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Estimating inequality aversion from subjective assessments of the just noticeable differences in welfare

JEL Classification: *C18;D11; D31; I31*

Keywords: *inequality; aversion; income; utility estimation*

Abstract
Research background: In Economics, the concept of inequality aversion corresponds with the concept of risk aversion in the literature on making decision under uncertainty. The risk aversion is estimated on the basis of subjective reactions of people to various lottery prospects. In Economics, however, an efficient method of estimating inequality aversion has not been developed yet.
Purpose of the article: The main aim of this paper is to develop the method of estimating inequality aversion.
Methods: The method is based on two income thresholds which are subjectively assessed by surveyed respondents. Given the level of household income, just noticeable worsening of household welfare is perceived below the first threshold, whereas just noticeable improvement of household welfare is perceived above the second threshold. The thresholds make possible effective calculations of the parameter of the Arrow-Pratt's constant inequality aversion utility function. In this way, an individual utility of income becomes an empirically observable economic phenomenon.
Findings & Value added: In this paper, two theorems are proved which provide the guidance on how to identify a proper version of the above function. The proposed method is tested on the basis of statistical data from the archival survey conducted among Polish households in 1999. The statistical analysis of those data reveals the appearance of convex utility functions as well as concave ones. Nevertheless, the prevailing part of the Polish society exhibited inequality aversion in the year 1999. Another result of this paper is that inequality aversion diminishes as income increases.

Introduction

This paper proposes a method of estimating household utility function. We assume the Arrow-Pratt's form of the utility function with parameter ε (Pratt, 1964; Arrow, 1965). This utility function is commonly used in many branches of Economics.

In the literature on making decision under uncertainty, parameter ε is interpreted as the risk aversion. Hence the utility function is called the constant risk aversion function[1]. Many other parametric forms of utility function are applied in this field. Parameters of those functions are estimated on the basis of subjective reactions of people to various lottery prospects (Levy & Levy, 2001; LiCalzi & Sorato, 2006). Lambert (2001, p. 129) reviewed various studies where inequality aversion was estimated. However, the results of those studies seem ambiguous.

In the literature on economic growth with consumer optimisation[2], the constant risk aversion function represents household's utility function. The elasticity of marginal utility equals $\varepsilon>0$ and $1/\varepsilon$ is the elasticity of substitution, Hence this function is called the *constant intertemporal elasticity of substitution utility function* (Barro & Sala-i-Martin , 2004, p. 91).

In Income Distribution Economics, inequality plays the role of a risk. For this reason ε is interpreted as the measure of *inequality aversion* and the Arrow-Pratt's utility function is called the *constant inequality aversion function* (CIAF). Parameter ε describes an impartial observer's attitude towards inequality when he/she judges welfare in income distributions. It is evident in the Kolm-Atkinson's concept of *equally distributed equivalent income* which is the ethical measure of the social welfare (Kolm, 1969; Atkinson, 1970). Also ε parameterises the Atkinson's family of economic inequality indices as well as the family of the generalized entropy indices of inequality (see, among others, Sen & Foster, 1997; Creedy, 1998; Lambert, 2001, p. 112). Moreover, the CIAF enables developing the parametric probability distribution of welfare where the parametric form of the density function of incomes is known (Kot, 2012).

Although parameter ε of inequality aversion plays such an important role in economic research, its values are unknown because economists traditionally conceive utility functions as empirically unobservable phenomena. In Economics, the long-lasting tradition is to set ε at an arbitrary level. Obviously, the arbitrariness in setting ε results in arbitrariness of research findings.

[1] In general, the Arrow-Pratt utility function can exhibit varying risk aversion.

[2] Consumer behaviour is a key element in the Ramsey growth model, as constructed by Ramsey (1928), and refined by Cass (1965) and Koopmans (1965).

Our method of estimating the parameter ε of inequality aversion seems to remove this arbitrariness. The method is based on two income thresholds which are subjectively assessed by surveyed respondents. Given the level of a household's monthly income, the barely noticeable worsening of household welfare is perceived below the first threshold, whereas barely noticeable improvement of household welfare is perceived above the second threshold. We check the proposed method using archival statistical data from a survey conducted among Polish households in 1999 (Kot, 2000).

We are not the first in estimating utility functions of income on the basis of subjective assessments of welfare. For instance, researchers from the "Leyden group" propose a system of income evaluation questions (IEQ) and apply the lognormal distribution function for modelling utility functions. The IEQ provides an empirical base for the estimation of the cardinal utility function. This approach has been adopted by Kapteyn and Van Praag (1976), Kapteyn *et al.* (1988), Hagenaars (1986), Dubnoff (1979), Vaughan (1984), Danzinger *et al*. (1984), Colasanto *et al.* (1984), De Vos and Garner (1986).

However, economists have reservations with regard to the Leyden approach. For instance, Seidl (1994) maintained that the IEQ fails to fulfil well-established requirements of psychological measurement. Moreover, he argued that the 'correct' measurement leads to unbounded utility functions, amongst which only the power function and the logarithmic are acceptable. On the Leyden approach, he stated that: 'this edifice is not built on solid ground, neither from the point of economic theory nor of experimental psychology'. In the response to Seidl, Van Praag and Kapteyn (1994), rejected his principal arguments as 'ill-founded'.

Kot (1998, 2000) proposed the set of income evaluation questions which is an alternative to the Leyden approach. In this paper, we show that only two Kot's questions are enough for estimating the CRAF.

Recently, Ravallion and Lokshin (2002) evaluate subjective economic welfare in Russia. The Authors' data on subjective perceptions use survey responses to a question in which respondents say what their level of welfare from 'poor' to 'rich' is on a nine-point ladder (see also: Ravallion, 2011, 2012)

The rest of this paper is organised as follows. In Section II, the methodology of research is presented. Section III presents the results of empirical findings. