Electric-field-induced magnetic quadrupole moment in the ground state of the relativistic hydrogenlike atom: Application of the Sturmian expansion of the generalized Dirac-Coulomb Green function

Radosław Szmytkowski* and Patrycja Stefańska
Atomic Physics Division, Department of Atomic, Molecular and Optical Physics, Faculty of Applied Physics and Mathematics, Gdańsk University of Technology, Narutowicza 11/12, 80-233 Gdańsk, Poland
(Received 28 May 2013; published 6 January 2014)

We consider a Dirac one-electron atom placed in a weak, static, uniform electric field. We show that, to the first order in the strength of the perturbing field, the only magnetic multipole moment induced in the ground state of the atom is the quadrupole one. The tensorial structure of that moment is resolved. Using the Sturmian expansion of the generalized Dirac-Coulomb Green function [Szmytkowski, J. Phys. B **30**, 825 (1997)**30**, 2747(E) (1997)], we derive a closed-form expression for an $E1 \to M2$ cross-susceptibility of the atom in the ground state.

DOI: 10.1103/PhysRevA.89.012501 PACS number(s): 31.15.ap, 31.15.aj, 31.30.jc, 32.10.Dk

I. INTRODUCTION

Some years ago, we undertook a research program with the goal to carry out systematic analytical calculations of various parameters characterizing a response of a relativistic hydrogenlike atom to external electric and magnetic fields. The main tool used in these calculations has been a Sturmian series representation of the Dirac-Coulomb Green function found by one of us in Ref. [1]. Thus far, our calculations have been restricted to the case when the atom is in its ground state. We have succeeded to find expressions for the atomic static and dynamic electric dipole polarizabilities [1–3], the induced magnetic anapole moment [4], the dipole magnetizability [5], the electric and magnetic dipole shielding factors [6,7], and, most recently, the *electric* quadrupole moment induced in the electronic cloud of the atom by a weak, static, uniform *magnetic* field [8]. In the present work, which is a natural extension of the research reported in Ref. [8], we present calculations of the *magnetic* quadrupole moment induced in the ground state of the Dirac one-electron atom by a weak, time-independent, uniform *electric* field. We are not aware of any previous calculations of that quantity.

The plan of the paper is as follows. In Sec. II, we recall basic relevant facts concerning the ground state of the Dirac one-electron atom perturbed by a weak, static, uniform electric field. In Sec. III, an analysis of atomic magnetic multipole moments is carried out. We show that in an unperturbed state the only nonvanishing magnetic multipole moment is the dipole one, while the electric field induces the magnetic quadrupole moment only. In Sec. IV, first, we resolve a structure of the induced quadrupole moment tensor, and then we calculate an atomic $E1 \to M2$ cross-susceptibility.

II. PRELIMINARIES

It has been already stated in the introduction that the system to be studied in this work is the Dirac one-electron atom placed in a weak, time-independent, uniform electric field $\boldsymbol{F}$. In what follows, we shall be assuming that the atomic nucleus is motionless, pointlike and spinless, and that its electric charge is $+Ze$. Before the field has been switched on, the atom was in its ground state. The external electric field is assumed to be so weak that the probability that the field ionization of the atom occurs is negligibly small.

The time-independent Dirac equation for the atomic electron is then

$$\left[-ic\hbar\boldsymbol{\alpha}\cdot\nabla+\beta mc^2-\frac{Ze^2}{(4\pi\epsilon_0)r}+e\boldsymbol{F}\cdot\boldsymbol{r}-E\right]\Psi(\boldsymbol{r})=0, \tag{2.1}$$

where $\boldsymbol{\alpha}$ and β are the standard Dirac matrices. To the lowest order in the perturbing field, the energy eigenvalue is

$$E \simeq E^{(0)} + E^{(1)}. \tag{2.2}$$

Here

$$E^{(0)} = mc^2\gamma_1, \tag{2.3}$$

with

$$\gamma_\kappa = \sqrt{\kappa^2-(\alpha Z)^2} \tag{2.4}$$

(α, not to be confused with the Dirac matrix $\boldsymbol{\alpha}$, is the Sommerfeld fine-structure constant), is the ground-state energy level of the isolated atom, whereas the first-order correction to energy appears to vanish:

$$E^{(1)} = 0. \tag{2.5}$$

To the same order in the perturbation, the electronic wave function is

$$\Psi(\boldsymbol{r}) \simeq \Psi^{(0)}(\boldsymbol{r}) + \Psi^{(1)}(\boldsymbol{r}), \tag{2.6}$$

with the unperturbed component given by

$$\Psi^{(0)}(\boldsymbol{r}) = a_{1/2}\Psi^{(0)}_{1/2}(\boldsymbol{r}) + a_{-1/2}\Psi^{(0)}_{-1/2}(\boldsymbol{r}), \tag{2.7}$$

the coefficients $a_{\pm 1/2}$ being subjected to the constraint

$$|a_{1/2}|^2 + |a_{-1/2}|^2 = 1. \tag{2.8}$$

The two basis states appearing in Eq. (2.7), orthonormal in the sense of

$$\int_{\mathbb{R}^3} d^3\boldsymbol{r}\ \Psi^{(0)\dagger}_{\mu}(\boldsymbol{r})\Psi^{(0)}_{\mu'}(\boldsymbol{r}) = \delta_{\mu\mu'} \quad \left(\mu,\mu' = \pm\frac{1}{2}\right), \tag{2.9}$$

*radek@mif.pg.gda.pl

are given by

$$\Psi_{\mu}^{(0)}(\boldsymbol{r}) = \frac{1}{r}\begin{pmatrix} P^{(0)}(r)\Omega_{-1\mu}(\boldsymbol{n}_r) \\ i\,Q^{(0)}(r)\Omega_{1\mu}(\boldsymbol{n}_r)\end{pmatrix} \quad \left(\mu = \pm\frac{1}{2}\right), \tag{2.10}$$

where $\Omega_{\kappa\mu}(\boldsymbol{n}_r)$ are the spherical spinors [9] with the quantization axis chosen along the electric field direction, while the radial functions are

$$P^{(0)}(r) = -\sqrt{\frac{Z}{a_0}\frac{1+\gamma_1}{\Gamma(2\gamma_1+1)}}\left(\frac{2Zr}{a_0}\right)^{\gamma_1} e^{-Zr/a_0}, \tag{2.11a}$$

$$Q^{(0)}(r) = \sqrt{\frac{Z}{a_0}\frac{1-\gamma_1}{\Gamma(2\gamma_1+1)}}\left(\frac{2Zr}{a_0}\right)^{\gamma_1} e^{-Zr/a_0}. \tag{2.11b}$$

It is easy to verify that the radial functions (2.11) are normalized to unity in the sense of

$$\int_0^{\infty} dr\left\{[P^{(0)}(r)]^2 + [Q^{(0)}(r)]^2\right\} = 1. \tag{2.12}$$

