Opuscula Math. 36, no. 5 (2016), 575–588
http://dx.doi.org/10.7494/OpMath.2016.36.5.575 Opuscula Mathematica

EDGE SUBDIVISION AND EDGE MULTISUBDIVISION VERSUS SOME DOMINATION RELATED PARAMETERS IN GENERALIZED CORONA GRAPHS

Magda Dettlaff, Joanna Raczek, and Ismael G. Yero

Communicated by Dalibor Fronček

Abstract. Given a graph $G = (V, E)$, the subdivision of an edge $e = uv \in E(G)$ means the substitution of the edge e by a vertex x and the new edges ux and xv. The domination subdivision number of a graph G is the minimum number of edges of G which must be subdivided (where each edge can be subdivided at most once) in order to increase the domination number. Also, the domination multisubdivision number of G is the minimum number of subdivisions which must be done in one edge such that the domination number increases. Moreover, the concepts of paired domination and independent domination subdivision (respectively multisubdivision) numbers are defined similarly. In this paper we study the domination, paired domination and independent domination (subdivision and multisubdivision) numbers of the generalized corona graphs.

Keywords: domination, paired domination, independent domination, edge subdivision, edge multisubdivision, corona graph.

Mathematics Subject Classification: 05C69, 05C70, 05C76.

1. INTRODUCTION

Studies about the influence of edge subdivisions over a parameter in graphs can be found in several areas of graph theory. Just for mentioning some of them we can refer to [10] (influence of edge subdivision over the independence number) and [12] (influence of edge subdivision over the total domination number) since are close to the topic of our work.

The domination subdivision number was defined by Velammal in 1997 (see [15]) and since then it is widely studied in graph theory. This parameter was studied in trees by Aram, Sheikholeslami and Favaron [1] and also by Benecke and Mynhardt [2]. General bounds and properties has been studied for example by Haynes, Hedetniemi and Hedetniemi [10], by Bhattacharya and Vijayakumar [3], by Favaron, Haynes and Hedetniemi [5] and by Favaron, Karami and Sheikholeslami [6].

For domination problems, multiple edges and loops are irrelevant, so we forbid them. Additionally, in this paper we only consider connected graphs. We use $V(G)$ and $E(G)$ for the vertex set and the edge set of a graph G and denote $|V(G)| = n$ and $|E(G)| = m$.

A subset D of $V(G)$ is a *dominating set* of G, if every vertex of $V(G) - D$ has at least one neighbor in D. Let $\gamma(G)$ be the minimum cardinality among all dominating sets in G. Moreover, the set D is a $\gamma(G)$-set if it is a dominating set of cardinality $\gamma(G)$.

Given a graph $G = (V, E)$, the subdivision of the edge $e = uv \in E(G)$ with a vertex x leads to a graph with vertex set $V \cup \{x\}$ and edge set $(E(G) - \{uv\}) \cup \{ux, xv\}$. We call x a subdivision vertex. We denote by $G_{uv,k} = G_{e,k}$ the graph obtained from G by subdividing the edge uv with k vertices (instead of the edge $e = uv$ we put a path $(u, x_1, x_2, \ldots, x_k, v)$). We call $x_1, x_2, \ldots, x_k$ the subdivision vertices. For $k = 1$ we write G_e instead of $G_{e,1}$. We denote by $G_{e_1,e_2,\ldots,e_k}$, the graph obtained from G by subdividing the edges $e_1, e_2, \ldots, e_k$, where each edge is subdivided once.

The *domination subdivision number* of a graph G, denoted by $\mathrm{sd}(G)$, is the minimum number of edges which must be subdivided (where each edge can be subdivided at most once) in order to increase the domination number. Since the domination number of the graph K_2 does not increase when its only edge is subdivided, we consider domination subdivision number for connected graphs of order at least three.

Similarly, let $\mathrm{msd}(uv)$ be the minimum number of subdivisions of the edge uv such that the domination number increases. In this sense, the *domination multisubdivision number* of a graph G, denoted by $\mathrm{msd}(G)$, is defined as

$$\mathrm{msd}(G) = \min\{\mathrm{msd}(uv) : \ uv \in E(G)\}.$$

This parameter was introduced by Dettlaff, Raczek and Topp in [4] and is well defined for all graphs having at least one edge. In their paper, they have also studied some complexity aspects regarding the domination subdivision and domination multisubdivision numbers of graphs. That is, they studied the following decision problems. Given a graph $G = (V, E)$ with domination number $\gamma(G)$: Is $\mathrm{sd}(G) > 1$? and, Is $\mathrm{msd}(G) > 1$? As a result, in [4], it was obtained that these decision problems for the domination subdivision number, as well as for the domination multisubdivision number, are NP-complete even for bipartite graphs. In this sense, it is desirable to find or describe some families of graphs in which is possible to give the exact value for these parameters. The general bounds for this parameter are as follows

Theorem 1.1 ([4]). *For any connected graph G with at least one edge,*

$$1 \leq \mathrm{msd}(G) \leq 3.$$

In particular for trees and cycles we have the following result.

Theorem 1.2 ([4])**.** *For $n \geq 3$,*

$$\operatorname{msd}(C_n) = \operatorname{msd}(P_n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 3, \\ 2 & \text{if } n \equiv 2 \pmod 3, \\ 3 & \text{if } n \equiv 1 \pmod 3. \end{cases}$$

In this paper we study the domination subdivision number and the domination multisubdivision number in the generalized corona product of graphs. Given a connected graph G of order $n \geq 2$ with a set of vertices $V(G) = \{u_1, u_2, \ldots, u_n\}$, and a sequence of n graphs $\mathcal{H} = (H_1, H_2, \ldots, H_n)$, the *generalized corona graph* $H = G \odot \mathcal{H}$ is obtained by joining each vertex of H_i to the vertex u_i of G. Figure 1 shows an example of a generalized corona graph. In order to simplify the notation, in some cases we denote by $Z\mathcal{H}$ the set of elements of $\mathcal{H}$. In this sense, if for instance, $\mathcal{H} = (K_1, K_2, K_1, K_2, K_1, K_2)$, then we just write $Z\mathcal{H} = \{K_1, K_2\}$.

