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Equivalence of equicontinuity concepts for Markov operators derived from a Schur-like property for spaces of measures[☆]

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ABSTRACT

Various equicontinuity properties for families of Markov operators have been – and still are – used in the study of existence and uniqueness of invariant probability for these operators, and of asymptotic stability. We prove a general result on equivalence of equicontinuity concepts. It allows comparing results in the literature and switching from one view on equicontinuity to another, which is technically convenient in proofs. More precisely, the characterisation is based on a 'Schur-like property' for measures: if a sequence of finite signed Borel measures on a Polish space is such that it is bounded in total variation norm and such that for each bounded Lipschitz function the sequence of integrals of this function with respect to these measures converges, then the sequence converges in dual bounded Lipschitz norm to a measure.

1. Introduction

The mathematical study of dynamical systems in discrete or continuous time on spaces of probability measures has a long-lasting history in probability theory (as Markov operators and Markov semigroups, see e.g. Jamison, 1964; Meyn and Tweedie, 2009) and the field of Iterated Function Systems (Barnsley et al., 1988; Czapla, 2018; Lasota and Yorke, 1994) in particular. In analysis there is a growing interest in solutions to evolution equations in spaces of positive or signed measures, e.g. in the study of structured population models (Ackleh and Ito, 2005; Carrillo et al., 2012), crowd dynamics (Piccoli and Tosin, 2011) or interacting particle systems (Evers et al., 2016).

Equicontinuous families of Markov operators were introduced in relation to conditions for existence and uniqueness of invariant probability measures for these operators, in particular to asymptotic stability: the convergence of the law of a Markov process to a unique invariant measure, see e.g. Czapla (2018), Hille et al. (2016), Komorowski et al. (2010),

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Szarek and Worm (2012) and Worm (2010). One has considered concepts like *e-chains* (Meyn and Tweedie, 2009), the *e-property* (Czapla, 2018; Czapla and Horbacz, 2014; Komorowski et al., 2010; Lasota and Szarek, 2006; Szarek and Worm, 2012), *Cesàro-e-property* (Worm, 2010), Chapter 7, or *uniform equicontinuity on balls* (Hille et al., 2016); see also Jamison (1964). Hairer and Mattingly introduced the so-called asymptotic strong Feller property for that purpose (Hairer and Mattingly, 2006). Our main result, Theorem 4.1, rigorously connects dual viewpoints concerning equicontinuity concepts: those for Markov operators acting on measures (laws) and those for Markov operators acting on functions (observables).

The difficult step in proving the equivalences in Theorem 4.1 is getting from a point-wise result to one that holds uniformly on the unit ball in the space of bounded Lipschitz functions on a Polish space S, $\mathrm{BL}(S)$. It is enabled by Theorem 3.2 that allows to deduce norm convergence of suitable sequences of finite signed Borel measures $\mathcal{M}(S)$ from $\mathrm{BL}(S)$-weak convergence. This is a 'Schur-*like*' property. Recall that a Banach space X has the Schur property if every weakly convergent sequence in X is norm convergent (e.g. Albiac and Kalton (2006), Definition 2.3.4). For example, the sequence space ℓ^1 has the Schur property (cf. Albiac and Kalton, 2006, Theorem 2.3.6). Although the dual space of $(\mathcal{M}(S), \|\cdot\|^*_{\mathrm{BL}})$, where $\|\cdot\|^*_{\mathrm{BL}}$ is the *Dudley norm* or *bounded Lipschitz norm* on $\mathcal{M}(S)$ (see Dudley, 1966), is isometrically isomorphic to $\mathrm{BL}(S)$ (cf. Hille and Worm, 2009b, Theorem 3.7), the (completion of the) space $(\mathcal{M}(S), \|\cdot\|^*_{\mathrm{BL}})$ is not a Schur space, generally (see Counterexample 3.1). The condition of bounded total variation cannot be omitted.

After having introduced the necessary notation and basic concepts, we show in Section 3 how Theorem 4.1 derives from functional analytic results that originate in the theory of uniform measures on uniform spaces. This topic has been well-researched in the past (see e.g. Cooper and Schachermayer, 1981; Pachl, 1979, 2013), but awareness of the multitude of results in that field has diminished nowadays, unfortunately. An independent proof of Theorem 4.1 can be found in Hille et al. (2018) or Ziemlańska (2020), Chapter 2. In Section 4 we prove the main result. This section also contains a characterisation of the e-property for the Markov semigroup obtained by lifting a flow on the underlying state space S to $\mathcal{M}(S)$ by push-forward. This uses Theorem 4.1.

Although an extensive body of functional analytic results on the space of finite signed Borel measures and subsets thereof have been obtained within probability theory (e.g. see Billingsley, 1968; Bogachev, 2007; Dudley, 1966; LeCam, 1957), there is still need for further results, driven for example by the topic of evolution equations in space of measures in which there is no conservation of mass, or in settings where positivity is not preserved. This paper contributes to these developments, by considering the pairing of $\mathcal{M}(S)$ with $\mathrm{BL}(S)$. The latter space of 'test functions' is intimately tied to most of the equicontinuity concepts defined so far.

Further applications of the characterisations of equicontinuous families of Markov operators in Theorem 4.1 and the fundamental Theorem 3.2 have been investigated in follow-up research, see e.g. Worm (2010). Recent results will appear in Ziemlańska (2020), e.g. Chapter 3.

2. Notation and basic concepts

If X and Y are vector spaces in duality $\langle\cdot,\cdot\rangle$, then by $\sigma(X, Y)$ we denote the weak topology on X defined by Y, i.e. the locally convex topology on X defined by the seminorms $p_y(\cdot) := |\langle\cdot, y\rangle|$. We shall also speak of the 'Y-*weak topology on* X' for this topology. For a normed space $(X, \|\cdot\|)$, X^* will be the norm-dual, equipped with the dual norm $\|\cdot\|^*$.

Let (S, d) be a metric space. There are three natural vector spaces of functions to consider on the space (S, d): the continuous and bounded functions, $\mathrm{BC}(S)$, in which one takes into consideration only the topology generated by the open balls for the metric d, the uniformly continuous and bounded functions, $\mathrm{BU}(S)$, corresponding to the uniformity defined by d (see e.g. Pachl, 2013) and finally the space of bounded Lipschitz functions $\mathrm{BL}(S)$ (we suppress d in the notation, unless strictly needed). The latter is a subspace of the space of Lipschitz functions $\mathrm{Lip}(S)$, which consists of $f : S \to \mathbb{R}$ such that the Lipschitz constant

$$|f|_L := \sup\left\{ \frac{|f(x) - f(y)|}{d(x, y)} \;:\; x, y \in S,\ x \neq y \right\}.$$

is finite. $\mathrm{BC}(S)$, $\mathrm{BU}(S)$ are Banach spaces for the supremum norm $\|\cdot\|_\infty$. $\mathrm{BL}(S)$ is a Banach space for the bounded Lipschitz norm

$$\|f\|_{\mathrm{BL}} := \|f\|_\infty + |f|_L.$$

One has

$$\mathrm{BL}(S) \subset \mathrm{BU}(S) \subset \mathrm{BC}(S),$$

where each smaller space is typically a proper subset of the larger and the respective inclusions are continuous. $\mathrm{BL}(S)$ is dense in $\mathrm{BU}(S)$ (cf. Dudley, 1966 Lemma 8).

The space $\mathcal{M}(S)$ of finite signed Borel measures on S embeds into the norm-dual spaces $\mathrm{BC}(S)^*$, $\mathrm{BU}(S)^*$ and $\mathrm{BL}(S)^*$ by means of integration: $\mu \mapsto I_\mu$, where

$$I_\mu(f) = \langle \mu, f\rangle := \int_S f\, d\mu.$$

The dual norm $\|\cdot\|_{BL}^*$ on $BL(S)^*$ introduces a norm on $\mathcal{M}(S)$ through the map $\mu \mapsto I_\mu$, which is given by

$$\|\mu\|_{BL}^* = \sup\left\{|\langle \mu, f\rangle| : f \in BL(S), \ \|f\|_{BL} \le 1\right\}.$$

This is called the *bounded Lipschitz norm*, or *Dudley norm*. $\mathcal{M}(S)$ equipped with the norm topology is denoted by $\mathcal{M}(S)_{BL}$. $(\mathcal{M}(S), \|\cdot\|_{BL}^*)$ is not complete generally. We write $\|\cdot\|_{TV}$ for the total variation norm on $\mathcal{M}(S)$:

$$\|\mu\|_{TV} = |\mu|(S) = \mu^+(S) + \mu^-(S),$$

where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ. $\mathcal{M}^+(S)$ is the convex cone of positive measures in $\mathcal{M}(S)$. $\mathcal{P}(S)$ is the convex set of probability measures in $\mathcal{M}^+(S)$. $\mathcal{P}(S)_{BL}$ and $\mathcal{M}^+(S)_{BL}$ are $\mathcal{P}(S)$ and $\mathcal{M}(S)^+$ equipped with the metric induced by the norm $\|\cdot\|_{BL}^*$. One has

$$\|\mu\|_{TV} = \|\mu\|_{BL}^* \qquad \text{for all } \mu \in \mathcal{M}^+(S). \tag{2.1}$$

In general, for $\mu \in \mathcal{M}(S)$, $\|\mu\|_{BL}^* \le \|\mu\|_{TV}$. Moreover, the dual norm in $BC(S)^*$ and $BU(S)^*$ of I_μ is equal to $\|\mu\|_{TV}$, see e.g. Dudley (1966) Lemma 2. The dual space of $(\mathcal{M}(S), \|\cdot\|_{BL}^*)$ is isometrically linearly isomorphic to $(BL(S), \|\cdot\|_{BL})$ (cf. Hille and Worm, 2009b, Theorem 3.7).

From this point on, we will assume that (S, d) is complete and separable.

A *Markov operator* on (measures on) S is a map $P : \mathcal{M}^+(S) \to \mathcal{M}^+(S)$ such that:

(i) $P(\mu + \nu) = P\mu + P\nu$ and $P(r\mu) = rP\mu$ for all $\mu, \nu \in \mathcal{M}^+(S)$ and $r \ge 0$,
(ii) $(P\mu)(S) = \mu(S)$ for all $\mu \in \mathcal{M}^+(S)$.

In particular, a Markov operator leaves invariant the convex set $\mathcal{P}(S)$ of probability measures in $\mathcal{M}^+(S)$. Let $BM(S)$ be the vector space of bounded Borel measurable real-valued functions on S. A Markov operator is called *regular* if there exist a linear map $U : BM(S) \to BM(S)$, the *dual operator*, such that

$$\langle P\mu, f\rangle = \langle \mu, Uf\rangle \qquad \text{for all } \mu \in \mathcal{M}^+(S), \ f \in BM(S).$$

A regular Markov operator P is *Feller* if its dual operator maps $BC(S)$ into itself. Equivalently, P is continuous for the $\|\cdot\|_{BL}^*$-norm topology (cf. e.g. Hille and Worm, 2009a Lemma 3.3 or Worm, 2010 Lemma 3.3.2).

Regular Markov operators on measures appear naturally e.g. in the theory of Iterated Function Systems (Barnsley et al., 1988; Lasota and Yorke, 1994) and the study of deterministic flows by their lift to measures (Piccoli and Tosin, 2011), see also Section 4. Dual Markov operators on $BC(S)$ (or a suitable linear subspace) are encountered naturally in the study of stochastic differential equations (e.g. Komorowski et al., 2010). Which specific viewpoint in this duality is used, is often determined by technical considerations and the mathematical problems that are considered.

3. Preliminary functional analytic results of the space of measures

In probability theory one has since long studied on $\mathcal{M}(S)$ the *weak topology on measures*, which is the $\sigma(\mathcal{M}(S), BC(S))$-weak topology, and its relative topology on $\mathcal{M}^+(S)$ and $\mathcal{P}(S)$ in particular, see e.g. Bogachev (2007), Chapter 8, for an extensive overview. For example, it is well known that the topology of weak convergence of measures on $\mathcal{M}^+(S)$ and $\mathcal{P}(S)$ coincides with that induced by the Dudley norm $\|\cdot\|_{BL}^*$ (see Dudley, 1966, or Bogachev, 2007 Theorem 8.3.2).

The weak topology on $\mathcal{M}(S)$ induced by the pairing with $BU(S)$ has been applied less in probability theory. The topology has been studied well though, in the general setting of '*uniform measures on uniform spaces*'. Results can be found in e.g. Cooper and Schachermayer (1981), Pachl (1979) and in Pachl (2013) — with applications to probability theory even. Two results, due to Pachl, are crucial for this paper (slightly reformulated, see also Pachl, 2013, Theorem 5.41 and Theorem 5.45):

Theorem 3.1 (*Pachl, 1979, Theorem 3.2*)**.** *Let (S, d) be a complete separable metric space.*

(a) $\mathcal{M}(S)$ is $BU(S)$-weakly sequentially complete:
if $(\mu_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}(S)$ such that for every f in $BU(S)$, $\langle \mu_n, f\rangle$ is convergent, then there exists $\mu \in \mathcal{M}(S)$ such that $\langle \mu_n, f\rangle \to \langle \mu, f\rangle$ for all $f \in BU(S)$.
(b) Let $M \subset \mathcal{M}(S)$ be such that $\sup_{\mu\in M} \|\mu\|_{TV} < \infty$. Then,
M is relatively $\|\cdot\|_{BL}^*$-compact if and only if M is relatively $\sigma(\mathcal{M}(S), BU(S))$-weakly countably compact.

(A set A in a topological space is countably compact if every infinite subset of A has a limit point $x \in A$, i.e. every neighbourhood of x contains infinitely many points of A).

In probability theory, the weak topology on $(\mathcal{M}(S), \|\cdot\|_{BL}^*)$ in strict functional analytic sense, i.e. the $\sigma(\mathcal{M}(S), BL(S))$-weak topology, has not been very much looked at or used in depth, as far as we know. A reason for this may be that in various applications of interest one considers a set $M \subset \mathcal{M}(S)$ that is bounded in $\|\cdot\|_{TV}$-norm. Because $BL(S)$ is dense in $BU(S)$, one has

Lemma 3.1. *Let $M \subset \mathcal{M}(S)$. If $\sup_{\mu\in M} \|\mu\|_{TV} < \infty$, then the $\sigma(\mathcal{M}(S), BU(S))$-weak and $\sigma(\mathcal{M}(S), BL(S))$-weak topologies coincide on M.*

Thus, consideration of the BU(S)-weak topology on $\mathcal{M}(S)$ has sufficed. However, we would like to point out and prove a consequence of Theorem 3.1 that is itself worth noting, and that will be fundamental to the main results on equicontinuity in this paper:

Theorem 3.2 (*'Weak Implies Strong Convergence'*)**.** *Let (S,d) be a complete separable metric space. Let $(\mu_n) \subset \mathcal{M}(S)$ be such that $\sup_n \|\mu_n\|_{\mathrm{TV}} < \infty$. If for every $f \in \mathrm{BL}(S)$ the sequence $\langle \mu_n, f\rangle$ converges, then there exists $\mu \in \mathcal{M}(S)$ such that $\|\mu_n - \mu\|_{\mathrm{BL}}^* \to 0$ as $n \to \infty$.*

Proof. Since the sequence $\langle \mu_n, f\rangle$ converges for every $f \in \mathrm{BL}(S)$ and (μ_n) is bounded in total variation norm, also $\langle \mu_n, g\rangle$ converges as $n \to \infty$ for every $g \in \mathrm{BU}(S)$. According to BU(S)-weak sequential completeness of $\mathcal{M}(S)$ (cf. Theorem 3.1(a)), there exists $\mu \in \mathcal{M}(S)$ such that $\langle \mu_n, g\rangle \to \langle \mu, g\rangle$ for all $g \in \mathrm{BU}(S)$.

Hence, $M := \{\mu_n : n \in \mathbb{N}\}$ is relatively $\sigma(\mathcal{M}(S), \mathrm{BU}(S))$-weakly countably compact. According to Theorem 3.1(b), M is relatively compact for the $\|\cdot\|_{\mathrm{BL}}^*$-norm topology.

Suppose that (μ_n) does not converge to μ in $\mathcal{M}(S)_{\mathrm{BL}}$. Then there exist $\varepsilon > 0$ and a subsequence $(\mu_{n_k})_k$ of (μ_n) such that

$$\|\mu_{n_k} - \mu\|_{\mathrm{BL}}^* \geq \varepsilon \quad \text{for all } k \in \mathbb{N}. \tag{3.1}$$

Because of the relative $\|\cdot\|_{\mathrm{BL}}^*$-compactness of M, $(\mu_{n_k})_k$ has a further subsequence (for which we shall use the same notation) that converges to some φ in the closure of $\mathcal{M}(S)$ in $\mathrm{BL}(S)^*$. Then for every $f \in \mathrm{BL}(S)$,

$$\varphi(f) = \lim_{k\to\infty} \langle \mu_{n_k}, f\rangle = \langle \mu, f\rangle .$$