Since the perturbation $e\boldsymbol{F}\cdot\boldsymbol{r}$ does not couple the two states $\Psi_{\pm 1/2}^{(0)}(\boldsymbol{r})$, the coefficients $a_{\pm 1/2}$ in Eq. (2.7) remain undetermined except for being forced to obey the normalization constraint (2.8). The correction $\Psi^{(1)}(\boldsymbol{r})$ is of the form

$$\Psi^{(1)}(\boldsymbol{r}) = a_{1/2}\Psi_{1/2}^{(1)}(\boldsymbol{r}) + a_{-1/2}\Psi_{-1/2}^{(1)}(\boldsymbol{r}), \tag{2.13}$$

where the coefficients $a_{\pm 1/2}$ are the same as in Eq. (2.7), while the functions $\Psi_{\pm 1/2}^{(1)}(\boldsymbol{r})$ solve the inhomogeneous equation

$$\left[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta mc^2 - \frac{Ze^2}{(4\pi\epsilon_0)r} - E^{(0)}\right]\Psi_{\mu}^{(1)}(\boldsymbol{r}) = -\left[e\boldsymbol{F}\cdot\boldsymbol{r} - E_{\mu}^{(1)}\right]\Psi_{\mu}^{(0)}(\boldsymbol{r}) \quad \left(\mu = \pm\frac{1}{2}\right) \tag{2.14}$$

(with $E_{\mu}^{(1)} = 0$) and are subjected to the orthogonality constraints

$$\int_{\mathbb{R}^3} d^3\boldsymbol{r}\;\Psi_{\mu}^{(0)\dagger}(\boldsymbol{r})\Psi_{\mu'}^{(1)}(\boldsymbol{r}) = 0 \quad \left(\mu,\mu' = \pm\frac{1}{2}\right). \tag{2.15}$$

Explicitly the functions $\Psi_{\pm 1/2}^{(1)}(\boldsymbol{r})$ are given by

$$\Psi_{\mu}^{(1)}(\boldsymbol{r}) = -\int_{\mathbb{R}^3} d^3\boldsymbol{r}'\;\bar{G}^{(0)}(\boldsymbol{r},\boldsymbol{r}')e\boldsymbol{F}\cdot\boldsymbol{r}'\Psi_{\mu}^{(0)}(\boldsymbol{r}') \quad \left(\mu = \pm\frac{1}{2}\right), \tag{2.16}$$

where $\bar{G}^{(0)}(\boldsymbol{r},\boldsymbol{r}')$ is the generalized Dirac-Coulomb Green function associated with the ground-state energy level (2.3) of the isolated atom.

III. MAGNETIC MULTIPOLE MOMENTS OF THE ATOM IN THE ELECTRIC FIELD

A. Decomposition of the atomic multipole magnetic moments into the permanent and the first-order induced components

For a given electric current distribution of density $\boldsymbol{j}(\boldsymbol{r})$, spherical components of the magnetic 2^L-pole moment tensor due to that current are defined as[1]

$$\mathcal{M}_{LM} = -\frac{1}{L+1}\sqrt{\frac{4\pi}{2L+1}}\int_{\mathbb{R}^3} d^3\boldsymbol{r}\; r^L Y_{LM}(\boldsymbol{n}_r)\boldsymbol{\nabla}\cdot[\boldsymbol{r}\times\boldsymbol{j}(\boldsymbol{r})], \tag{3.1}$$

where $Y_{LM}(\boldsymbol{n}_r)$ is the normalized spherical harmonic. (As various phase conventions for the spherical harmonics are used in the literature, we emphasize that in this paper we adopt the Condon-Shortley phase convention; Ref. [11], Chap. 5.) It is easy to see that Eq. (3.1) can be rewritten as

$$\mathcal{M}_{LM} = \frac{i}{L+1}\sqrt{\frac{4\pi}{2L+1}}\int_{\mathbb{R}^3} d^3\boldsymbol{r}\; r^L Y_{LM}(\boldsymbol{n}_r)\boldsymbol{\Lambda}\cdot\boldsymbol{j}(\boldsymbol{r}), \tag{3.2}$$

with

$$\boldsymbol{\Lambda} = -i\boldsymbol{r}\times\boldsymbol{\nabla}. \tag{3.3}$$

For the purposes of the present paper, the representation of $\mathcal{M}_{LM}$ given by Eq. (3.2) is superior to the one in Eq. (3.1).

In the case under consideration when the system is the Dirac one-electron atom in the static, uniform, and weak electric field, characterized briefly in the preceding section, the current density is

$$\boldsymbol{j}(\boldsymbol{r}) = -ec\Psi^{\dagger}(\boldsymbol{r})\boldsymbol{\alpha}\Psi(\boldsymbol{r}), \tag{3.4}$$

provided the wave function $\Psi(\boldsymbol{r})$ is normalized to unity:

$$\int_{\mathbb{R}^3} d^3\boldsymbol{r}\;\Psi^{\dagger}(\boldsymbol{r})\Psi(\boldsymbol{r}) = 1. \tag{3.5}$$

Equations (3.4) and (2.6), together with the orthogonality constraint (2.15), imply that to the first order in the perturbing electric field the current $\boldsymbol{j}(\boldsymbol{r})$ can be approximated as

$$\boldsymbol{j}(\boldsymbol{r}) \simeq \boldsymbol{j}^{(0)}(\boldsymbol{r}) + \boldsymbol{j}^{(1)}(\boldsymbol{r}), \tag{3.6}$$

where

$$\boldsymbol{j}^{(0)}(\boldsymbol{r}) = -ec\Psi^{(0)\dagger}(\boldsymbol{r})\boldsymbol{\alpha}\Psi^{(0)}(\boldsymbol{r}) \tag{3.7}$$

and

$$\boldsymbol{j}^{(1)}(\boldsymbol{r}) = -2ec\,\mathrm{Re}\left[\Psi^{(0)\dagger}(\boldsymbol{r})\boldsymbol{\alpha}\Psi^{(1)}(\boldsymbol{r})\right] \tag{3.8}$$

are the unperturbed and the first-order induced current distributions, respectively. The decomposition (3.6) gives rise to the analogous splitting

$$\mathcal{M}_{LM} \simeq \mathcal{M}_{LM}^{(0)} + \mathcal{M}_{LM}^{(1)} \tag{3.9}$$

of components of the multipole moment into the unperturbed

$$\mathcal{M}_{LM}^{(0)} = \frac{i}{L+1}\sqrt{\frac{4\pi}{2L+1}}\int_{\mathbb{R}^3} d^3\boldsymbol{r}\; r^L Y_{LM}(\boldsymbol{n}_r)\boldsymbol{\Lambda}\cdot\boldsymbol{j}^{(0)}(\boldsymbol{r}) \tag{3.10}$$

[1] There is not a unique definition of the magnetic multipole moments in the physical literature. In some definitions the factor $\sqrt{4\pi/(2L+1)}$ is omitted, in other the harmonic $Y_{LM}(\boldsymbol{n}_r)$ is replaced by its complex conjugate [in Ref. [10], Sec. 16.5, both these deviations from the definition (3.1) occur simultaneously].

and the first-order induced

$$\mathcal{M}^{(1)}_{LM} = \frac{i}{L+1}\sqrt{\frac{4\pi}{2L+1}}\int_{\mathbb{R}^3} d^3\boldsymbol{r}\, r^L Y_{LM}(\boldsymbol{n}_r)\boldsymbol{\Lambda}\cdot\boldsymbol{j}^{(1)}(\boldsymbol{r}) \tag{3.11}$$

constituents.