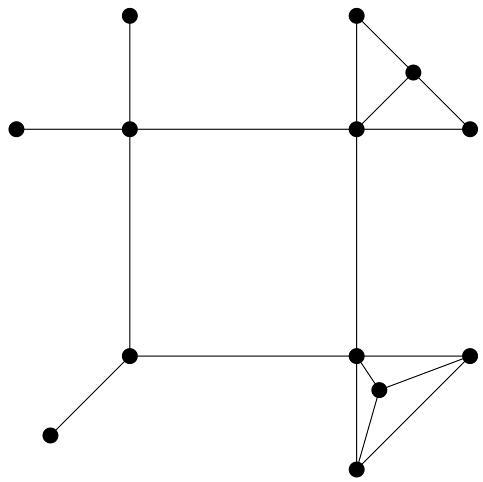

Fig. 1. Graph $C_4 \odot \mathcal{H}$, where $\mathcal{H} = (\overline{K_2}, P_3, K_3, K_1)$

The standard corona graph is a particular case of the generalized corona graph when all the graphs of the family $\mathcal{H}$ are isomorphic to a graph H, in which case we will just write $G \odot H$. Clearly, $\gamma(G \odot K_1) = \gamma(G \odot \mathcal{H}) = |V(G)|$.

In this paper we first present novel and interesting results concerning the domination subdivision number and the domination multisubdivision number of generalized corona graphs. Further we study similar (subdivision and multisubdivision) parameters related to domination for generalized corona graphs, which are the following ones: paired domination and independent domination.

2. STANDARD DOMINATION

The following lemma presented independently in [7,13] will be useful in this section to present results about domination subdivision numbers and domination multisubdivision numbers of generalized corona graphs.

Lemma 2.1 ([7,13])**.** *For any graph G with even order n and without isolated vertices, $\gamma(G) = \frac{n}{2}$ if and only if the components of G are the cycle graph C_4 or the corona graph $H \odot K_1$ for any connected graph H.*

2.1. DOMINATION SUBDIVISION NUMBER

We begin the study with the domination subdivision number of generalized corona graphs $G \odot \mathcal{H}$ for the case when every $H_i \in \mathcal{H}$ is isomorphic to K_1, i.e., the graph $G \odot K_1$.

Proposition 2.2. *If G is a connected graph of order $n \geq 2$, then*

$$\mathrm{sd}(G \odot K_1) = 3.$$

Proof. Let G be a connected graph of order $n \geq 2$. We will denote by $V(G) = \{u_i : i = 1, \ldots, n\}$ the set of vertices of G, and by v_i the corresponding pendant vertex of u_i, where $u_i v_i$ is an edge of $G \odot K_1$. Clearly $V(G)$ is a $\gamma(G \odot K_1)$-set.

Haynes *et al.* [11], proved that if a graph G has adjacent support vertices, then $\mathrm{sd}(G) \leq 3$. Hence, $\mathrm{sd}(G \odot K_1) \leq 3$.

Now we show that $\mathrm{sd}(G \odot K_1) \geq 3$, what means that subdivision of any two edges e_1, e_2 does not change the domination number of a graph $G \odot K_1$. Let us subdivide these edges with vertices x and y, respectively.

If $e_1, e_2 \in E(G)$, then let $D' = V(G)$, if $e_1 \in E(G)$, $e_2 = u_i v_i$ (the case $e_1 = u_i v_i$, $e_2 \in E(G)$ is similar), then let $D' = (V(G) - \{u_i\}) \cup \{y\}$, and if $e_1 = u_i v_i$, $e_2 = u_j v_j$ for some $i \neq j$, then let $D' = (V(G) - \{u_i, u_j\}) \cup \{x, y\}$. In all cases, D' is a dominating set of $(G \odot K_1)_{e_1,e_2}$ of size $\gamma(G \odot K_1)$ and this implies that $\mathrm{sd}(G \odot K_1) = 3$. □

Theorem 2.3. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a family of n graphs. Then*

$$\mathrm{sd}(G \odot \mathcal{H}) = \begin{cases} 3, & \text{if } Z\mathcal{H} = \{K_1\}, \\ 2, & \text{if } Z\mathcal{H} = \{K_1, K_2\}, \\ 1, & \text{for other cases.} \end{cases}$$

Proof. If $Z\mathcal{H} = \{K_1\}$, then the result follows by Proposition 2.2. Now let $Z\mathcal{H} = \{K_1, K_2\}$ and let without loss of generality that $H_1 = K_2 = (w_1, w_2)$. Assume H' is the graph obtained from $G \odot \mathcal{H}$ by subdividing the edges $u_1 w_1$, $u_1 w_2$ with vertices x, y. If D' is a $\gamma(H')$-set, then to dominate the vertices x, y, w_1, w_2 we must have $|D' \cap \{u_1, x, y, w_1, w_2\}| \geq 2$ and to dominate the vertices of H_i, we must have $|D' \cap (V(H_i) \cup \{u_i\})| \geq 1$ for $i \geq 2$. It follows that $|D'| \geq |V(G)| + 1$ and so $\mathrm{sd}(G \odot \mathcal{H}) \leq 2$. Now we show that $\mathrm{sd}(G \odot \mathcal{H}) \geq 2$. Let $e = uv \in E(G \odot \mathcal{H})$ and H' be the graph obtained from $G \odot \mathcal{H}$ by subdividing the edge e with vertex x. If $u, v \in V(G)$, then let $D' = V(G)$, if $u, v \in V(H_i)$ for some i, then let $D' = (V(G) - \{u_i\}) \cup \{x\}$, and if $u \in V(G)$, $v \in V(H_i)$ for some i, then let $D' = (V(G) - \{u_i\}) \cup \{v\}$. In all cases, D' is a dominating set of H' of size $\gamma(G \odot \mathcal{H})$ and this implies that $\mathrm{sd}(G \odot \mathcal{H}) \geq 2$.

