEMMA 5.5. *A power series $\sum_{n\geqslant 0} c_n x^n$ represents a rational function if and only if the Kronecker–Hankel determinants*

$$\Delta_m = \det \begin{pmatrix} c_0 & c_1 & \dots & c_m \\ c_1 & c_2 & \dots & c_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_m & c_{m+1} & \dots & c_{2m} \end{pmatrix} \tag{28}$$

are all zero for m large enough. In fact (c_n) satisfies a linear recurrence of order p if and only if Δ_m vanishes for $m \geqslant p$.

REMARK 5.6. The determinant (28) in Lemma 5.5 is an example of a Hankel determinant; that is, a determinant of a square matrix in which each ascending anti-diagonal is constant.

Proof of Lemma 5.5. We will only use — and so will only prove — one implication, namely the fact that if the series is rational then its Hankel determinants of sufficiently high order must vanish. The relation

$$c_{n+p} = \alpha_{p-1} c_{n+p-1} + \dots + \alpha_0 c_n$$

shows that the $(p+1)$-st column of the Hankel matrix is a linear combination of the preceding p columns. □

THEOREM 5.7. *Let (a_n) be a sequence of Lefschetz numbers of the form*

$$a_n = \operatorname{Tr} A^n - \operatorname{Tr} B^n$$

for all $n \in \mathbb{N}$, where $A \in \mathrm{Mat}_{k,k}(\mathbb{C})$ and $B \in \mathrm{Mat}_{\ell,\ell}(\mathbb{C})$. If (c_n) is the generating sequence of the sequence (a_n), then the determinant of a Hankel matrix Δ_m vanishes for $m \geqslant \max(k, \ell)$.

Proof. Let λ_i for $1 \leqslant i \leqslant k$ and μ_j for $1 \leqslant j \leqslant \ell$ be the eigenvalues of A and B, respectively (with multiplicities). Computing as in the proof of Theorem 5.4, we obtain

$$\exp\left(-\sum_{n=1}^{\infty} a_n \frac{x^n}{n}\right) = \frac{\prod_{j=1}^{\ell}(1-\mu_j z)}{\prod_{i=1}^{k}(1-\lambda_i z)}.$$

By Lemma 5.3, the generating sequence satisfies a linear recurrence of order no more than $\max(k, \ell)$, and hence, by Lemma 5.5, the corresponding Hankel determinants Δ_m vanish for $m \geqslant \max(k, \ell)$. □

EXAMPLE 5.8. The sequence $(1, 3, 2, 4, 5, 7, 6, 8, 9, \dots)$ is not a generating sequence of a Lefschetz sequence obtained from matrices of dimensions not more than 4 since

$$\det \begin{pmatrix} 1 & 3 & 2 & 4 & 5 \\ 3 & 2 & 4 & 5 & 7 \\ 2 & 4 & 5 & 7 & 6 \\ 4 & 5 & 7 & 6 & 8 \\ 5 & 7 & 6 & 8 & 9 \end{pmatrix} = -256 \neq 0.$$

LEMMA 5.9. *Let (a_n) be a sequence of integers. Then the following conditions are equivalent:*

(a) *The sequence (a_n) is a periodic Lefschetz sequence.*
(b) *The sequence (a_n) is a bounded Dold sequence.*

Proof. It is clear that a periodic Lefschetz sequence is a bounded Dold sequence. By Lemma 2.8, a bounded Dold sequence is a sum of integer multiples of elementary periodic sequences reg_k. Any such sequence reg_k can be written as the sequence of traces of powers of matrices of the form

$$M = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}_{k\times k} \tag{29}$$

since $\operatorname{Tr} M^n = \mathsf{reg}_k(n)$. Thus any bounded Dold sequence is a Lefschetz sequence. □

5.2. *Asymptotic properties*

One may pose several questions concerning possible growth rates of sequences that lie in more than one of the various classes. For example, what are the possible rates of growth of realisable sequences that are linear recurrence sequences? The relationship between growth and arithmetic properties in a topological setting for Lefschetz sequences is explored in many places. We cite below an interesting alternative due to Babenko and Bogatyĭ (we also refer to the monograph of Jezierski and Marzantowicz [**59**] for results in this direction).