B. Analysis of permanent multipole magnetic moments of the atom

At first, we shall focus our interest on the unperturbed moment $\mathcal{M}^{(0)}_{LM}$ given in Eq. (3.10). In view of Eqs. (3.7) and (2.7), it can be written as

$$\mathcal{M}^{(0)}_{LM} = \sum_{\mu,\mu'=\pm 1/2} a^*_\mu a_{\mu'}\mathcal{M}^{(0)}_{LM,\mu\mu'}, \tag{3.12}$$

with

$$\mathcal{M}^{(0)}_{LM,\mu\mu'} = -\frac{i}{L+1}\sqrt{\frac{4\pi}{2L+1}}\, ec\int_{\mathbb{R}^3} d^3\boldsymbol{r}\, r^L Y_{LM}(\boldsymbol{n}_r)\boldsymbol{\Lambda} \cdot\big[\Psi^{(0)\dagger}_\mu(\boldsymbol{r})\boldsymbol{\alpha}\Psi^{(0)}_{\mu'}(\boldsymbol{r})\big]. \tag{3.13}$$

Now, recalling the definition (3.3) and exploiting the fact that the matrix $\boldsymbol{\alpha}$ is Hermitian, we have

$$\boldsymbol{\Lambda}\cdot\big[\Psi^{(0)\dagger}_\mu(\boldsymbol{r})\boldsymbol{\alpha}\Psi^{(0)}_{\mu'}(\boldsymbol{r})\big] = -\big[\boldsymbol{\alpha}\cdot\boldsymbol{\Lambda}\Psi^{(0)}_\mu(\boldsymbol{r})\big]^\dagger\Psi^{(0)}_{\mu'}(\boldsymbol{r}) + \Psi^{(0)\dagger}_\mu(\boldsymbol{r})\boldsymbol{\alpha}\cdot\boldsymbol{\Lambda}\Psi^{(0)}_{\mu'}(\boldsymbol{r}). \tag{3.14}$$

The expression on the right-hand side of the above equation can be simplified with the aid of Eq. (2.10) and of the identity [Ref. [9], Eq. (3.2.3)]

$$\boldsymbol{\sigma}\cdot\boldsymbol{\Lambda}\Omega_{\kappa\mu}(\boldsymbol{n}_r) = -(\kappa+1)\Omega_{\kappa\mu}(\boldsymbol{n}_r). \tag{3.15}$$

This casts Eq. (3.14) into

$$\boldsymbol{\Lambda}\cdot\big[\Psi^{(0)\dagger}_\mu(\boldsymbol{r})\boldsymbol{\alpha}\Psi^{(0)}_{\mu'}(\boldsymbol{r})\big] = -2ir^{-2}P^{(0)}(r)Q^{(0)}(r)[\Omega^\dagger_{1\mu}(\boldsymbol{n}_r)\Omega_{-1\mu'}(\boldsymbol{n}_r) + \Omega^\dagger_{-1\mu}(\boldsymbol{n}_r)\Omega_{1\mu'}(\boldsymbol{n}_r)]. \tag{3.16}$$

Hence, it follows that

$$\mathcal{M}^{(0)}_{LM,\mu\mu'} = -\frac{2}{L+1}\sqrt{\frac{4\pi}{2L+1}}\, ec[\langle\Omega_{1\mu}|Y_{LM}\Omega_{-1\mu'}\rangle + \langle\Omega_{-1\mu}|Y_{LM}\Omega_{1\mu'}\rangle]\int_0^\infty dr\, r^L P^{(0)}(r)Q^{(0)}(r), \tag{3.17}$$

where the shorthand bracket notation

$$\langle\Omega_{\kappa\mu}|Y_{LM}\Omega_{\kappa'\mu'}\rangle \equiv \oint_{4\pi} d^2\boldsymbol{n}_r\, \Omega^\dagger_{\kappa\mu}(\boldsymbol{n}_r)Y_{LM}(\boldsymbol{n}_r)\Omega_{\kappa'\mu'}(\boldsymbol{n}_r) \tag{3.18}$$

has been used for the angular integrals. Exploiting the identity [Ref. [9], Eq. (3.1.3)]

$$\boldsymbol{n}_r\cdot\boldsymbol{\sigma}\Omega_{\kappa\mu}(\boldsymbol{n}_r) = -\Omega_{-\kappa\mu}(\boldsymbol{n}_r), \tag{3.19}$$

it is easy to prove that

$$\langle\Omega_{-\kappa\mu}|Y_{LM}\Omega_{-\kappa'\mu'}\rangle = \langle\Omega_{\kappa\mu}|Y_{LM}\Omega_{\kappa'\mu'}\rangle, \tag{3.20}$$

and therefore Eq. (3.17) can be rewritten in the form

$$\mathcal{M}^{(0)}_{LM,\mu\mu'} = -\frac{4}{L+1}\sqrt{\frac{4\pi}{2L+1}}\, ec\langle\Omega_{1\mu}|Y_{LM}\Omega_{-1\mu'}\rangle \times\int_0^\infty dr\, r^L P^{(0)}(r)Q^{(0)}(r). \tag{3.21}$$