Finally let $H_i \in \mathcal{H}$ be a complete graph of order at least three or a non complete graph of order at least two for some i, say $i = 1$. If H_1 is a complete graph of order at least three, then let H' be a graph obtained from $G \odot \mathcal{H}$ by subdividing an edge $uv \in E(H_1)$, and if H_1 is a non complete graph of order at least two, then let u, v be two non adjacent vertices in H_1 and H' be the graph obtained from $G \odot \mathcal{H}$ by subdividing an edge $u_1 u$, with a vertex x. Assume D' is a $\gamma(H')$-set. To dominate the vertices u, v, x we must have $|D' \cap (V(H_1) \cup \{u_1\})| \geq 2$ and to dominate the vertices of $V(H_i)$, we must have $|D' \cap (V(H_i) \cup \{u_i\})| \geq 1$ for $i \geq 2$. This implies that $|D'| \geq |V(G)| + 1$ and so $\mathrm{sd}(G \odot \mathcal{H}) = 1$. □

2.2. DOMINATION MULTISUBDIVISION NUMBER

The following observation (from [4]) and the lemma are useful to present our results.

Observation 2.4 ([4])**.** Let G be a graph. Then $\mathrm{sd}(G) = 1$ if and only if $\mathrm{msd}(G) = 1$.

Lemma 2.5. *Let G be a graph of order n such that G does not have K_2 as a component. If $\gamma(G) = \frac{n}{2}$, then*

$$\mathrm{msd}(G) = 3.$$

Proof. If $\gamma(G) = \frac{n}{2}$, then by Lemma 2.1 we have that the components of G are C_4 or the corona graph $H \odot K_1$ for some connected graph H. Since the domination multisubdivision number of G is the minimum of the domination multisubdivision numbers of the graphs induced by its components, we study each component of G separately. By Theorem 1.2 we have that $\mathrm{msd}(C_4) = 3$.

Now, let H be the subgraph of G induced by a component isomorphic to a corona graph $H' \odot K_1$. Since G does not have K_2 as a component and $\gamma(G) = \frac{n}{2}$, $|V(H)| \geq 4$. Suppose that $\mathrm{msd}(H) \leq 2$. So, there exists an edge uv of H such that $\gamma(H_{uv,2}) > \gamma(H)$. We have the following cases.

Case 1. uv is an edge of H'. Since the whole vertex set of H' is a $\gamma(H)$-set, we have that the vertices used to obtain the graph $H_{uv,2}$ are still dominated by the whole vertex set of H'. Thus $\gamma(H_{uv,2}) = \gamma(H)$, a contradiction.

Case 2. uv is a pendant edge of $H' \circ K_1$ with $u \in V(H')$ and $v \in V(K_1)$. Let x_1, x_2 be the vertices used to subdivide the edge uv such that $u \sim x_1 \sim x_2 \sim v$. Notice that $(V(H') - \{u\}) \cup \{x_2\}$ is a dominating set of H with cardinality $\gamma(H)$. Thus, $\gamma(H_{uv,2}) = \gamma(H)$, a contradiction again.

Therefore, for any edge xy of H, $\gamma(H_{xy,t}) = \gamma(H)$ and $t \in \{1, 2\}$ we have $\gamma(H_{xy,t}) = \gamma(H)$ and, as a consequence, $\mathrm{msd}(G) = 3$. □

Theorem 2.6. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a family of n graphs. Then*

$$\mathrm{msd}(G \odot \mathcal{H}) = \begin{cases} 3, & \text{if } Z\mathcal{H} = \{K_1\}, \\ 2, & \text{if } Z\mathcal{H} = \{K_1, K_2\}, \\ 1, & \text{for other cases.} \end{cases}$$

Proof. First we notice that, as a consequence of Observation 2.4 and Theorem 2.3 we have that $\mathrm{msd}(G \odot \mathcal{H}) = 1$ if and only if there exists $H_l \in \mathcal{H}$ which is neither isomorphic to K_1 nor to K_2.

Now, assume that every $H_i \in \mathcal{H}$ is isomorphic to K_1. Hence, by Lemma 2.1, $\gamma(G \odot \mathcal{H}) = \gamma(G \odot K_1) = n$. Thus, Lemma 2.5 leads to $\mathrm{msd}(G \odot \mathcal{H}) = 3$.

On the other hand, we consider a generalized corona graph $G \odot \mathcal{H}$ with $\mathrm{msd}(G \odot \mathcal{H}) = 3$. Suppose there exists $H_i \in \mathcal{H}$ such that $H_i \ncong K_1$. So, there exist at least two vertices $v_{i1}, v_{i2} \in V_i$ in the graph H_i. Let u_i be the vertex of G such that $u_i \sim v_{i1}$ and $u_i \sim v_{i2}$. Now we subdivide the edge $u_i v_{i1}$ with the vertices x_1, x_2 such that $u_i \sim x_1 \sim x_2 \sim v_{i1}$. We consider two cases.

Case 1. $v_{i1} \sim v_{i2}$. Since any $\gamma((G \odot \mathcal{H})_{u_i v_{i1},2})$-set contains at least one vertex in $V_i \cup \{x_1, x_2, u_i\}$, to obtain a $\gamma((G \odot \mathcal{H})_{u_i v_{i1},2})$-set we need to dominate the vertices $u_i, x_1, x_2, v_{i1}, v_{i2}$, which induce a cycle of order five in $(G \odot \mathcal{H})_{u_i v_{i1},2}$. Thus, any $\gamma((G \odot \mathcal{H})_{u_i v_{i1},2})$-set contains at least two vertices of $V_i \cup \{x_1, x_2, u_i\}$. As a consequence we have that $\gamma((G \odot \mathcal{H})_{u_i v_{i1},2}) > n = \gamma(G \odot \mathcal{H})$, a contradiction.

