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DOMINATION NUMBERS IN GRAPHS WITH REMOVED EDGE OR SET OF EDGES

Magdalena Lemańska

Department of Mathematics
Gdańsk University of Technology
Narutowicza 11/12, 80–952 Gdańsk, Poland

e-mail: magda@mif.pg.gda.pl

Abstract

It is known that the removal of an edge from a graph G cannot decrease a domination number $\gamma(G)$ and can increase it by at most one. Thus we can write that $\gamma(G) \leq \gamma(G-e) \leq \gamma(G)+1$ when an arbitrary edge e is removed. Here we present similar inequalities for the weakly connected domination number γ_w and the connected domination number γ_c, i.e., we show that $\gamma_w(G) \leq \gamma_w(G-e) \leq \gamma_w(G)+1$ and $\gamma_c(G) \leq \gamma_c(G-e) \leq \gamma_c(G)+2$ if G and $G-e$ are connected. Additionally we show that $\gamma_w(G) \leq \gamma_w(G-E_p) \leq \gamma_w(G)+p-1$ and $\gamma_c(G) \leq \gamma_c(G-E_p) \leq \gamma_c(G)+2p-2$ if G and $G-E_p$ are connected and $E_p = E(H_p)$ where H_p of order p is a connected subgraph of G.

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1. Introduction

Let $G=(V,E)$ be a connected undirected graph. The *neighbourhood* $N_G(v)$ of a vertex $v \in V$ is the set of all vertices adjacent to v. For a set $X \subseteq V$, the *open neighbourhood* $N_G(X)$ is defined to be $\bigcup_{v \in X} N_G(v)$ and the *closed neighbourhood* $N_G[X] = N_G(X) \cup X$. A set $D \subseteq V$ is a *dominating set* if

$N_G[D] = V$. Further, D is a *connected dominating set* if D is dominating and $\langle D \rangle$, the subgraph induced by D, is connected.

The *domination number* of G, denoted $\gamma(G)$, is $\min\{|D| : D$ is a dominating set of $G\}$, while the *connected domination number* of G, denoted $\gamma_c(G)$, is $\min\{|D| : D$ is a connected dominating set of $G\}$.
A dominating set D is a *weakly connected dominating set* if the subgraph weakly induced by D, $\langle D \rangle_w = (N(D), E_w)$, is connected, where E_w is the set of all edges having at least one vertex in D. The *weakly connected domination number* of G, denoted $\gamma_w(G)$, is $\min\{|D| : D$ is a weakly connected dominating set of $G\}$. For unexplained terms and symbols see [1].

Let H_p be a connected subgraph of G with p vertices for $p \geq 2$ and $E_p = E(H_p)$ the set of edges of H_p. By $G - e$ we denote the graph formed by removing an edge e from G and by $G - E_p$ the graph formed by removing the set of edges E_p from G.

It is known [2] that the removal of an edge from G cannot decrease $\gamma(G)$ and can increase it by at most one. Thus we can write that $\gamma(G) \leq \gamma(G - e) \leq \gamma(G) + 1$ when an arbitrary edge e is removed. Here we present similar inequalities for numbers $\gamma_c(G)$ and $\gamma_w(G)$, i.e., we show that $\gamma_c(G) \leq \gamma_c(G - e) \leq \gamma_c(G) + 2$ and $\gamma_w(G) \leq \gamma_w(G - e) \leq \gamma_w(G) + 1$ if G and $G - e$ are connected.

We also prove that $\gamma_c(G) \leq \gamma_c(G - E_p) \leq \gamma_c(G) + 2p - 2$ and $\gamma_w(G) \leq \gamma_w(G - E_p) \leq \gamma_w(G) + p - 1$ if G and $G - E_p$ are connected.

2. Connected Domination Number

We study the behavior the connected domination number, with respect to edge or set of edges deletion. First we show that removing an edge cannot decrease the connected domination number and can increase it by at most two.

Theorem 1. *If e is an edge of G and if G and $G - e$ are connected, then* $\gamma_c(G) \leq \gamma_c(G - e) \leq \gamma_c(G) + 2$.