In view of Eqs. (2.11a) and (2.11b), the radial integral in the above equation is elementary. In turn, the angular integral can be evaluated from the general formula

$$\sqrt{\frac{4\pi}{2L+1}}\langle\Omega_{\kappa\mu}|Y_{LM}\Omega_{\kappa'\mu'}\rangle = (-)^{\mu+1/2}2\sqrt{|\kappa\kappa'|}\begin{pmatrix}|\kappa|-\frac{1}{2} & L & |\kappa'|-\frac{1}{2}\\ \frac{1}{2} & 0 & -\frac{1}{2}\end{pmatrix} \times\begin{pmatrix}|\kappa|-\frac{1}{2} & L & |\kappa'|-\frac{1}{2}\\ -\mu & M & \mu'\end{pmatrix}\Pi(l,L,l'), \tag{3.22}$$

where $\begin{pmatrix}j_a & j_b & j_c\\ m_a & m_b & m_c\end{pmatrix}$ denotes the Wigner's $3j$ coefficient and

$$\Pi(l,L,l') = \begin{cases}1 & \text{for } l+L+l' \text{ even}\\ 0 & \text{for } l+L+l' \text{ odd,}\end{cases} \tag{3.23}$$

with

$$l = \left|\kappa+\tfrac{1}{2}\right| - \tfrac{1}{2} \tag{3.24}$$

and similarly for l'. The result is

$$\sqrt{\frac{4\pi}{2L+1}}\langle\Omega_{1\mu}|Y_{LM}\Omega_{-1\mu'}\rangle = \frac{1}{3}\delta_{L1}(\sqrt{2}\,\delta_{M1}\delta_{\mu,1/2}\delta_{\mu',-1/2} - \delta_{M0}\delta_{\mu,1/2}\delta_{\mu',1/2} + \delta_{M0}\delta_{\mu,-1/2}\delta_{\mu',-1/2} - \sqrt{2}\,\delta_{M,-1}\delta_{\mu,-1/2}\delta_{\mu',1/2}). \tag{3.25}$$

Inserting this into Eq. (3.21), evaluating the radial integral with the aid of Eqs. (2.11) and plugging the final expression for $\mathcal{M}^{(0)}_{LM,\mu\mu'}$ into the double sum on the right-hand side of Eq. (3.12) yields

$$\mathcal{M}^{(0)}_{LM} = \mathcal{M}^{(0)}_{1M}\delta_{L1}, \tag{3.26}$$

with

$$\mathcal{M}^{(0)}_{10} = -\frac{1}{3}(2\gamma_1+1)\mu_{\mathrm{B}}(|a_{1/2}|^2 - |a_{-1/2}|^2), \tag{3.27a}$$

$$\mathcal{M}^{(0)}_{1,\pm1} = \pm\frac{\sqrt{2}}{3}(2\gamma_1+1)\mu_{\mathrm{B}}a^*_{\pm1/2}a_{\mp1/2}, \tag{3.27b}$$

where $\mu_{\mathrm{B}} = e\hbar/2m$ is the Bohr magneton. We have thus proved that in the ground state of the unperturbed atom the only nonvanishing magnetic multipole moment is the dipole one; its spherical components are given by Eqs. (3.27a) and (3.27b).

If the coefficients $a_{\pm1/2}$ are parametrized as

$$a_{1/2} = e^{i(\chi-\phi)/2}\cos(\vartheta/2), \quad a_{-1/2} = e^{i(\chi+\phi)/2}\sin(\vartheta/2) \quad (0\leq\chi,\phi<2\pi,\ 0\leq\vartheta\leq\pi), \tag{3.28}$$

then Eqs. (3.27) become

$$\mathcal{M}^{(0)}_{10} = -\frac{2\gamma_1+1}{3}\mu_{\mathrm{B}}\cos\vartheta, \tag{3.29a}$$

$$\mathcal{M}^{(0)}_{1,\pm1} = \pm\frac{2\gamma_1+1}{3\sqrt{2}}\mu_{\mathrm{B}}e^{\pm i\phi}\sin\vartheta. \tag{3.29b}$$

Switching to the Cartesian components

$$\mathcal{M}_{1x}^{(0)} = \frac{1}{\sqrt{2}}\left(\mathcal{M}_{1,-1}^{(0)} - \mathcal{M}_{11}^{(0)}\right), \tag{3.30a}$$

$$\mathcal{M}_{1y}^{(0)} = \frac{i}{\sqrt{2}}\left(\mathcal{M}_{1,-1}^{(0)} + \mathcal{M}_{11}^{(0)}\right), \tag{3.30b}$$

$$\mathcal{M}_{1z}^{(0)} = \mathcal{M}_{10}^{(0)}, \tag{3.30c}$$

we arrive at the well-known result that the magnetic dipole moment vector of the unperturbed atom can be written as

$$\boldsymbol{\mathcal{M}}_1^{(0)} = -\frac{2\gamma_1+1}{3}\mu_{\mathrm{B}}\boldsymbol{\nu}, \tag{3.31}$$

$\boldsymbol{\nu}$ being the unit vector

$$\boldsymbol{\nu} = \sin\vartheta\cos\phi\,\boldsymbol{n}_x + \sin\vartheta\sin\phi\,\boldsymbol{n}_y + \cos\vartheta\,\boldsymbol{n}_z. \tag{3.32}$$

C. Analysis of the first-order induced multipole magnetic moments

Next we proceed to the analysis of the first-order induced magnetic multipole moments. Insertion of Eqs. (3.8), (2.7), and (2.13) into Eq. (3.11) yields $\mathcal{M}_{LM}^{(1)}$ as the sum

$$\mathcal{M}_{LM}^{(1)} = \sum_{\mu,\mu'=\pm1/2} a_\mu^* a_{\mu'} \mathcal{M}_{LM,\mu\mu'}^{(1)}, \tag{3.33}$$

where

$$\mathcal{M}_{LM,\mu\mu'}^{(1)} = \widetilde{\mathcal{M}}_{LM,\mu\mu'}^{(1)} + (-)^M \widetilde{\mathcal{M}}_{L,-M,\mu'\mu}^{(1)*}, \tag{3.34}$$

with

$$\widetilde{\mathcal{M}}_{LM,\mu\mu'}^{(1)} = -\frac{i}{L+1}\sqrt{\frac{4\pi}{2L+1}}\,ec\int_{\mathbb{R}^3} d^3\boldsymbol{r}\, r^L Y_{LM}(\boldsymbol{n}_r)\boldsymbol{\Lambda} \cdot \left[\Psi_\mu^{(0)\dagger}(\boldsymbol{r})\boldsymbol{\alpha}\Psi_{\mu'}^{(1)}(\boldsymbol{r})\right]. \tag{3.35}$$

A further use of Eq. (2.16) casts Eq. (3.35) into the more explicit one

$$\widetilde{\mathcal{M}}_{LM,\mu\mu'}^{(1)} = \frac{i}{L+1}\frac{4\pi}{\sqrt{3(2L+1)}}e^2cF\int_{\mathbb{R}^3} d^3\boldsymbol{r}\int_{\mathbb{R}^3} d^3\boldsymbol{r}'\, r^L Y_{LM}(\boldsymbol{n}_r)\boldsymbol{\Lambda}\cdot\left[\Psi_\mu^{(0)\dagger}(\boldsymbol{r})\boldsymbol{\alpha}\bar{G}^{(0)}(\boldsymbol{r},\boldsymbol{r}')\right]r' Y_{10}(\boldsymbol{n}_r')\Psi_{\mu'}^{(0)}(\boldsymbol{r}'). \tag{3.36}$$

To tackle the integral on the right-hand side of Eq. (3.36), we exploit the identity

$$\boldsymbol{\Lambda}\cdot\left[\Psi_\mu^{(0)\dagger}(\boldsymbol{r})\boldsymbol{\alpha}\bar{G}^{(0)}(\boldsymbol{r},\boldsymbol{r}')\right] = -\left[\boldsymbol{\alpha}\cdot\boldsymbol{\Lambda}\Psi_\mu^{(0)}(\boldsymbol{r})\right]^\dagger\bar{G}^{(0)}(\boldsymbol{r},\boldsymbol{r}') + \Psi_\mu^{(0)\dagger}(\boldsymbol{r})\boldsymbol{\alpha}\cdot\boldsymbol{\Lambda}\bar{G}^{(0)}(\boldsymbol{r},\boldsymbol{r}'), \tag{3.37}$$

akin to the one in Eq. (3.14), and the following partial-wave expansion of the generalized Dirac-Coulomb Green function:

$$\bar{G}^{(0)}(\boldsymbol{r},\boldsymbol{r}') = \frac{4\pi\epsilon_0}{e^2}\sum_{\substack{\kappa=-\infty\\(\kappa\neq0)}}^{\infty}\sum_{m=-|\kappa|+1/2}^{|\kappa|-1/2}\frac{1}{rr'}\begin{pmatrix} \bar{g}_{\kappa,(++)}^{(0)}(r,r')\Omega_{\kappa m}(\boldsymbol{n}_r)\Omega_{\kappa m}^\dagger(\boldsymbol{n}_r') & -i\bar{g}_{\kappa,(+-)}^{(0)}(r,r')\Omega_{\kappa m}(\boldsymbol{n}_r)\Omega_{-\kappa m}^\dagger(\boldsymbol{n}_r') \\ i\bar{g}_{\kappa,(-+)}^{(0)}(r,r')\Omega_{-\kappa m}(\boldsymbol{n}_r)\Omega_{\kappa m}^\dagger(\boldsymbol{n}_r') & \bar{g}_{\kappa,(--)}^{(0)}(r,r')\Omega_{-\kappa m}(\boldsymbol{n}_r)\Omega_{-\kappa m}^\dagger(\boldsymbol{n}_r') \end{pmatrix}. \tag{3.38}$$

With some labor, this yields

$$\widetilde{\mathcal{M}}_{LM,\mu\mu'}^{(1)} = \frac{1}{L+1}\frac{4\pi}{\sqrt{3(2L+1)}}(4\pi\epsilon_0)cF\sum_{\kappa m}(-\kappa+1) \times R_\kappa^{L1}\langle\Omega_{1\mu}|Y_{LM}\Omega_{\kappa m}\rangle\langle\Omega_{\kappa m}|Y_{10}\Omega_{-1\mu'}\rangle, \tag{3.39}$$

where we define

$$R_\kappa^{LL'} = \int_0^\infty dr\int_0^\infty dr'\,\begin{pmatrix}Q^{(0)}(r) & P^{(0)}(r)\end{pmatrix} r^L \bar{\mathsf{G}}_\kappa^{(0)}(r,r')r'^{L'} \times \begin{pmatrix}P^{(0)}(r')\\ Q^{(0)}(r')\end{pmatrix}, \tag{3.40}$$

with

$$\bar{\mathsf{G}}_\kappa^{(0)}(r,r') = \begin{pmatrix}\bar{g}_{\kappa,(++)}^{(0)}(r,r') & \bar{g}_{\kappa,(+-)}^{(0)}(r,r')\\ \bar{g}_{\kappa,(-+)}^{(0)}(r,r') & \bar{g}_{\kappa,(--)}^{(0)}(r,r')\end{pmatrix} \tag{3.41}$$

being the radial generalized Dirac-Coulomb Green function associated with the combined angular momentum and parity symmetry κ. The angular integrals appearing in Eq. (3.39) can be taken with the aid of the formula in Eq. (3.22); this yields

$$\begin{aligned}\frac{4\pi(-\kappa+1)}{\sqrt{3(2L+1)}}&\langle\Omega_{1\mu}|Y_{LM}\Omega_{\kappa m}\rangle\langle\Omega_{\kappa m}|Y_{10}\Omega_{-1\mu'}\rangle\\ &= \frac{\sqrt{2}}{5}\delta_{L2}\delta_{\kappa,-2}(\sqrt{3}\,\delta_{M1}\delta_{m,-1/2}\delta_{\mu,1/2}\delta_{\mu',-1/2}\\ &\quad-\sqrt{2}\,\delta_{M0}\delta_{m,1/2}\delta_{\mu,1/2}\delta_{\mu',1/2}\\ &\quad+\sqrt{2}\,\delta_{M0}\delta_{m,-1/2}\delta_{\mu,-1/2}\delta_{\mu',-1/2}\\ &\quad-\sqrt{3}\,\delta_{M,-1}\delta_{m,1/2}\delta_{\mu,-1/2}\delta_{\mu',1/2}).\end{aligned} \tag{3.42}$$

Inserting Eq. (3.42) into Eq. (3.39) and combining the result with Eqs. (3.33) and (3.34) yields $\mathcal{M}_{LM}^{(1)}$ in the form

$$\mathcal{M}_{LM}^{(1)} = \mathcal{M}_{2M}^{(1)}\delta_{L2}, \tag{3.43}$$

where

$$\mathcal{M}_{20}^{(1)} = -\frac{4}{15}(4\pi\epsilon_0)cR_{-2}^{21}F(|a_{1/2}|^2 - |a_{-1/2}|^2), \tag{3.44a}$$

$$\mathcal{M}_{2,\pm1}^{(1)} = \pm\frac{2\sqrt{6}}{15}(4\pi\epsilon_0)cR_{-2}^{21}Fa_{\pm1/2}^*a_{\mp1/2}, \tag{3.44b}$$

$$\mathcal{M}_{2,\pm2}^{(1)} = 0. \tag{3.44c}$$

In this way, we have shown that the only magnetic 2^L-pole moment induced in the atom by a weak, static, uniform electric field is the quadrupole ($L=2$) one, spherical components of $\mathcal{M}_{2M}^{(1)}$ being given by Eqs. (3.44a)–(3.44c). A detailed analysis of the induced magnetic quadrupole moment will be carried out in the next section.