THEOREM 5.10 (Babenko and Bogatyĭ [**10**]). *Let f be a map of a space with finitely-generated real homology spaces (for example, a map of a compact Euclidean neighbourhood retract) and write $\mathrm{sp}_{\mathrm{es}}$ for the essential spectral radius of the induced map in real homology. Then exactly one of the following three possibilities holds.*

(a) *$L(f^n) = 0$ for $m = 1, 2, \dots,$ which happens if and only if $\mathrm{sp}_{\mathrm{es}}(f) = 0$.*
(b) *The sequence $(\frac{L(f^n)}{\mathrm{sp}_{\mathrm{es}}(f)^n})$ has the same set of limit points as a periodic sequence of the form $(\sum_i \alpha_i \varepsilon_i^n)$, where $\alpha_i \in \mathbb{Z}, \varepsilon_i \in \mathbb{C}$, and $\varepsilon_i^k = 1$ for some $k \in \mathbb{N}$.*
(c) *The set of limit points of the sequence $(\frac{|L(f^n)|}{\mathrm{sp}_{\mathrm{es}}(f)^n})$ contains an interval.*

6. *Topological invariants of iterated maps as Dold sequences: topological and dynamical consequences*

In this section we describe some dynamical and topological consequences of the Dold congruences. In particular, we show that they are valid for many different topological invariants, and show how the congruences may be transferred into information about the dynamical properties of maps, or about the structure of periodic points.

6.1. *Fixed-point indices and applications*

Fixed-point indices of iterations were the original motivation for the definition of Dold sequences. We first recall their definition and then indicate some consequences of the Dold congruences.

Consider a Euclidean neighbourhood retract Y and a continuous map

$$f\colon V \longrightarrow Y,$$

where $V \subseteq Y$ is an open subset, and assume that $\mathscr{F}_f(1) \subseteq V$ is compact. Then there is a well-defined *fixed-point index* $\operatorname{ind}(f) = \operatorname{ind}(f, V) \in \mathbb{Z}$, which is a topological invariant. We refer to the monograph of Jezierski and Marzantowicz [**59**, Section 2.2] for the formal definition.

Definition 6.1 (Dold [**27**]). We define the iterations $f^n\colon V_n \to Y$ for V_n defined as follows. We first set $V_1 = V$, and then inductively define

$$V_n = f^{-1}(V_{n-1})$$

for $n > 1$. Under the assumption that $\mathscr{F}_f(n)$ is compact, the fixed-point index $\operatorname{ind}(f^n) = \operatorname{ind}(f^n, V_n)$ is a well-defined integer for each $n \in \mathbb{N}$.

Theorem 6.2 was shown by Dold, although it was known in certain cases earlier. In the terminology we have adopted (which of course is not that used by Dold) we have the following result.

Theorem 6.2 (Dold [**27**, Theorem 1.1]). *The sequence* $(\operatorname{ind}(f^n))_n$ *is a Dold sequence.*

The recent survey by Steinlein [**119**] with an emphasis on topology and the earlier survey of Nussbaum [**97**] with an emphasis on non-linear functional analysis, contain much of the interesting history of proofs that $(\operatorname{ind}(f^n))$ and $(L(f^n))$ are Dold sequences, described in the more general context of the Leray–Schauder degree.

6.1.1. *Existence of broken orbits.* Definition 6.3. Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a continuous map and let $\Omega \subsetneq \mathbb{R}^m$ be a bounded open set. A periodic orbit with the property that at least one of its points lies inside Ω and at least one of its points lies outside of the closure $\overline{\Omega}$ of Ω will be called an *Ω-broken orbit*.

We define

$$b_n(f, \Omega) = \frac{1}{n} \sum_{d \mid n} \mu(n/d) \operatorname{ind}(f^d, \Omega)$$

for $n \in \mathbb{N}$.

Theorem 6.4 (Krasnosel'skiĭ and Zabreĭko [**71**]; Pokrovskii and Rasskazov [**105**]). *Let n be an integer, let*

$$f\colon \mathbb{R}^m \to \mathbb{R}^m$$

be a continuous map with $\partial\Omega \cap \mathscr{F}_f(n) = \varnothing$, and assume that $b_n(f, \Omega) \not\equiv 0$ modulo n. Then there exists an Ω-broken orbit whose minimal period is a divisor of n.

Proof. Fix n and let $G_k = \mathscr{F}_f(k) \cap \Omega$ for $k \mid n$. Assume for the purposes of a contradiction that for all k dividing n and for all $x \in G_k$ we have $f^i(x) \in \Omega$ for $1 \leqslant i \leqslant k$. Define V_n inductively, in the same way as in Definition 6.1, so

$$V_1 = \Omega, \ldots, V_n = f^{-1}(V_{n-1}).$$

Note that we then have

$$V_n = \{x \in V \mid x, f(x), \ldots, f^{n-1}(x) \in V\} \supseteq G_k.$$

By the localisation property of the fixed-point index (see [59, Section 2.2.1]), we have $\operatorname{ind}(f^n,\Omega)=\operatorname{ind}(f^n,V_n)$ and so $b_n(f,\Omega)=b_n(f,V_n)$, but by Theorem 6.2 we know that $b_n(f,V_k)\equiv 0$ modulo n, which is a contradiction to the assumption. □

We have stated Theorem 6.4 for maps on $\mathbb{R}^m$ because that is the context considered by Pokrovskii and Rasskazov [105], who use this assumption to draw additional conclusions. A similar proof should give the result in the setting of a Euclidean neighbourhood retract and compact set of fixed points.

6.1.2. *Planar homeomorphisms.* The Dold relations are useful in finding restrictions on the form of indices of iterations.

Let us recall that by a local fixed-point index at an isolated fixed-point q, written $\operatorname{ind}(f,q)$, we understand the index $\operatorname{ind}(f,V)$ for a neighbourhood V of q that is small enough to have $\mathsf{F}_f\cap V=\{q\}$. The following theorem in this direction was proved by Brown in 1990.

THEOREM 6.5 (Brown [18, Theorem 4]). *Let $f\colon\mathbb{R}^2\to\mathbb{R}^2$ be a planar orientation preserving homeomorphism with an isolated fixed point at 0 for each iteration. Then there is an integer $p\neq 1$ such that*

$$\operatorname{ind}(f^n,0)=\begin{cases}\operatorname{ind}(f,0) & \text{if } \operatorname{ind}(f,0)\neq 1;\\ 1 \text{ or } p & \text{if } \operatorname{ind}(f,0)=1\end{cases}\tag{30}$$

for all $n\in\mathbb{N}$.

REMARK 6.6. Brown conjectured that if $\operatorname{ind}(f,0)=1$, then every integer p can appear as an index of some iteration in the formula (30) (cf. [18, Remark after Theorem 4]), and gave examples of realisations for all values of p except for $p=0$ and $p=2$. The Dold congruences easily exclude these two cases, by showing that $p=0$ and $p=2$ cannot occur as indices of any iteration if $\operatorname{ind}(f,0)=1$ (cf. [53]). To see how this works, assume that $p=0$ and let n be the first iteration for which $\operatorname{ind}(f^n,0)=0$. Then

$$\sum_{k|n}\mu(n/k)\operatorname{ind}(f^k,0)=\sum_{k|n,k\neq n}\mu(n/k)=\sum_{k|n}\mu(n/k)-1=-1\not\equiv 0$$

modulo n, where in the last equality we used well-known identities for the Möbius function contained in Lemma 3.3.

Note that by the formula (30) and Lemma 2.8 the sequence $(\operatorname{ind}(f^n,0))$ must be periodic.

Let us recall that an isolated fixed-point p is non-accumulated if $\operatorname{Per}(f)\cap V=\{p\}$ for some neighbourhood V of p. Using subtle topological analysis, Ruiz del Portal and Salazar showed later in [111] that if 0 is not an accumulated fixed point, then

$$\operatorname{ind}(f^n,0)=\mathsf{reg}_1(n)+a_d\cdot\mathsf{reg}_d(n),$$

where $d\geqslant 1$ and a_d is an integer.

6.1.3. *Periodic sequences of indices of iterations.* Consider a compact Euclidean neighbourhood retract X and continuous map $f\colon X\to X$ such that the two following conditions are satisfied:

(a) the set $\mathscr{F}_f(n)$ is compact for each $n\geqslant 1$ and consists of isolated fixed points of f^n;
(b) for each $x\in\mathscr{P}(f)$, the set of periodic points of f, the sequence $(\operatorname{ind}(f^n,x))_n$ is bounded.

Note that the condition (a) is equivalent to the fact that the number of n-periodic point is finite for each $n\geqslant 1$, while (b) means that $(\operatorname{ind}(f^n,x))_n$ is periodic (by Lemma 2.8).

In this class of maps, the fact that the Lefschetz numbers of iterations of f are unbounded implies the existence of infinitely many periodic points, by the following result.

THEOREM 6.7. *Let $f\colon X \to X$ satisfy the conditions* (a) *and* (b) *above. If* $(L(f^n))$ *is unbounded, then f has infinitely many periodic points with distinct periods.*

Proof. By the Lefschetz–Hopf formula, we have

$$L(f^n) = \sum_{x\in\mathscr{F}(f^n)} \operatorname{ind}(f^n, x) \tag{31}$$

for each $k \geqslant 1$.

As a consequence of the formula (31), the sequence $(L(f^n))$ is a sum of $(\operatorname{ind}(f^n, x))$ over all $x \in \bigcup_{n\geqslant 1} \mathscr{F}_f(n)$. By assumption $(L(f^n))$ is unbounded, while by (b) each $(\operatorname{ind}(f^n, x))_n$ is bounded and by (a) we know $\mathscr{F}_f(n)$ is finite for each $n \geqslant 1$. Thus f must have infinitely many periodic points of distinct minimal periods. □

REMARK 6.8. One of the directions of recent research is to identify the classes of maps for which $(\operatorname{ind}(f^n, x))$ is bounded. Among such maps there are:

- planar maps and homeomorphisms of $\mathbb{R}^3$ at a fixed point which is an isolated invariant set (see Hernández-Corbato and Ruiz del Portal [57], Le Calvez, Ruiz del Portal, and Salazar [73], and Le Calvez and Yoccoz [74]);
- C^1 maps in work of Chow, Mallet-Paret, and Yorke [24];
- holomorphic maps in work of Bogatyĭ [14], Fagella and Llibre [37], and Zhang [135];
- simplicial maps of smooth type in work of Graff [45].

For such maps the structure of the indices of iterations allows one to detect information about periodic points and dynamical behaviour in the neighbourhood of periodic points, and in certain cases some features of the global dynamics as well. The determination of the exact form of possible indices of iterations for smooth maps (see [50]) turned out to have many topological consequences. In particular, based on that result Graff and Jezierski constructed a smooth branch of Nielsen periodic point theory, obtaining invariants that allow the minimal number of periodic points in a smooth homotopy class to be computed [47–49].

6.1.4. *Detecting periodic points using dold congruences.* This is again a large area of research, which we illustrate with a sample of the type of result that may be expected.

PROPOSITION 6.9 (Dugundji and Granas [30]). *Let W be a connected polyhedron and let $f\colon W \to W$ be a continuous map with the property that f^n is homotopic to a constant map for some $n \geqslant 1$. Then f has a fixed point.*

Proof. Writing $\sim$ for homotopy equivalence, observe first that

$$f^n \sim c \Longrightarrow f^{n+1} \sim c,$$

where c is a constant map. In particular, $f^p \sim c$ for any prime number $p \geqslant n$. Hence

$$f^p_{*i}\colon H_i(W, \mathbb{Q}) \longrightarrow H_i(W, \mathbb{Q})$$

is the zero homomorphism for all $i > 0$, so $L(f^p) = 1$. On the other hand,

$$L(f) \equiv L(f^p)$$

modulo p by the Dold congruences, which implies that $L(f) \neq 0$. This proves there must be a fixed point by the Lefschetz fixed-point theorem. □

6.2. *Nielsen and Reidemeister numbers*

Let K be a connected, compact polyhedron with a continuous map $f\colon K \to K$. Let $p\colon \widetilde{K} \to K$ be the universal cover of K, and let $\widetilde{f}\colon \widetilde{K} \to \widetilde{K}$ be a lifting of f, so $p \circ \widetilde{f} = f \circ p$. Liftings $\widetilde{f}$ and $\widetilde{f}'$ are said to be *conjugate* if there is a $\gamma \in \pi_1(K)$ with

$$\widetilde{f}' = \gamma \circ \widetilde{f} \circ \gamma^{-1}. \tag{32}$$

We call the subset $p(\mathscr{F}(\tilde{f})) \subseteq \mathscr{F}(f)$ the *fixed-point class* of f determined by the lifting class $[\tilde{f}]$. A fixed-point class is called *essential* if its fixed-point index is non-zero. In this setting we can introduce the *Reidemeister number* $R(f)$ of f and the *Nielsen number* $N(f)$ of f. In our setting, $R(f)$ is the number of lifting classes of f, and $N(f)$ is the number of essential fixed-point classes. Note that $R(f)$ is equal to the number of fixed-point classes, and is a positive integer or infinity. Both $R(f)$ and $N(f)$ are topological invariants.

The importance of fixed-point theory in topology goes back to Poincaré and the origins of topology itself. Lefschetz (see [**76**] and its references to his earlier works in the area) found a way to count fixed points (with a multiplicity given by the fixed-point index) of continuous maps on compact topological spaces in terms of traces of induced maps on the homology groups of the space. Nielsen [**95**] studied the minimal number of fixed points in an isotopy class of homeomorphisms of the torus, a result extended by Brouwer to continuous maps of the torus, and went on to publish an influential study [**96**] on homeomorphisms of hyperbolic surfaces in which fixed points are classified in terms of behaviour on the universal covering space. In the case of a compact manifold of dimension at least 3 (or a polyhedron satisfying some additional natural hypotheses), this lower bound is the best one possible in that $N(f) = \min\{\mathsf{F}_g(1) \mid g{\sim}f\}$ by work of Jiang [**60**]. We refer to that monograph for an extensive treatment of Nielsen fixed-point theory, and to a survey by Jiang and Zhao [**61**] and a historical survey by Brown [**19**] for thorough treatments.

In general the sequence $(N(f^n))$ is not a Dold sequence. However, this does happen for all maps on a given space if certain topological restrictions on the space are imposed. We mention here that there is a minor error in [**39**, Example 11.3], where it is claimed that for an orientation-reversing homeomorphism of $\mathbb{S}^1$ the Dold congruence fails (the term corresponding to the even factor $10 = 2 \cdot 5$ was omitted from a sum over the divisors of 90). In fact for a continuous map $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ of the circle, it is known that

$$N(f^n) = |L(f^n)| = |1 - d^n|,$$

where d is the degree of the map. Clearly $a_n = d^n$ for all $n \geqslant 1$ defines a Dold sequence (a_n), as it is the sequence of traces of the 1×1 matrices $[d]^n$. Consider now the sequence with nth term given by $N(f^n) = |1 - d^n|$. If $d \geqslant 1$, then $|1 - d^n| = d^n - 1$ is a Dold sequence. If $d < 0$ then $d = -b$, where $b > 0$, and

$$|1 - d^n| = |1 - (-b)^n| = b^n - (-1)^n = b^n + \mathsf{reg}_1(n) - \mathsf{reg}_2(n)$$

is a Dold sequence as it is a sum of Dold sequences.

On the other hand, let us observe that $(|L(f^n)|)$ is not a Dold sequence in general. For example, $(a_n) = (-\mathsf{reg}_2 + \mathsf{reg}_3)$ is a Lefschetz sequence by Lemma 5.9, but $(|a_n|) = (0, 2, 3, 2, 0, 1, \ldots)$ does not satisfy the Dold congruence modulo 6.