Case 2. $v_{i1} \nsim v_{i2}$. To dominate the vertex v_{i2} we need a vertex in $\{v_{i2}, u_i\}$ and to dominate the vertex v_{i1} we need a vertex in $\{v_{i1}, x_2\}$. Thus, any $\gamma((G \odot \mathcal{H})_{u_i v_{i1},2})$-set contains at least two vertices of $V_i \cup \{x_1, x_2, u_i\}$, and again we have that $\gamma((G \odot \mathcal{H})_{u_i v_{i1},2}) > n = \gamma(G \odot \mathcal{H})$, which is a contradiction.

Therefore, as a consequence of the above cases, $H_i \cong K_1$ for every $H_i \in \mathcal{H}$.

Finally, if there exists $H_j \in \mathcal{H}$ isomorphic to K_2 and every $H_i \in \mathcal{H} - H_j$ is isomorphic to K_1 or K_2, then from the above we have that $\mathrm{msd}(G \odot \mathcal{H}) > 1$ and $\mathrm{msd}(G \odot \mathcal{H}) < 3$. Thus, it is clear that $\mathrm{msd}(G \odot \mathcal{H}) = 2$, and the proof is done. □

As a consequence of the subsections above, we can see that every generalized corona graph has equal domination subdivision and domination multisubdivision numbers.

3. PAIRED DOMINATION

A set of pairwise non-adjacent edges of a graph G is called a *matching* in G. If M is a matching in a graph G such that every vertex of G is incident with an edge of M, then M is a *perfect matching* in G. By $V(M)$ we denote the set of vertices of the graph induced by M. If M is a matching, $u \in V(G)$ and there is an edge $uv \in M$ for some $v \in V(G)$, then u is a *matched vertex*. Otherwise, u is *unmatched*.

A *paired dominating set*, introduced by Haynes and Slater in [9], is a dominating set whose induced subgraph contains at least one perfect matching. The minimum cardinality of a paired dominating set in G is the *paired domination number* and is denoted by $\gamma_{pr}(G)$. A set D is a $\gamma_{pr}(G)$-set if it is a paired dominating set of cardinality $\gamma_{pr}(G)$.

A *support vertex* is a vertex adjacent to a vertex of degree one. It is worth to observe that every support vertex belongs to a paired dominating set of a graph G.

Now, the *paired domination subdivision number*, $\mathrm{sd}_{pr}(G)$, of a nonempty graph G is the minimum number of edges which must be subdivided (where each edge can be subdivided at most once) in order to increase the paired domination number. Moreover, the *paired domination multisubdivision number* of G, $\mathrm{msd}_{pr}(G)$, is the minimum k such that there exists an edge uv of G satisfying that $\gamma_{pr}(G_{uv,k}) > \gamma_{pr}(G)$. This parameter was introduced by Raczek and Dettlaff in [14]. There was proven the general bound for it.

Theorem 3.1 ([14])**.** *For any connected graph G with at least one edge,*

$$1 \leq \mathrm{msd}_{pr}(G) \leq 4.$$

To continue with our results, we first determine the paired domination number of generalized corona graphs.

Lemma 3.2. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a sequence of n graphs. Then*

$$\gamma_{pr}(G \odot \mathcal{H}) = 2n - |V(M)|,$$

where M is a maximum matching in G.

Proof. Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a family of n graphs. Since every support vertex belongs to every paired dominating set, $V(G)$ is a subset of every minimum paired dominating set of $G \odot K_1$. Hence $\gamma_{pr}(G \odot K_1) \geq n$ and any minimum paired dominating set of $G \odot K_1$ may be formed by the vertices of a graph induced by the maximum matching in G, denoted $M(G)$, together with unmatched vertices of $V(G)$ paired with any neighboring vertex of $V(G \odot K_1) - V(G)$. Therefore,

$$\gamma_{pr}(G \odot K_1) = |V(M)| + 2(n - |V(M)|) = 2n - |V(M)|.$$

Since $G \odot K_1$ is an induced subgraph of $G \odot \mathcal{H}$, $\gamma_{pr}(G \odot K_1) \leq \gamma_{pr}(G \odot \mathcal{H})$. On the other hand, any minimum paired dominating set of $G \odot K_1$ is a paired dominating set of $G \odot \mathcal{H}$, implying that

$$\gamma_{pr}(G \odot K_1) = \gamma_{pr}(G \odot \mathcal{H}) = 2n - |V(M)|.$$

□

Theorem 3.3. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a sequence of n graphs. If there exists a vertex $u \in V(G)$ such that u is matched in every maximum matching in G, then*

$$\mathrm{sd}_{pr}(G \odot \mathcal{H}) = \mathrm{msd}_{pr}(G \odot \mathcal{H}) = 1.$$

Proof. Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a sequence of n graphs. Assume there exists a vertex $u \in V(G)$ such that u is matched in every maximum matching in G. Then Lemma 3.2 and its proof imply that u belongs to each minimum paired dominating set of $G \odot \mathcal{H}$ and u is always paired in a graph induced by a minimum paired dominating set of $G \odot \mathcal{H}$ with a vertex of $V(G)$.

Consider the graph $(G \odot \mathcal{H})_{uu',1}$, where $u' \in V(H_i)$ for some $i = 1, \ldots, n$ and denote by s the subdivision vertex. Let D' be a minimum paired dominating set of $(G \odot \mathcal{H})_{uu',1}$. Then to dominate vertices of $V(H_i)$, D' should contain at least one vertex of $V(H_i) \cup \{s\}$. If $|D'| = \gamma_{pr}(G \odot \mathcal{H})$, then we would find in $G \odot \mathcal{H}$ a minimum paired dominating set containing vertices of $V(H_i)$, which is impossible.

Hence, $|D'| > \gamma_{pr}(G \odot \mathcal{H})$. Therefore, if there exists a vertex $u \in V(G)$ such that u is matched in every maximum matching in G, then $\mathrm{sd}_{pr}(G \odot \mathcal{H}) = \mathrm{msd}_{pr}(G \odot \mathcal{H}) = 1$. $\square$

Theorem 3.4. *Let G be a connected graph of order $n \geq 2$ such that for each vertex $u \in V(G)$ there exists a maximum matching of G with u unmatched and let $\mathcal{H}$ be a sequence of n graphs. Then*

$$\mathrm{msd}_{pr}(G \odot \mathcal{H}) = \begin{cases} 4, & \text{if } Z\mathcal{H} = \{K_1\}, \\ 3, & \text{if } Z\mathcal{H} = \{K_1, K_2\}, \\ 2, & \text{for other cases.} \end{cases}$$

Proof. Let G be a connected graph of order $n \geq 2$ such that for each vertex $u \in V(G)$ there exist a maximum matching of G with u unmatched and let $\mathcal{H}$ be a family of n graphs.