Thus, $\varphi = \mu$ in $\mathrm{BL}(S)^*$ and $\|\mu_{n_k} - \mu\|_{\mathrm{BL}}^* \to 0$ as $k \to \infty$. This contradicts (3.1). □

Remark 3.1. (1) Theorem 3.2 resembles the so-called Schur property for a Banach space X: every weakly convergent sequence in X is norm-convergent. It differs in two aspects however. First, it is valid for a subset M of the space $(\mathcal{M}(S), \|\cdot\|_{\mathrm{BL}}^*)$ that is bounded in total variation norm, not for the whole space. Second, it is not assumed a priori that the sequence (μ_n) is weakly convergent. It is a consequence of the assumptions and Theorem 3.1(a) instead.
(2) The boundedness in total variation norm cannot be omitted. Thus, $(\mathcal{M}(S), \|\cdot\|_{\mathrm{BL}}^*)$ (or its completion) does not have the Schur property. This is shown by the following:

Counterexample 3.1. *Let $S = [0,1]$ with the Euclidean metric. Let $d\mu_n := n\sin(2\pi nx)\,dx$, where dx is Lebesgue measure on S. Then $\|\mu_n\|_{\mathrm{TV}}$ is unbounded, since*

$$\int_0^1 \left|n\sin(2\pi nx)\right| dx = \frac{2n}{\pi}.$$

Let $g \in \mathrm{BL}(S)$ with $|g|_L \leq 1$. According to Rademacher's Theorem, g is differentiable Lebesgue almost everywhere. Since $|g|_L \leq 1$, there exists $f \in L^\infty([0,1])$ such that for all $0 \leq a < b \leq 1$,

$$\int_a^b f(x)\,dx = g(b) - g(a).$$

This yields

$$\langle \mu_n, g\rangle = \frac{1}{2\pi}\int_0^1 \cos(2\pi nx) f(x)\,dx.$$

Since $f \in L^2([0,1])$, it follows from Bessel's Inequality that

$$\lim_{n\to\infty} \int_0^1 \cos(2\pi nx) f(x)\,dx = 0.$$

So $\langle \mu_n, g\rangle \to 0$ for all $g \in \mathrm{BL}(S)$. Now, let $g_n \in \mathrm{BL}(S)$ be the piecewise linear function that satisfies $g_n(0) = 0 = g_n(1)$,

$$g_n\left(\tfrac{1+4i}{4n}\right) = \tfrac{1}{4n}, \qquad g_n\left(\tfrac{3+4i}{4n}\right) = -\tfrac{1}{4n}, \qquad \text{for } i \in \mathbb{N},\ 0 \leq i \leq n-1.$$

Then $|g|_L = 1$ and $\|g_n\|_\infty = \frac{1}{4n}$. An easy calculation shows that $\langle \mu_n, g_n\rangle = \frac{1}{\pi^2}$ for all $n \in \mathbb{N}$. Therefore $\|\mu_n\|_{\mathrm{BL}}^*$ cannot converge to zero as $n \to \infty$.

Remark 3.2. (1) Theorem 3.2 does not seem to have appeared as stated in the literature. It was first observed in Worm (2010), Theorem 2.3.24, p. 32, and proven to be a consequence of the results obtained by Pachl, Theorem 3.1, as above. In Ziemlańska (2020), Chapter 2, a proof is given of Theorem 3.2 that is independent of Theorem 3.1 and Pachl (1979, 2013).

Thus, one obtains through Theorem 3.2 and Ziemlańska (2020) an alternative proof of BU(S)-weak sequential completeness of $\mathcal{M}(S)$.

(2) Theorem 3.2 is related to results on asymptotic proximity of sequences of distributions, e.g. see Davydov and Rotar (2009), Theorem 4. In that setting $\mu_n = P_n - Q_n$, where P_n and Q_n are probability measures. These are asymptotically proximate (for the $\|\cdot\|^*_{\rm BL}$-norm; other norms are considered in Davydov and Rotar, 2009 too) if $\|P_n - Q_n\|^*_{\rm BL} \to 0$. So one knows in advance that $\langle \mu_n, f\rangle \to 0$. That is, the limit measure μ exists: $\mu = 0$. Combining such a result with the BU(S)-weak sequential completeness of $\mathcal{M}(S)$ implies Theorem 3.2.