IV. EVALUATION OF THE FIRST-ORDER INDUCED MAGNETIC QUADRUPOLE MOMENT

The knowledge of the spherical components of the induced magnetic quadrupole moment tensor $\boldsymbol{\mathcal{M}}_2^{(1)}$ enables one to express that tensor in a coordinate-free form. To achieve this, we use the fact that the Cartesian components of $\boldsymbol{\mathcal{M}}_2^{(1)}$ are defined as

$$\mathcal{M}_{2,ij}^{(1)} = -\frac{1}{6}\int_{\mathbb{R}^3} d^3\boldsymbol{r}\,(3r_ir_j - r^2\delta_{ij})\nabla\cdot\left[\boldsymbol{r}\times\boldsymbol{j}^{(1)}(\boldsymbol{r})\right] \qquad (i,j\in\{x,y,z\}) \tag{4.1}$$

or, equivalently, as

$$\mathcal{M}_{2,ij}^{(1)} = \frac{1}{6}\,i\int_{\mathbb{R}^3} d^3\boldsymbol{r}\,(3r_ir_j - r^2\delta_{ij})\boldsymbol{\Lambda}\cdot\boldsymbol{j}^{(1)}(\boldsymbol{r}) \qquad (i,j\in\{x,y,z\}). \tag{4.2}$$

Making use of Eqs. (3.11) and (4.2), and of the explicit forms of the spherical harmonics $Y_{2M}(\boldsymbol{n}_r)$ with $M=0,\pm1,\pm2$, the matrix representation of $\boldsymbol{\mathcal{M}}_2^{(1)}$ in the Cartesian basis, written in terms of the spherical components, is found to be

$$\boldsymbol{\mathcal{M}}_2^{(1)} = \begin{pmatrix} -\frac{1}{2}\mathcal{M}_{20}^{(1)} + \sqrt{\frac{3}{8}}\left(\mathcal{M}_{2,-2}^{(1)} + \mathcal{M}_{22}^{(1)}\right) & i\sqrt{\frac{3}{8}}\left(\mathcal{M}_{2,-2}^{(1)} - \mathcal{M}_{22}^{(1)}\right) & \sqrt{\frac{3}{8}}\left(\mathcal{M}_{2,-1}^{(1)} - \mathcal{M}_{21}^{(1)}\right) \\ i\sqrt{\frac{3}{8}}\left(\mathcal{M}_{2,-2}^{(1)} - \mathcal{M}_{22}^{(1)}\right) & -\frac{1}{2}\mathcal{M}_{20}^{(1)} - \sqrt{\frac{3}{8}}\left(\mathcal{M}_{2,-2}^{(1)} + \mathcal{M}_{22}^{(1)}\right) & i\sqrt{\frac{3}{8}}\left(\mathcal{M}_{2,-1}^{(1)} + \mathcal{M}_{21}^{(1)}\right) \\ \sqrt{\frac{3}{8}}\left(\mathcal{M}_{2,-1}^{(1)} - \mathcal{M}_{21}^{(1)}\right) & i\sqrt{\frac{3}{8}}\left(\mathcal{M}_{2,-1}^{(1)} + \mathcal{M}_{21}^{(1)}\right) & \mathcal{M}_{20}^{(1)} \end{pmatrix}. \tag{4.3}$$

With the aid of Eqs. (3.44a)–(3.44c), this can be simplified to the form

$$\boldsymbol{\mathcal{M}}_2^{(1)} = \frac{2}{15}(4\pi\epsilon_0)cR_{-2}^{21}F\begin{pmatrix} |a_{1/2}|^2 - |a_{-1/2}|^2 & 0 & -3\,\mathrm{Re}(a_{1/2}^*a_{-1/2}) \\ 0 & |a_{1/2}|^2 - |a_{-1/2}|^2 & -3\,\mathrm{Im}(a_{1/2}^*a_{-1/2}) \\ -3\,\mathrm{Re}(a_{1/2}^*a_{-1/2}) & -3\,\mathrm{Im}(a_{1/2}^*a_{-1/2}) & -2(|a_{1/2}|^2 - |a_{-1/2}|^2) \end{pmatrix}. \tag{4.4}$$

Plugging here the expressions (3.28) for $a_{\pm 1/2}$ and combining the result with Eq. (3.32), we see that the sought coordinate-free representation of $\boldsymbol{\mathcal{M}}_2^{(1)}$ is

$$\boldsymbol{\mathcal{M}}_2^{(1)} = (4\pi\epsilon_0)c\alpha_{E1\to M2}\left[\tfrac{3}{4}(\boldsymbol{\nu}\boldsymbol{F} + \boldsymbol{F}\boldsymbol{\nu}) - \tfrac{1}{2}(\boldsymbol{\nu}\cdot\boldsymbol{F})\boldsymbol{\mathcal{I}}\right], \tag{4.5}$$

where $\boldsymbol{\mathcal{I}}$ is the unit dyadic. The proportionality factor

$$\alpha_{E1\to M2} = -\tfrac{4}{15}R_{-2}^{21}, \tag{4.6}$$

expressed here in terms of the radial integral R_{-2}^{21}, is the electric-dipole-to-magnetic-quadrupole ($E1\to M2$) cross-susceptibility of the atom.

The point concerning the geometric properties of the tensor $\boldsymbol{\mathcal{M}}_2^{(1)}$ being now clarified, we proceed to the calculation of the cross-susceptibility $\alpha_{E1\to M2}$. This requires the integral R_{-2}^{21} is evaluated. To accomplish the goal, we use as a tool the following Sturmian expansion of the radial generalized Dirac-Coulomb Green function for the ground state of the atom:

$$\bar{\mathsf{G}}_\kappa^{(0)}(r,r') = \sum_{n_r=-\infty}^{\infty}\frac{1}{\mu_{n_r\kappa}^{(0)}-1}\begin{pmatrix} S_{n_r\kappa}^{(0)}(r) \\ T_{n_r\kappa}^{(0)}(r)\end{pmatrix} \times\left(\mu_{n_r\kappa}^{(0)}S_{n_r\kappa}^{(0)}(r') \quad T_{n_r\kappa}^{(0)}(r')\right) \quad (\kappa\neq-1), \tag{4.7}$$

found by one of us in Ref. [1]. Here

$$S_{n_r\kappa}^{(0)}(r) = \sqrt{\frac{(1+\gamma_1)(|n_r|+2\gamma_\kappa)|n_r|!}{2ZN_{n_r\kappa}(N_{n_r\kappa}-\kappa)\Gamma(|n_r|+2\gamma_\kappa)}} \times\left(\frac{2Zr}{a_0}\right)^{\gamma_\kappa}e^{-Zr/a_0}\left[L_{|n_r|-1}^{(2\gamma_\kappa)}\left(\frac{2Zr}{a_0}\right) + \frac{\kappa-N_{n_r\kappa}}{|n_r|+2\gamma_\kappa}L_{|n_r|}^{(2\gamma_\kappa)}\left(\frac{2Zr}{a_0}\right)\right] \tag{4.8a}$$

and

$$T_{n_r\kappa}^{(0)}(r) = \sqrt{\frac{(1-\gamma_1)(|n_r|+2\gamma_\kappa)|n_r|!}{2ZN_{n_r\kappa}(N_{n_r\kappa}-\kappa)\Gamma(|n_r|+2\gamma_\kappa)}} \times\left(\frac{2Zr}{a_0}\right)^{\gamma_\kappa}e^{-Zr/a_0}\left[L_{|n_r|-1}^{(2\gamma_\kappa)}\left(\frac{2Zr}{a_0}\right) - \frac{\kappa-N_{n_r\kappa}}{|n_r|+2\gamma_\kappa}L_{|n_r|}^{(2\gamma_\kappa)}\left(\frac{2Zr}{a_0}\right)\right] \tag{4.8b}$$