First assume $Z\mathcal{H} = \{K_1\}$. Since the paired multisubdivision number for any graph is equal either 1, 2, 3 or 4, it suffices to justify that subdividing any edge of $G \odot \mathcal{H}$, where $\mathcal{H} = \{K_1\}$ with three vertices does not increase its paired domination number. Subdivide any uu' edge with three new vertices, where $u \in V(G)$ and u' is the neighbor of u with degree one in $G \odot \mathcal{H}$. Then the set formed by the vertices of the graph induced by a maximum matching in G not containing u, altogether with two adjacent subdivision vertices, where one of them is a support vertex in $(G \odot \mathcal{H})_{uu',3}$ and with unmatched vertices of $V(G) - \{u\}$ paired with any neighboring vertex of $V(G \odot \mathcal{H}) - V(G)$, is a paired dominating set of $(G \odot \mathcal{H})_{uu',3}$ of cardinality $\gamma_{pr}(G \odot \mathcal{H})$.

Now subdivide any edge $uw \in E(G \odot \mathcal{H})$ with three subdivision vertices, where $u, w \in V(G)$. Then the set formed by the vertices of the graph induced by a maximum matching in G not containing u, altogether with u paired with the subdivision vertex adjacent to u and with unmatched vertices of $V(G) - \{u\}$ paired with any neighboring vertex of $V(G \odot \mathcal{H}) - V(G)$, is a paired dominating set of $(G \odot \mathcal{H})_{uw,3}$ of cardinality $\gamma_{pr}(G \odot \mathcal{H})$.

Since subdividing any edge of $G \odot \mathcal{H}$ with three vertices does not increase the paired domination number of $G \odot \mathcal{H}$, we conclude that if for each vertex $u \in V(G)$ the graph G has a maximum matching with u unmatched, then $\mathrm{msd}_{pr}(G \odot \mathcal{H}) = 4$ when every $H_i \in \mathcal{H}$ is isomorphic to K_1 (recall that $\mathrm{msd}_{pr}(G) \leq 4$ for any graph G).

Now consider the case when $Z\mathcal{H} = \{K_1, K_2\}$. Let $e = uv$ be any edge of $G \odot \mathcal{H}$. If $u, v \in V(G)$ or if $u \in V(G)$ and v has degree one in $G \odot \mathcal{H}$, then by similar reasoning as in previous case we obtain that $\gamma_{pr}((G \odot \mathcal{H})_{uv,2}) = \gamma_{pr}(G \odot \mathcal{H})$. Assume now u is adjacent to a graph $K_2 \in \mathcal{H}$ in $G \odot \mathcal{H}$. Denote $V(K_2) = \{x, y\}$. Observe that the graphs $(G \odot \mathcal{H})_{ux,2}$, $(G \odot \mathcal{H})_{xy,2}$ and $(G \odot \mathcal{H})_{yu,2}$ are isomorphic. In each such a graph the subdivision vertices altogether with x, y induce a path P_4. Therefore it is possible to construct a paired dominating set D of each such a graph which contains the vertices of the maximum matching in G with u unmatched altogether with two inner vertices of the path P_4. Then $|D| = \gamma_{pr}(G \odot \mathcal{H})$. We conclude that in this case $\mathrm{msd}_{pr}(G \odot \mathcal{H}) \geq 3$.

On the other hand, consider again a vertex $u \in V(G)$ adjacent to K_2 in $G \odot \mathcal{H}$ and denote $V(K_2) = \{x, y\}$. Observe that the graphs $(G \odot \mathcal{H})_{ux,3}$, $(G \odot \mathcal{H})_{xy,3}$ and $(G \odot \mathcal{H})_{yu,3}$ are isomorphic and, in each such a graph the subdivision vertices altogether with x, y induce a path P_5. Thus, no two paired vertices may dominate the subdivision vertices plus x, y. Therefore the minimum paired dominating set of $(G \odot \mathcal{H})_{yu,3}$ is greater than $\gamma_{pr}(G \odot \mathcal{H})$, implying that $\mathrm{msd}_{pr}(G \odot \mathcal{H}) = 3$ for the case when there exists $H_j \in \mathcal{H}$ isomorphic to K_2 and every $H_i \in \mathcal{H} - H_j$ is isomorphic to K_1 or K_2.

At last consider the case when at least one element of $\mathcal{H}$ contains at least three vertices or when $\mathcal{H}$ contains $\overline{K_2}$. Let $e = uv$ be any edge of $G \odot \mathcal{H}$. If $u, v \in V(G)$ or if $u \in V(G)$ and v is a vertex of a K_1 or K_2 of $\mathcal{H}$, then by similar reasoning as in previous cases we obtain that $\gamma_{pr}((G \odot \mathcal{H})_{uv,1}) = \gamma_{pr}(G \odot \mathcal{H})$. Thus assume $u \in V(G)$ is adjacent to $H_i \in \mathcal{H}$ with at least three vertices or $H_i = \overline{K_2}$. It is no problem to verify that the paired domination number does not increase when we subdivide once any edge incident with a vertex of $V(H_i)$. We conclude that in this case $\mathrm{msd}_{pr}(G \odot \mathcal{H}) \geq 2$.