(3) If the sequence of measures μ_n in Theorem 3.2 consists of positive measures, then the result is an easy consequence of Pachl (2013) Theorem 5.36 and the BU(S)-weak sequential completeness of $\mathcal{M}(S)$.

(4) The general case for signed measures cannot be reduced straightforwardly to the result for positive measures mentioned in point (3). This is mainly caused by the complication, that for a sequence (μ_n) of signed measures such that $\langle \mu_n, f\rangle$ is convergent for every $f \in {\rm BL}(S)$, it need not hold that $\left\langle \mu_n^+, f\right\rangle$ and $\left\langle \mu_n^-, f\right\rangle$ converge for every $f \in {\rm BL}(S)$. Take for example on $S = \mathbb{R}$ with the usual Euclidean metric $\mu_n := \delta_n - \delta_{n+\frac{1}{n}}$. Then $\langle \mu_n, f\rangle \to 0$ for every $f \in {\rm BL}(\mathbb{R})$. However, $\mu_n^+ = \delta_n$ and $\mu_n^- = \delta_{n+\frac{1}{n}}$, so $\left\langle \mu_n^\pm, f\right\rangle$ will not converge for every $f \in {\rm BL}(\mathbb{R})$.

4. Equicontinuous families of Markov operators

We start by recalling the concept of equicontinuity, for clarity. Let T be a topological space and (X, ρ) a metric space. A family of functions $\mathcal{E} \subset C(T, X)$ is *equicontinuous at* $t_0 \in T$ if for every $\varepsilon > 0$ there exists an open neighbourhood U_{t_0} of t_0 such that

$$\rho(f(t), f(t_0)) < \varepsilon \quad \text{for all } f \in \mathcal{E},\ t \in U_{t_0}.$$

$\mathcal{E}$ is *equicontinuous* if it is equicontinuous at every point of T. Any subset of an equicontinuous family of maps is itself equicontinuous.

Following Szarek et al. (Komorowski et al., 2010), a family $(P_\lambda)_{\lambda\in\Lambda}$ of regular Markov operators has the *e-property* if for each $f \in {\rm BL}(S)$ the family $\{U_\lambda f : \lambda \in \Lambda\}$ is equicontinuous in $C(S, \mathbb{R})$, where U_λ is the dual operator of P_λ. Necessarily, the operators P_λ are Feller. In particular one may consider the family of iterates of a single Markov operator P: $(P^n)_{n\in\mathbb{N}}$, or Markov semigroups $(P_t)_{t\in\mathbb{R}^+}$, where each P_t is a regular Markov operator and $P_0 = I$, $P_tP_s = P_{t+s}$. Worm defined in Worm (2010), Chapter 7, that a Markov operator P has the *Cesàro e-property*, if the family $\{\frac{1}{n}\sum_{k=0}^{n-1} U^k f : n \in \mathbb{N}\}$ is equicontinuous in $C(S, \mathbb{R})$ for every $f \in {\rm BL}(S)$. Thus, a Markov operator with the e-property has the Cesàro e-property. One may also encounter Markov operators that are *non-expansive* for the $\|\cdot\|^*_{\rm BL}$ norm, or that satisfy the so-called *spectral gap condition*:

$$\|P^n\mu - P^n\nu\|^*_{\rm BL} \le C\theta^n \|\mu - \nu\|^*_{\rm BL} \quad \text{for all } n \in \mathbb{N},\ \mu, \nu \in \mathcal{P}(S) \tag{4.1}$$

for some $0 \le \theta < 1$ and $C > 0$ (cf. Gulgowksi et al., 2019). Both classes of Markov operators have the e-property, as can be verified from the properties $\|P\mu - P\nu\|^*_{\rm BL} \le \|\mu - \nu\|^*_{\rm BL}$ or (4.1).