The simplest examples of maps whose Nielsen numbers do not satisfy the Dold congruences are found in the class of maps of simply connected spaces.

Example 6.10. Let f be a map of a simply-connected compact space X. Then

$$N(f) = \begin{cases} 0 & \text{if } L(f) = 0; \\ 1 & \text{if } L(f) \neq 0. \end{cases}$$

Thus, to find an example of map f for which $(N(f^n))_n$ is not a Dold sequence, it is enough to find a map f such that

$$\left.\begin{aligned} L(f) &= 0 \\ L(f^2) &\neq 0 \end{aligned}\right\} \tag{33}$$

and for this we may take a homeomorphism f of the 2-sphere S^2 that changes the orientation. Then $L(f^n) = 1 + (-1)^n$, and the Lefschetz numbers satisfy (33).

On the other hand, for many other classes of spaces beyond the circle the sequence $(N(f^n))$ is a Dold sequence for all continuous maps.

Proposition 6.11. *If f is a map of a Klein bottle K, then $(N(f^n))$ is a Dold sequence.*

Proof. The Nielsen numbers of iterations in the case of Klein bottle may be expressed in terms of generators of the fundamental group (see the works of Kim, Kim, and Zhao [**68**] or Llibre [**78**] for the details). A consequence of these calculations is that there exist integers u and v such that

$$N(f^n) = \begin{cases} |u^n \cdot (v^n - 1)| & \text{if } |u| > 1; \\ |v^n - 1| & \text{if } |u| \leqslant 1. \end{cases}$$

As we mentioned above, the sequences (a_n) and (b_n), defined by the relations $a_n = |u|^n$ and $b_n = |1 - v^n|$ for all $n \geqslant 1$ are Dold sequences. This shows that $(N(f^n)$ is a Dold sequence since the property is closed under products (as discussed in the remark at the end of Section 4.2). □

Although $(|L(f^n)|)$ is not always a Dold sequence, nevertheless it belongs to that class if

$$L(f^n) = \det(I - A^n), \tag{34}$$

for some $k \times k$ integer-valued matrix A. This follows from the fact that it is realised by a self-map of a torus. Indeed, taking $f\colon \mathbb{T}^k \to \mathbb{T}^k$, a toral map induced by the linear map A, we know that f^n has exactly $|\det(I - A^n)|$ fixed points for each $n \geqslant 1$ (see, for example, [**59**]). The spaces for which $N(f^n)$ and $|L(f^n)|$ agree and Lefschetz numbers satisfy (34) include, among others, self-maps of nilmanifolds and some solvmanifolds. Thus, in these cases, Nielsen numbers of iterations are Dold sequences. Similar arguments may be applied to so-called infra-solvmanifold of type R by work of Fel'shtyn and Lee [**39**, Theorem 11.4] (moreover, in this setting $N(f^n) = R(f^n)$ for all $n \geqslant 1$, showing that $(R(f^n))$ is also a Dold sequence). Various topological consequences of these facts are discussed by Fel'shtyn and Troitsky [**43**] and by Fel'shtyn and Lee [**40, 41**].

A similar theorem holds for solv- and infra-nilmanifolds under additional assumptions.

Theorem 6.12 (Kwasik & Lee [**72**, Theorem 2; **75**]). *Let f be a continuous map of a solvmanifold or infra-nilmanifold, and assume that f is homotopically periodic (that is, f^k is homotopic to the identity for some $k > 1$). Then $(N(f^n))$ is a Dold sequence.*

This follows from the fact that $L(f^n) = N(f^n)$ for all $n \geqslant 1$ for any map of this type.

Remark 6.13. The Reidemeister number may be also defined in a purely algebraic way for a group endomorphism ϕ (as the number of ϕ-conjugacy classes). In many cases the sequence of Reidemeister number of ϕ^n is a Dold sequence (for details, we refer to the works of Fel'shtyn and Troitsky [**38, 42, 43**] and the references therein).

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