On the other hand, consider the edge uu', where $u \in V(G)$ is adjacent to each vertex of $H_i \in \mathcal{H}$, such that H_i has least three vertices or $H_i = \overline{K_2}$. Then every minimum paired dominating set of $(G \odot \mathcal{H})_{uu',2}$ contains u, a vertex paired with u and the vertices of the graph induced by a maximum matching in G with u unmatched. However, in this case u' does not have any neighbor in such a set. Therefore, the minimum paired dominating set of $(G \odot \mathcal{H})_{uu',2}$ has cardinality greater than $\gamma_{pr}(G \odot \mathcal{H})$, implying that $\mathrm{msd}_{pr}(G \odot \mathcal{H}) = 2$ for $\mathcal{H}$ containing a graph with at least three vertices or containing $\overline{K_2}$. $\square$

Theorem 3.5. *Let G be a connected graph of order $n \geq 2$ such that for each vertex $u \in V(G)$, there exists a maximum matching in G with u unmatched and let $\mathcal{H}$ be a sequence of n graphs. Then,*

$$\mathrm{sd}_{pr}(G \odot \mathcal{H}) = 2.$$

Proof. Let G be a connected graph of order $n \geq 2$ such that for each vertex $u \in V(G)$ there exists a maximum matching in G with u unmatched and let $\mathcal{H}$ be a family of n graphs. By Theorem 3.4, $\mathrm{msd}_{pr}(G \odot \mathcal{H}) > 1$, so $\mathrm{sd}_{pr}(G \odot \mathcal{H}) > 1$. By Lemma 3.2, there exists a minimum paired dominating set of $G \odot \mathcal{H}$ with u unmatched.

Let $u, v \in V(G)$ and let u', v' be vertices not belonging to $V(G)$ and adjacent to u and v, respectively, in $G \odot \mathcal{H}$. Denote by $(G \odot \mathcal{H})_{uu',vv'}$ the graph obtained from $G \odot \mathcal{H}$ by subdividing edges uu', vv'. Then every minimum paired dominating set of $(G \odot \mathcal{H})_{uu',vv'}$ contains every subdivision vertex. Thus, $\gamma_{pr}((G \odot \mathcal{H})_{uu',vv'}) > \gamma_{pr}(G \odot \mathcal{H})$. Therefore $\mathrm{sd}_{pr}(G \odot \mathcal{H}) = 2$. $\square$

4. INDEPENDENT DOMINATION

A set D is an *independent set* if the subgraph induced by D has no edges. The maximum cardinality of an independent set in G is the *independence number* and it is denoted by $\alpha(G)$. In this sense, the set D is an *independent dominating set*, if D is independent and dominating in G. The minimum cardinality of an independent dominating set in G is the *independent domination number* and is denoted by $i(G)$. The set D is a $i(G)$-set, if it is an independent dominating set of cardinality $i(G)$.

Now, the *independent domination subdivision number*, $\mathrm{sd}_i(G)$, of a graph G is the minimum number of edges which must be subdivided (where each edge can be subdivided at most once) in order to increase the independent domination number. Analogously, we only consider independent domination subdivision number for connected graphs of order at least three. The *independent domination multisubdivision number* of G, $\mathrm{msd}_i(G)$, is the minimum k such that there exists an edge uv of G satisfying that $i(G_{uv,k}) > i(G)$.

The following result will be useful in this section.

Lemma 4.1. *Let G be a graph of order n and let $\mathcal{H}$ be a family of n graphs. Then*

$$i(G \odot \mathcal{H}) \geq \alpha(G) + (n - \alpha(G)) \min\{i(H_i) \,:\, H_i \in \mathcal{H}\}$$

and

$$i(G \odot \mathcal{H}) \leq \alpha(G) + (n - \alpha(G)) \max\{i(H_i) \,:\, H_i \in \mathcal{H}\}.$$

Proof. We consider an $i(G \odot \mathcal{H})$-set S. Hence, if $u_i \in V(G) \cap S$, then for every $v \in V(H_i)$ we have that $v \notin S$. Also, if $u_i \notin V(G) \cap S$, then there exists $S_i \subset V(H_i)$, such that $S_i \subset S$ and $|S_i| \geq i(H_i) \geq \min\{i(H_i) \,:\, H_i \in \mathcal{H}\}$. As a consequence, there exists an independent set $A \subset V(G)$ and t independent dominating sets $S_i \subset V(H_i)$ in H_i, such that $n = t + |A|$, $|A| \leq \alpha(G)$ and $S = (\bigcup_{i=1}^{t} S_i) \cup A$. Thus, $t \geq n - \alpha(G)$ and the lower bound is obtained as follows.

$$\begin{aligned}
|S| &= |A| + \sum_{i=1}^{t} |S_i| = n - t + \sum_{i=1}^{t} |S_i| \\
&\geq n - t + t \cdot \min\{i(H_i) \,:\, H_i \in \mathcal{H}\} \\
&= n + t(\min\{i(H_i) \,:\, H_i \in \mathcal{H}\} - 1) \\
&\geq n + (n - \alpha(G))(\min\{i(H_i) \,:\, H_i \in \mathcal{H}\} - 1) \\
&= \alpha(G) + (n - \alpha(G)) \min\{i(H_i) \,:\, H_i \in \mathcal{H}\}.
\end{aligned}$$

On the other hand, let A be an independent set of maximum cardinality in G and for every $u_i \in \overline{A}$, let $S_i \subset V(H_i)$ be an independent dominating set in H_i. It is straightforward to observe that the set $S = A \cup (\bigcup_{u_i \in \overline{A}} S_i)$ is independent and dominating in $G \odot \mathcal{H}$. Therefore,

$$i(G \odot \mathcal{H}) \leq |A| \cup \left(\bigcup_{u_i \in \overline{A}} |S_i| \right) \leq \alpha(G) + (n - \alpha(G)) \max\{i(H_i) \,:\, H_i \in \mathcal{H}\},$$

and the upper bound follows. $\square$

Notice that the above result leads to the following consequence, for the case in which the whole graphs of the family $\mathcal{H}$ are isomorphic between them. Such a result was previously presented in [8].

Theorem 4.2 ([8])**.** *For any connected graph G of order n and for any graph H,*

$$i(G \odot H) = \alpha(G) + (n - \alpha(G))i(H).$$

Corollary 4.3. *For any connected graph G of order n and any graph H having one simplicial vertex* (*its neighbors form a clique*), *$i(G \odot H) = n$.*

Some examples of the above result are $i(G \odot K_r) = n$ and $i(G \odot S_{1,r}) = n$.