[with $L_n^{(\alpha)}(\rho)$ denoting the generalized Laguerre polynomial [12]; we define $L_{-1}^{(\alpha)}(\rho)\equiv 0$] are the radial Dirac-Coulomb Sturmian functions associated with the hydrogenic ground-state energy level, while

$$\mu_{n_r\kappa}^{(0)} = \frac{|n_r|+\gamma_\kappa+N_{n_r\kappa}}{\gamma_1+1}, \tag{4.9}$$

with

$$N_{n_r\kappa} = \pm\sqrt{(|n_r| + \gamma_\kappa)^2 + (\alpha Z)^2} = \pm\sqrt{|n_r|^2 + 2|n_r|\gamma_\kappa + \kappa^2} \tag{4.10}$$

being the "apparent principal quantum number" (notice that it can assume positive as well as negative values); the following sign convention applies to the definition (4.10): the plus sign should be chosen for $n_r > 0$ and the minus sign for $n_r < 0$; for $n_r = 0$ one chooses the plus sign if $\kappa < 0$ and the minus sign if $\kappa > 0$. Combining Eqs. (4.6), (3.40), and (4.7), we obtain

$$\alpha_{E1\to M2} = -\frac{4}{15}\sum_{n_r=-\infty}^{\infty}\frac{1}{\mu^{(0)}_{n_r,-2} - 1}\int_0^\infty dr\, r^2\big[Q^{(0)}(r)S^{(0)}_{n_r,-2}(r) + P^{(0)}(r)T^{(0)}_{n_r,-2}(r)\big] \times \int_0^\infty dr'\, r'\big[\mu^{(0)}_{n_r,-2}P^{(0)}(r')S^{(0)}_{n_r,-2}(r') + Q^{(0)}(r')T^{(0)}_{n_r,-2}(r')\big]. \tag{4.11}$$

The two radial integrals appearing in Eq. (4.11) can be taken with the aids of Eqs. (2.11a), (2.11b), (4.8a), (4.8b), and (4.9), the integral formula [Ref. [13], Eq. (7.414.11)]

$$\int_0^\infty dx\, x^\gamma e^{-x} L_n^{(\alpha)}(x) = \frac{\Gamma(\gamma+1)\Gamma(n+\alpha-\gamma)}{n!\Gamma(\alpha-\gamma)} = (-)^n\frac{\Gamma(\gamma+1)\Gamma(\gamma-\alpha+1)}{n!\Gamma(\gamma-\alpha-n+1)} \qquad (\mathrm{Re}\,\gamma > -1) \tag{4.12}$$

and the trivial but extremely useful identity

$$\gamma_2^2 = \gamma_1^2 + 3. \tag{4.13}$$

This yields

$$\int_0^\infty dr\, r^2\big[Q^{(0)}(r)S^{(0)}_{n_r,-2}(r) + P^{(0)}(r)T^{(0)}_{n_r,-2}(r)\big] = -\alpha Z\left(\frac{a_0}{2Z}\right)^3\frac{\Gamma(\gamma_2+\gamma_1+3)}{\Gamma(\gamma_2-\gamma_1-2)}\frac{\sqrt{2}(N_{n_r,-2}+2)\Gamma(|n_r|+\gamma_2-\gamma_1-2)}{\sqrt{a_0|n_r|!N_{n_r,-2}(N_{n_r,-2}+2)\Gamma(|n_r|+2\gamma_2+1)\Gamma(2\gamma_1+1)}} \tag{4.14}$$

and

$$\int_0^\infty dr\, r\big[\mu^{(0)}_{n_r,-2}P^{(0)}(r)S^{(0)}_{n_r,-2}(r) + Q^{(0)}(r)T^{(0)}_{n_r,-2}(r)\big] = -\left(\frac{a_0}{2Z}\right)^2\frac{\Gamma(\gamma_2+\gamma_1+2)}{\Gamma(\gamma_2-\gamma_1-1)}\frac{(N_{n_r,-2}+2)\Gamma(|n_r|+\gamma_2-\gamma_1-2)}{\sqrt{2a_0|n_r|!N_{n_r,-2}(N_{n_r,-2}+2)\Gamma(|n_r|+2\gamma_2+1)\Gamma(2\gamma_1+1)}} \times [(|n_r|+\gamma_2-\gamma_1-2+N_{n_r,-2})(2\gamma_1-1)+3]. \tag{4.15}$$

Consequently, the cross-susceptibility $\alpha_{E1\to M2}$ can be written as

$$\alpha_{E1\to M2} = \frac{\alpha a_0^4}{Z^4}\frac{(\gamma_1+1)\Gamma^2(\gamma_2+\gamma_1+3)}{120(4\gamma_1+1)\Gamma(2\gamma_1+1)\Gamma^2(\gamma_2-\gamma_1-2)}\sum_{n_r=-\infty}^{\infty}\frac{\Gamma^2(|n_r|+\gamma_2-\gamma_1-2)}{|n_r|!\Gamma(|n_r|+2\gamma_2+1)} \times \frac{N_{n_r,-2}+2}{N_{n_r,-2}}\frac{(|n_r|+\gamma_2-\gamma_1-2+N_{n_r,-2})(2\gamma_1-1)+3}{|n_r|+\gamma_2-\gamma_1-1+N_{n_r,-2}}. \tag{4.16}$$

To express $\alpha_{E1\to M2}$ in terms of known special functions, in the above series we collect together terms with the same absolute value of the summation index n_r. Proceeding in this way, we obtain

$$\alpha_{E1\to M2} = \frac{\alpha a_0^4}{Z^4}\frac{\Gamma^2(\gamma_2+\gamma_1+3)}{60(4\gamma_1+1)\Gamma(2\gamma_1+1)\Gamma^2(\gamma_2-\gamma_1-2)}\sum_{n_r=0}^{\infty}\frac{\Gamma^2(n_r+\gamma_2-\gamma_1-2)}{n_r!(n_r+\gamma_2-\gamma_1)\Gamma(n_r+2\gamma_2+1)} \times\big[\big(2\gamma_1^2+2\gamma_1-3\big)(n_r+\gamma_2-\gamma_1) - 3(\gamma_1-2)\big]. \tag{4.17}$$

Now, it is known from the theory of the generalized hypergeometric functions ${}_pF_q$ [14,15] that

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+a_1)\Gamma(n+a_2)}{n!\Gamma(n+b)} = \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(b)}\,{}_2F_1\left(\begin{matrix}a_1,a_2\\ b\end{matrix};1\right) \tag{4.18}$$

and

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+a_1)\Gamma(n+a_2)\Gamma(n+a_3)}{n!\Gamma(n+b_1)\Gamma(n+b_2)} = \frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}{\Gamma(b_1)\Gamma(b_2)}\,{}_3F_2\left(\begin{matrix}a_1,a_2,a_3\\ b_1,b_2\end{matrix};1\right). \tag{4.19}$$