Our main result on equicontinuous families of Markov operators is:

Theorem 4.1. *Let $\{P_\lambda : \lambda \in \Lambda\}$ be a family of regular Markov operators on a complete separable metric space (S, d). Let U_λ be the dual Markov operator of P_λ. The following statements are equivalent:*

(i) $\{P_\lambda : \lambda \in \Lambda\}$ *is equicontinuous in* $C\big(\mathcal{M}^+(S)_{\rm BL}, \mathcal{M}^+(S)_{\rm BL}\big)$.
(ii) $\{P_\lambda : \lambda \in \Lambda\}$ *is equicontinuous in* $C\big(\mathcal{P}(S)_{\rm BL}, \mathcal{P}(S)_{\rm BL}\big)$.
(iii) $\{U_\lambda f : \lambda \in \Lambda,\ f \in {\rm BL}(S), \|f\|_{\rm BL} \le 1\}$ *is equicontinuous in* $C(S, \mathbb{R})$.
(iv) $\{U_\lambda f : \lambda \in \Lambda\}$ *is equicontinuous in* $C(S, \mathbb{R})$ *for every* $f \in {\rm BL}(S)$.

Proof. *(i) ⇒ (ii).* Follows immediately by restriction of the Markov operators P_Λ to $\mathcal{P}(S)$.
(ii) ⇒ (iii). Let $x_0 \in S$. Let $\varepsilon > 0$. Since $\{P_\lambda : \lambda \in \Lambda\}$ is equicontinuous at δ_{x_0} there exists an open neighbourhood V of δ_{x_0} in $\mathcal{P}(S)_{\rm weak}$ such that

$$\|P_\lambda\delta_{x_0} - P_\lambda\mu\|^*_{\rm BL} < \varepsilon \quad \text{for all } \lambda \in \Lambda \text{ and } \mu \in V.$$

Since the map $x \mapsto \delta_x : S \to \mathcal{P}(S)_{\rm weak}$ is continuous, there exists an open neighbourhood V_0 of x_0 in S such that $\delta_x \in V$ for all $x \in V_0$. Then, uniformly for f in the unit ball of ${\rm BL}(S)$,

$$|U_\lambda f(x) - U_\lambda f(x_0)| = |\left\langle P_\lambda\delta_x - P_\lambda\delta_{x_0}, f\right\rangle| \le \varepsilon \cdot \|f\|_{\rm BL} < \varepsilon$$

for all $x \in V_0$ and $\lambda \in \Lambda$.
(iii) ⇒ (iv). Immediate.

(iv) ⇒ *(i)*. Assume on the contrary that $\{P_\lambda : \lambda \in \lambda\}$ is not an equicontinuous family of maps. Then there exists a point $\mu_0 \in \mathcal{M}^+(S)$ at which this family is not equicontinuous. Hence there exists $\varepsilon_0 > 0$ such that for every $k \in \mathbb{N}$ there are $\lambda_k \in \Lambda$ and $\mu_k \in \mathcal{M}^+(S)$ such that

$$\|\mu_k - \mu_0\|_{\mathrm{BL}}^* < \tfrac{1}{k} \quad \text{and} \quad \|P_{\lambda_k}\mu_k - P_{\lambda_k}\mu_0\|_{\mathrm{BL}}^* \geq \varepsilon_0 \qquad \text{for all } k \in \mathbb{N}. \tag{4.2}$$

Because the measures μ_k are positive and the $\|\cdot\|_{\mathrm{BL}}^*$-norm metrises the $\mathrm{BC}(S)$-weak topology on $\mathcal{M}^+(S)$ (cf. Dudley, 1966, Theorem 18), $\langle \mu_k, f\rangle \to \langle \mu_0, f\rangle$ for every $f \in \mathrm{BC}(S)$. According to Dudley (1966), Theorem 7, this convergence is uniform on any equicontinuous and uniformly bounded subset $\mathcal{E}$ of $\mathrm{BC}(S)$. By assumption, $\mathcal{M}_f := \{U_{\lambda_k} f : k \in \mathbb{N}\}$ is such a family for every $f \in \mathrm{BL}(S)$. Therefore

$$|\langle P_{\lambda_k}\mu_k - P_{\lambda_k}\mu_0, f\rangle| = |\langle \mu_k - \mu_0, U_{\lambda_k} f\rangle| \to 0 \tag{4.3}$$

as $k \to \infty$ for every $f \in \mathrm{BL}(S)$. Since for positive measures μ one has $\|\mu\|_{\mathrm{TV}} = \|\mu\|_{\mathrm{BL}}^*$, one obtains

$$\big|\|\mu_k\|_{\mathrm{TV}} - \|\mu_0\|_{\mathrm{TV}}\big| \leq \|\mu_k - \mu_0\|_{\mathrm{BL}}^* \to 0.$$