In contrast with the standard domination and the paired domination, the influence of the edge (subdivision or multisubdivision) over the independent domination number of a graph seems to be quite difficult to settle. This is related with the fact that the subdivision or multisubdivision of an edge in a generalized corona graph $G \odot \mathcal{H}$ can decrease or increase the independent domination number $i(G \odot \mathcal{H})$. For instance, take the graph $K_4 \odot P_4$. From 4.2 we have that $i(K_4 \odot P_4) = 7$. Now, the subdivision of any edge of any copy of P_4 or any edge connecting K_4 with a copy of P_4 does not change $i(K_4 \odot P_4) = 7$, while the subdivision of any edge of K_4 makes that the obtained graph has independent domination number equal to six. Hence, in this section, we are only centered in the case of corona graphs $G \odot K_1$.

4.1. INDEPENDENT DOMINATION SUBDIVISION NUMBER

A set X is a *clique* in a graph G, if the subgraph induced by X is isomorphic to a complete graph. A clique X is maximum if it has the maximum cardinality among all cliques in G and its cardinality is the *clique number*.

Proposition 4.4. *If G is a connected graph of order $n \geq 2$, then*

$$\mathrm{sd}_i(G \odot K_1) = \begin{cases} 3, & \text{if the clique number of } G \text{ is at most three,} \\ 2, & \text{otherwise.} \end{cases}$$

Proof. First notice that it is straightforward to check that $\mathrm{sd}_i(G \odot K_1) > 1$. Thus, $\mathrm{sd}_i(G \odot K_1) \geq 2$. Now, for every $i \in \{1, \ldots, n\}$, let u_i be a vertex of G and let v_i be the corresponding pendant vertex of u_i.

We consider the case that G has clique number greater than three. Let S be a maximum clique of G and let four different vertices $u_i, u_j, u_k, u_l \in S$. We will subdivide the edges u_iu_j and u_ku_l with vertices x and y, respectively. Notice that to independently dominate the vertex x we need at least one of the vertices u_i or u_j. Analogously, to independently dominate the vertex y we need at least one of the vertices u_k or u_l. Now, since the vertices v_i, v_j, v_k, v_l must be also independently dominated and u_i, u_j, u_k, u_l form a clique in G, it is clear that $i((G \odot K_1)_{u_iu_j,u_ku_l}) > i(G \odot K_1)$ and we have that $\mathrm{sd}_i(G \odot K_1) \leq 2$. Therefore, if G has clique number greater than three, then $\mathrm{sd}_i(G \odot K_1) = 2$.

Now assume that G has clique number at most three. We shall prove that $\mathrm{sd}_i(G \odot K_1) \leq 3$, Let e_1, e_2, e_3 be three different edges of $G \odot K_1$ such that $e_1 = u_i u_j$, $e_2 = u_i v_i$ and $e_3 = u_j v_j$, where $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$. We subdivide them with the vertices x, y, z, respectively. So, notice that the set $D = \{v_1, \ldots, v_n\} - \{v_i, v_j\} \cup \{x, y, z\}$ is an $i((G \odot K_1)_{e_1,e_2,e_3})$-set and $|D| = n + 1 > n = i(G \odot K_1)$.

Now, we will prove that subdividing any two different edges e_1, e_2 does not change the independent domination number of the graph $G \odot K_1$. Let us subdivide these edges with the vertices x and y, respectively. We consider the following cases.

Case 1. If $e_1 = u_i v_i$ and $e_2 = u_k v_k$, where $i \neq k$ and $i, k \in \{1, 2, \ldots, n\}$, then $D = \{v_1, \ldots, v_n\} - \{v_i, v_k\} \cup \{x, y\}$ is an $i((G \odot K_1)_{e_1,e_2})$-set and $|D| = i(G \odot K_1)$.

Case 2. If $e_1 = u_i v_i$ and $e_2 = u_k u_l$, where $i, k, l \in \{1, 2, \ldots, n\}$ and $k \neq l$, then $i \neq k$ or $i \neq l$. Without loss of generality suppose $i \neq k$. Hence the set $D = \{v_1, \ldots v_n\} - \{v_i, v_k\} \cup \{x, u_k\}$ is an $i((G \odot K_1)_{e_1,e_2})$-set and $|D| = i(G \odot K_1)$.

Case 3. If $e_1 = u_i u_j$ and $e_2 = u_k u_l$, where $i, j, k, l \in \{1, 2, \ldots, n\}$ and $i \neq j$, $k \neq l$, then we consider the following subcases.

Subcase 3.1. If $j = k$, then $i \neq l$ and $D = \{v_1, \ldots, v_n\} - \{v_j\} \cup \{u_j\}$ is an $i((G \odot K_1)_{e_1,e_2})$-set and $|D| = i(G \odot K_1)$. Analogously, if $i = l$, then $j \neq k$ and $D = \{v_1, \ldots, v_n\} - \{v_i\} \cup \{u_i\}$ is an $i((G \odot K_1)_{e_1,e_2})$-set and $|D| = i(G \odot K_1)$.

Subcase 3.2. If $i \neq l$ and $j \neq k$, then since G has clique number at most three, we have that $u_i \not\sim u_k$ or $u_i \not\sim u_l$ or $u_j \not\sim u_k$ or $u_j \not\sim u_l$; say for instance, $u_i \not\sim u_k$. Thus, the set $D = \{v_1, \ldots, v_n\} - \{v_i, v_k\} \cup \{u_i, u_k\}$ is an $i((G \odot K_1)_{e_1,e_2})$-set and $|D| = i(G \odot K_1)$.