Consequently, with some algebra Eq. (4.17) can be cast in the form

$$\alpha_{E1\to M2} = \frac{\alpha a_0^4}{Z^4}\frac{\Gamma^2(\gamma_2+\gamma_1+3)}{60(4\gamma_1+1)\Gamma(2\gamma_1+1)\Gamma(2\gamma_2+1)}\left[\left(2\gamma_1^2+2\gamma_1-3\right){}_2F_1\left(\begin{matrix}\gamma_2-\gamma_1-2,\,\gamma_2-\gamma_1-2\\2\gamma_2+1\end{matrix};1\right)\right.$$
$$\left.-(\gamma_1-2)(\gamma_2+\gamma_1)\,{}_3F_2\left(\begin{matrix}\gamma_2-\gamma_1-2,\,\gamma_2-\gamma_1-2,\,\gamma_2-\gamma_1\\\gamma_2-\gamma_1+1,\,2\gamma_2+1\end{matrix};1\right)\right]. \tag{4.20}$$

The ${}_2F_1$ function can be eliminated from Eq. (4.20) with the aid of the Gauss identity [Ref. [13], Eq. (9.122.1)]

$${}_2F_1\left(\begin{matrix}a_1,a_2\\b\end{matrix};1\right)=\frac{\Gamma(b)\Gamma(b-a_1-a_2)}{\Gamma(b-a_1)\Gamma(b-a_2)}\qquad[\mathrm{Re}(b-a_1-a_2)>0], \tag{4.21}$$

yielding

$$\alpha_{E1\to M2}=\frac{\alpha a_0^4}{Z^4}\frac{\Gamma(2\gamma_1+5)}{60(4\gamma_1+1)\Gamma(2\gamma_1+1)}\left[2\gamma_1^2+2\gamma_1-3\right.$$
$$\left.-\frac{(\gamma_1-2)(\gamma_2+\gamma_1)\Gamma^2(\gamma_2+\gamma_1+3)}{\Gamma(2\gamma_1+5)\Gamma(2\gamma_2+1)}\,{}_3F_2\left(\begin{matrix}\gamma_2-\gamma_1-2,\,\gamma_2-\gamma_1-2,\,\gamma_2-\gamma_1\\\gamma_2-\gamma_1+1,\,2\gamma_2+1\end{matrix};1\right)\right]. \tag{4.22}$$

Other equivalent representations of $\alpha_{E1\to M2}$ can be deduced from Eq. (4.22) using recurrence relations involving contiguous ${}_3F_2$ functions. For instance, exploiting the identity

$${}_3F_2\left(\begin{matrix}a_1,a_2,a_3\\a_3+1,b\end{matrix};1\right)=-\frac{a_3}{a_2-a_3}\frac{\Gamma(b)\Gamma(b-a_1-a_2)}{\Gamma(b-a_1)\Gamma(b-a_2)}+\frac{a_2}{a_2-a_3}\,{}_3F_2\left(\begin{matrix}a_1,a_2+1,a_3\\a_3+1,b\end{matrix};1\right)\qquad[\mathrm{Re}(b-a_1-a_2)>0] \tag{4.23}$$

(for its derivation, with a_1 and a_2 interchanged, see Ref. [4], Appendix D), we arrive at the expression

$$\alpha_{E1\to M2}=\frac{\alpha a_0^4}{Z^4}\frac{\Gamma(2\gamma_1+5)}{240\Gamma(2\gamma_1)}\left[1-\frac{(\gamma_1-2)(\gamma_2+\gamma_1)\Gamma(\gamma_2+\gamma_1+2)\Gamma(\gamma_2+\gamma_1+3)}{\gamma_1\Gamma(2\gamma_1+5)\Gamma(2\gamma_2+1)}\right.$$
$$\left.\times\,{}_3F_2\left(\begin{matrix}\gamma_2-\gamma_1-2,\,\gamma_2-\gamma_1-1,\,\gamma_2-\gamma_1\\\gamma_2-\gamma_1+1,\,2\gamma_2+1\end{matrix};1\right)\right]. \tag{4.24}$$

The ${}_3F_2$ function appearing in Eq. (4.24) is the same one which enters the expression for the magnetic-field-induced electric quadrupole moment given in Ref. [8], Eq. (4.20).

In the nonrelativistic limit one has

$$\gamma_1\xrightarrow{c\to\infty}1,\quad\gamma_2\xrightarrow{c\to\infty}2, \tag{4.25}$$

so that either of Eqs. (4.22) or (4.24) reduces to

$$\alpha_{E1\to M2}\xrightarrow{c\to\infty}\frac{9}{2}\frac{\alpha a_0^4}{Z^4}. \tag{4.26}$$

It has been verified by us that the limit on the right-hand side of Eq. (4.26) agrees with the expression for the $E1\to M2$ cross-susceptibility of the one-electron atom in the ground state obtained within the framework of the Schrödinger-Pauli approximation.

ACKNOWLEDGMENTS

We thank Grzegorz Łukasik for stimulating discussions, in particular for the suggestion to transform the expression in Eq. (4.22) into the one in Eq. (4.24).

[1] R. Szmytkowski, J. Phys. B **30**, 825 (1997); **30**, 2747(E) (1997); addendum, arXiv:physics/9902050.
[2] R. Szmytkowski and K. Mielewczyk, J. Phys. B **37**, 3961 (2004).
[3] R. Szmytkowski, Phys. Rev. A **65**, 012503 (2001).
[4] K. Mielewczyk and R. Szmytkowski, Phys. Rev. A **73**, 022511 (2006); **73**, 039908(E) (2006).
[5] R. Szmytkowski, J. Phys. B **35**, 1379 (2002).
[6] R. Szmytkowski and P. Stefańska, arXiv:1102.1811.
[7] P. Stefańska and R. Szmytkowski, Int. J. Quantum Chem. **112**, 1363 (2012).
[8] R. Szmytkowski and P. Stefańska, Phys. Rev. A **85**, 042502 (2012).
[9] R. Szmytkowski, J. Math. Chem. **42**, 397 (2007).
[10] J. D. Jackson, *Electrodynamics*, 2nd ed. (Wiley, New York, 1975).
[11] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Quantum Theory of Angular Momentum* (Nauka, Leningrad, 1975) (in Russian).
[12] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3rd ed. (Springer, Berlin, 1966).
[13] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed. (Elsevier, Amsterdam, 2007).
[14] W. N. Bailey, *Generalized Hypergeometric Series* (Cambridge University Press, Cambridge, 1935).
[15] L. J. Slater, *Generalized Hypergeometric Functions* (Cambridge University Press, Cambridge, 1966).