So $m_0 := \sup_{k\geq 1} \|\mu_k\|_{\mathrm{TV}} < \infty$. Moreover,

$$\|P_{\lambda_k}\mu_k - P_{\lambda_k}\mu_0\|_{\mathrm{TV}} \leq \|P_{\lambda_k}\mu_k\|_{\mathrm{TV}} + \|P_{\lambda_k}\mu_0\|_{\mathrm{TV}} \leq \|\mu_k\|_{\mathrm{TV}} + \|\mu_0\|_{\mathrm{TV}} \leq m_0 + \|\mu_0\|_{\mathrm{TV}}.$$

Theorem 3.2 and (4.3) yield that $\|P_{\lambda_k}\mu_k - P_{\lambda_k}\mu_0\|_{\mathrm{BL}}^* \to 0$ as $k \to \infty$. This contradicts the second property in (4.2). □

A particular class of examples of Markov operators or semigroups is furnished by the lift of a map or semigroup $(\phi_t)_{t\geq 0}$ of measurable maps $\phi_t : S \to S$ to measures on S by means of push-forward:

$$P_t^\phi \mu(E) := \mu\big(\phi_t^{-1}(E)\big) \qquad \mu \in \mathcal{M}^+(S),\ E \subset S \text{ Borel}.$$

Proposition 4.1. *Let (S, d) be a complete separable metric space and let $(\phi_t)_{t\geq 0}$ be a semigroup of Borel measurable transformations of S. Then P_t^ϕ is a regular Markov operator for each $t \geq 0$. Moreover, $(P_t^\phi)_{t\geq 0}$ is equicontinuous in $C\big(\mathcal{M}^+(S)_{\mathrm{BL}}, (\mathcal{M}^+(S), \|\cdot\|_{\mathrm{BL}}^*)\big)$ if and only if $(\phi_t)_{t\geq 0}$ is equicontinuous in $C(S, S)$.*

Proof. The regularity of P_t^ϕ is immediate, as $U_t^\phi f = f \circ \phi_t$.
'⇒': Let $x_0 \in S$ and $\varepsilon > 0$. Define $h : [0, \infty) \to [0, 2) : u \mapsto 2u/(2+u)$ and put $\varepsilon' := h(\varepsilon)$. By equicontinuity of $(P_t^\phi)_{t\geq 0}$ at δ_{x_0}, there exists and open neighbourhood U of δ_{x_0} in $\mathcal{M}^+(S)_{\mathrm{BL}}$ such that

$$\|P_t^\phi \mu - P_t^\phi \delta_{x_0}\|_{\mathrm{BL}}^* < \varepsilon'$$

for all $t \geq 0$ and $\mu \in U$. Because the map $\delta : x \mapsto \delta_x : S \to \mathcal{M}^+(S)_{\mathrm{BL}}$ is continuous, $U_0 := \delta^{-1}(U)$ is open in S. It contains x_0. Moreover,

$$\|P_t^\phi \delta_x - P_t^\phi \delta_{x_0}\|_{\mathrm{BL}}^* = \|\delta_{\phi_t(x)} - \delta_{\phi_t(x_0)}\|_{\mathrm{BL}}^* = h\big(d\phi_t(x), \phi_t(x_0)\big) < \varepsilon' \tag{4.4}$$

for all $x \in U_0$ and $t \geq 0$ (see Hille and Worm, 2009b Lemma 3.5 for a detailed proof of the last equality in Eq. (4.4)). Because h is monotone increasing,

$$d\big(\phi_t(x), \phi_t(x_0)\big) < \varepsilon \qquad \text{for all } x \in U_0,\ t \geq 0.$$

'⇐': Let $f \in \mathrm{BL}(S)$. Let U_t be the dual operator of P_t. Then for all $x, x_0 \in S$,

$$|U_t f(x) - U_t f(x_0)| = |f(\phi_t(x)) - f(\phi_t(x_0))| \leq |f|_L d(\phi_t(x), \phi_t(x_0)),$$

from which the equicontinuity of $\{U_t f : t \geq 0\}$ follows. Now apply Theorem 4.1. □

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