As a consequence of the above cases, we obtain that $\mathrm{sd}(G \odot K_1) = 3$. □

4.2. INDEPENDENCE DOMINATION MULTISUBDIVISION NUMBER

Theorem 4.5. *If G is a connected graph of order $n \geq 2$, then*

$$\mathrm{msd}_i(G \odot K_1) = 3.$$

Proof. First we prove that $\mathrm{msd}_i(G \odot K_1) \leq 3$. To do so, it is only necessary to subdivide three times a pendant edge uv of $G \odot K_1$ with the vertices x_1, x_2, x_3 and we obtain a graph $(G \odot K_1)_{uv,3}$ such that $i((G \odot K_1)_{uv,l}) = n + 1$, since to independently dominate the vertices u, x_1, x_2, x_3, v are necessary two vertices and we also need one vertex $n - 1$ vertices to independently dominate the rest $n - 1$ vertices of K_1.

On the other hand, suppose that $\mathrm{msd}_i(G \odot K_1) \leq 2$. So, there exists an edge uv of $G \odot K_1$ and an integer $l \in \{1, 2\}$ such that $i((G \odot K_1)_{uv,l}) > i(G \odot K_1)$. Notice that the whole set S of pendant vertices of $G \odot K_1$ (the vertices of all the copies of K_1) is an $i(G \odot K_1)$-set. We consider the following cases.

Case 1. uv is an edge of G. Let u' and v' be the pendant vertices of u and v, respectively. Hence, we have that the set $S' = S - \{u', v'\} \cup \{u, v\}$ is an independent dominating set in $(G \odot K_1)_{uv,l}$ with cardinality $i(G \odot K_1)$. Thus $i((G \odot K_1)_{uv,l}) = i(G \odot K_1)$, a contradiction.

Case 2. uv is a pendant edge of G, say u is a vertex of G and v is the vertex of K_1. We consider the following subcases.

Subcase 2.1. $l = 1$. Let x be the vertex used to subdivide the edge uv. Hence the set $S' = S - \{v\} \cup \{x\}$ is an independent dominating set in $(G \odot K_1)_{uv,l}$ with cardinality $i(G \odot K_1)$, a contradiction.
Subcase 2.2. $l = 2$. Let x_1, x_2 be the vertices used to subdivide the edge uv such that $u \sim x_1 \sim x_2 \sim v$. Let w be a neighbor of u in G and let w' be the pendant vertex adjacent to w. Hence the set $S' = S - \{w', v\} \cup \{x_2, w\}$ is an independent dominating set in $(G \odot K_1)_{uv,l}$ with cardinality $i(G \odot K_1)$, a contradiction again.

Therefore, for any edge xy of $G \odot K_1$, $i((G \odot K_1)_{xy,l}) = i(G \odot K_1)$, with $l \in \{1, 2\}$ and, as a consequence, $\mathrm{msd}_i(G \odot K_1) = 3$. □

Acknowledgments
The authors thank the referees for their valuable comments that helped significantly to improve the paper.

REFERENCES

[1] H. Aram, S.M. Sheikholeslami, O. Favaron, *Domination subdivision number of trees*, Discrete Math. **309** (2009), 622–628.

[2] S. Benecke, C.M. Mynhardt, *Trees with domination subdivision number one*, Australas. J. Combin. **42** (2008), 201–209.

[3] A. Bhattacharya, G.R. Vijayakumar, *Effect of edge–subdivision on vertex–domination in a graph*, Discuss. Math. Graph Theory **22** (2002), 335–347.

[4] M. Dettlaff, J. Raczek, J. Topp, *Domination subdivision and domination multisubdivision numbers of graphs*, arXiv:1310.1345 [math.CO], submitted.

[5] O. Favaron, T.W. Haynes, S.T. Hedetniemi, *Domination subdivision numbers in graphs*, Util. Math. **66** (2004), 195–209.

[6] O. Favaron, H. Karami, S.M. Sheikholeslami, *Disproof of a conjecture on the subdivision domination number of a graph*, Graphs Combin. **24** (2008), 309–312.

[7] J.F. Fink, M.S. Jacobson, L.F. Kinch, J. Roberts, *On graphs having domination number half their order*, Period. Math. Hungar. **16** (1985) 4, 287–293.

[8] I. González Yero, D. Kuziak, A. Rondón Aguilar, *Coloring, location and domination of corona graphs*, Aequationes Math. **86** (2013), 1–21.

[9] T.W. Haynes, P.J. Slater, *Paired-domination in graphs*, Networks **32** (1998), 199–206.

[10] T.W. Haynes, S.M. Hedetniemi, S.T. Hedetniemi, *Domination and independence subdivision numbers of graphs*, Discuss. Math. Graph Theory **20** (2000), 271–280.

[11] T.W. Haynes, S.M. Hedetniemi, S.T. Hedetniemi, L.C. van der Merwe, *Domination subdivision numbers*, Discuss. Math. Graph Theory **21** (2001), 239–253.

[12] T.W. Haynes, S.T. Hedetniemi, L.C. van der Merwe, *Total domination subdivision numbers*, J. Combin. Math. Combin. Comput. **44** (2003), 115–128.

[13] C. Payan, N.H. Xuong, *Domination-balanced graphs*, J. Graph Theory **6** (1982) 1, 23–32.

[14] J. Raczek, M. Dettlaff, *Paired domination subdivision and multisubdivision numbers of graphs*, submitted.

[15] S. Velammal, *Studies in graph theory: covering, independence, domination and related topics*, Ph.D. Thesis, Manonmaniam Sundaranar University, Tirunelveli, 1997.

Magda Dettlaff
mdettlaff@mif.pg.gda.pl

Gdańsk University of Technology
Faculty of Applied Physics and Mathematics
ul. Narutowicza 11/12, 80-233 Gdańsk, Poland

Joanna Raczek
Joanna.Raczek@pg.gda.pl

Gdańsk University of Technology
Faculty of Applied Physics and Mathematics
ul. Narutowicza 11/12, 80-233 Gdańsk, Poland

Ismael G. Yero
ismael.gonzalez@uca.es

Universidad de Cádiz,
Escuela Politécnica Superior de Algeciras
Departamento de Matemáticas
Av. Ramón Puyol, s/n, 11202 Algeciras, Spain

Received: December 2, 2015.
Revised: May 8, 2016.
Accepted: May 10, 2016.