Proof. First we show that $\gamma_c(G) \leq \gamma_c(G - e)$. Let D_0 be a minimum connected dominating set of $G - e$. Certainly, D_0 is a connected dominating set of G. Thus $\gamma_c(G) \leq |D_0| = \gamma_c(G - e)$.

Now we prove that $\gamma_c(G - e) \leq \gamma_c(G) + 2$. Let D be a minimum connected dominating set of G and let e, say $e = xy$, be an edge of G such that $G - e$ is connected. We consider three cases.

Case 1. $x, y \notin D$. It is easy to see that D is a connected dominating set of $G - e$ and $\gamma_c(G - e) \leq |D| = \gamma_c(G) \leq \gamma_c(G) + 2$.

Case 2. $|\{x, y\} \cap D| = 1$, say $x \in D, y \notin D$. If $N_{G-e}(y) \cap D \neq \emptyset$, then D is a connected dominating set of $G - e$ and we have $\gamma_c(G - e) \leq |D| = \gamma_c(G) \leq \gamma_c(G) + 2$.

If $N_{G-e}(y) \cap D = \emptyset$, then $N_{G-e}(y) \cap (V - D) \neq \emptyset$ as $G - e$ is connected. Thus, there exists a vertex $y' \in N_{G-e}(y) \cap (V - D)$ such that $N_{G-e}(y') \cap D \neq \emptyset$. In this case $D \cup \{y'\}$ is a connected dominating set of $G - e$ and $\gamma_c(G - e) \leq |D \cup \{y'\}| = \gamma_c(G) + 1 \leq \gamma_c(G) + 2$.

Case 3. $x, y \in D$. Let $\langle D \rangle_{G-e}$ be the subgraph induced by D in $G - e$. If $\langle D \rangle_{G-e}$ is connected, then D is a connected dominating set of $G - e$ and $\gamma_c(G - e) \leq |D| = \gamma_c(G) \leq \gamma_c(G) + 2$.

If $\langle D \rangle_{G-e}$ is not connected, then it has exactly two components with vertex sets, say D_1 and D_2. Since $G - e$ is connected, there exists a path connecting D_1 and D_2. Let $P = (x_1, \ldots, x_k)$ be a shortest path between D_1 and D_2, say $x_1 \in D_1, x_k \in D_2$. From the choice of P it follows that $x_2, \ldots, x_{k-1}$ belong to $V - D$ and $3 \leq k \leq 4$ (otherwise some of vertices from a path would not be dominated).

If $k = 3$, then $D \cup \{x_2\}$ is a connected dominating set of $G - e$ and $\gamma_c(G - e) \leq |D \cup \{x_2\}| = \gamma_c(G) + 1 \leq \gamma_c(G) + 2$.

If $k = 4$, then $D \cup \{x_2, x_3\}$ is a connected dominating set of $G - e$ and thus $\gamma_c(G - e) \leq |D \cup \{x_2, x_3\}| = \gamma_c(G) + 2$. ∎

Now we study the effects on the connected domination number when a graph is modified by deleting a set of edges.

Theorem 2. *Let H_p be a connected subgraph of order p in G, let E_p be the edge set of H_p and let $G - E_p$ be the graph obtained from G by deleting edges of E_p. If G and $G - E_p$ are connected, then $\gamma_c(G) \leq \gamma_c(G - E_p) \leq \gamma_c(G) + 2p - 2$.*

Proof. Let D_0 be a minimum connected dominating set of $G - E_p$. Then D_0 is a connected dominating set of G and obviously $\gamma_c(G) \leq |D_0| = \gamma_c(G - E_p)$.

We now prove the inequality $\gamma_c(G - E_p) \leq \gamma_c(G) + 2p - 2$. Let D be a minimum connected dominating set of G and let us denote $V(H_p) \cap D$ and $V(H_p) \cap (V - D)$ by S_1 and S_2, respectively. Certainly, $0 \leq |S_1| \leq p$ and $0 \leq |S_2| \leq p$. If H_p is not a tree, then let $\{C_1, \ldots, C_k\}$ be a fundamental basis of H_p. We sequently remove edges belonging to E_p from a graph G according to the algorithm.

INPUT: a graph G, a subgraph H_p
OUTPUT: a spanning tree H'_p of H_p
$H'_p := H_p$;
for $i = 1$ **to** k **do**
Let $\{C_1, \ldots, C_{k-i+1}\}$ be a fundamental basis of H'_p ;
  if $V(C_i) \subset S_1$ or $V(C_i) \subset S_2$ **then** remove from H'_p any edge e of C_i;
  else if there exists an edge e of C_i joining two vertices of S_2 **then**
    remove e from H'_p;
   else there exists a vertex v belonging to $V(C_i) \cap S_2$ such that its
    neighbours on C_i, say x and y, belong to S_1, then we remove
    from H'_p either the edge vx or vy
   fi;
  fi;
od;

Let E_s be the set of edges removed according to the above algorithm. Since $\{C_1, \ldots, C_k\}$ is a fundamental basis, the graph $H'_p = H_p - E_s$ is a spanning tree of H_p, so $|H_p - E_s| = p - 1$ and $|E_s| \leq \binom{p}{2} - p + 1$. Certainly, the set D is a minimum connected dominating set of the graph $G_0 = G - E_s$ and $\gamma_c(G_0) = \gamma_c(G - E_s) = \gamma_c(G)$.

Let $e_1, \ldots, e_{p-1}$ be the edges of $H_p - E_s$ and let $G_i = G_{i-1} - e_i = G_0 - \{e_1, \ldots, e_i\}$ for $i = 1, \ldots, p - 1$.

As $\gamma_c(G - E_p) = \gamma_c(G_{p-1})$, by Theorem 1 we have

$$\gamma_c(G - E_p) = \gamma_c(G_{p-1}) \leq \gamma_c(G_{p-2}) + 2 \leq \gamma_c(G_{p-3}) + 4$$

$$\leq \ldots \leq$$

$$\gamma_c(G_1) + 2p - 4 \leq \gamma_c(G_0) + 2p - 2.$$

Thus $\gamma_c(G - E_p) \leq \gamma_c(G) + 2p - 2$ since $\gamma_c(G_0) = \gamma_c(G)$. ■

Following theorem is an obvious generalisation of obtained results.

Theorem 3. *If G and $G - E_p$ are connected and H_p has k components, then $\gamma_c(G) \leq \gamma_c(G - E_p) \leq \gamma_c(G) + 2(p - k)$.*

3. Weakly Connected Domination Number

In this part we study the behavior the weakly connected domination number with respect to edge or set of edges deletion from a graph.

Theorem 4. *If e is an edge of a graph G and if G and $G-e$ are connected, then $\gamma_w(G) \leq \gamma_w(G-e) \leq \gamma_w(G)+1$.*

Proof. Let D_0 be a minimum weakly connected dominating set of $G-e$. Certainly, D_0 is also a weakly connected dominating set of G and $\gamma_w(G) \leq |D_0| = \gamma_w(G-e)$.

To prove the inequality $\gamma_w(G-e) \leq \gamma_w(G)+1$, let D be a minimum weakly connected dominating set of G, and let e, say $e = xy$, be an edge of G such that $G-e$ is connected. We consider three cases.

Case 1. If $x, y \in V-D$, then D is a weakly connected dominating set of $G-e$ and $\gamma_w(G-e) \leq |D| = \gamma_w(G) \leq \gamma_w(G)+1$.

Case 2. $x, y \in D$. Let F be the subgraph weakly induced by D in $G-e$. If F is connected, then D is a weakly connected dominating set of $G-e$ and $\gamma_w(G-e) \leq |D| = \gamma_w(G) \leq \gamma_w(G)+1$. If F is not connected, then it has exactly two components with vertex sets, say D_1, D_2, and suppose $x \in D_1, y \in D_2$.

Since F is disconnected and $G-e$ is connected, there are adjacent vertices $a, b \in V-D$ such that $a \in D_1$, $b \in D_2$.

Thus $D \cup \{a\}$ (and $D \cup \{b\}$) is a weakly connected dominating set of $G-e$ and $\gamma_w(G-e) \leq |D \cup \{a\}| = \gamma_w(G)+1$.

Case 3. $|\{x,y\} \cap D| = 1$, say $x \in D, y \in V-D$. As in Case 2, let F be the subgraph weakly induced by D in $G-e$. If F is connected and $N_{G-e}(y) \cap D \neq \emptyset$, then D is a weakly connected dominating set in $G-e$ and we have desired inequality.

If F is connected and $N_{G-e}(y) \cap D = \emptyset$, then, since $G-e$ is connected, $N_{G-e}(y) \cap (V-D) \neq \emptyset$. Thus, there is a vertex $y' \in N_{G-e}(y) \cap (V-D)$ such that $N_{G-e}(y') \cap D \neq \emptyset$. In this case $D \cup \{y\}$ is a weakly connected dominating set of $G-e$ and $\gamma_w(G-e) \leq |D \cup \{y\}| = \gamma_w(G)+1$.

If F is not connected, then it has exactly two components with vertex sets, say D_1, D_2 and assume that $x \in D_1, y \in D_2$. Then it is no problem to observe that $N_{G-e}(y) \cap D \neq \emptyset$, i.e., y has a neighbour in D in $G-e$. This implies, that D is a dominating set of $G-e$.

Since F is disconnected and $G - e$ is connected, there are adjacent vertices $a, b \in V - D$ such that $a \in D_1$, $b \in D_2$.

Thus $D \cup \{a\}$ (and $D \cup \{b\}$) is a weakly connected dominating set of $G - e$ and $\gamma_w(G - e) \leq |D \cup \{a\}| = \gamma_w(G) + 1$. ∎

Theorem 5. *Let H_p be a connected subgraph of order p in G, let E_p be the edge set of H_p and let $G - E_p$ be the graph formed by removing edges E_p from G. If G and $G - E_p$ are connected, then $\gamma_w(G) \leq \gamma_w(G - E_p) \leq \gamma_w(G) + p - 1$.*

Proof. Let D_0 be a minimum weakly connected dominating set of $G - E_p$. It is no problem to observe that D_0 is a weakly connected dominating set of G, so $\gamma_w(G) \leq |D_0| = \gamma_w(G - E_p)$.

Now we prove that $\gamma_w(G - E_p) \leq \gamma_w(G) + p - 1$. Let D be a minimum weakly connected dominating set of G. As in the proof of Theorem 2, let E_s be a subset of E_p such that $H_p - E_s$ is a spanning tree of H_p, let $e_1, \ldots, e_{p-1}$ be the edges of $H_p - E_s$ and $G_i = G_{i-1} - e_i = G_0 - \{e_1, \ldots, e_i\}$ for $i = 1, \ldots, p-1$. The set D is a minimum weakly connected dominating set of a graph $G_0 = G - E_s$. Thus $\gamma_w(G_0) = \gamma_w(G - E_s) = \gamma_w(G)$.

As $\gamma_w(G - E_p) = \gamma_w(G_{p-1})$, by Theorem 4 we have

$$\gamma_w(G - E_p) = \gamma_w(G_{p-1}) \leq \gamma_w(G_{p-2}) + 1 \leq \gamma_c(G_{p-3}) + 2$$

$$\leq \ldots \leq$$

$$\gamma_c(G_1) + p - 2 \leq \gamma_c(G_0) + p - 1.$$

Thus $\gamma_w(G - E_p) \leq \gamma_w(G) + p - 1$ as $\gamma_w(G_0) = \gamma_w(G)$. ∎

References

[1] T. Haynes, S. Hedetniemi and P. Slater, Fundamentals of domination in graphs (Marcel Dekker, Inc. 1998).

[2] J. Topp, Domination, independence and irredundance in graphs, Dissertationes Mathematicae 342 (PWN, Warszawa, 1995).

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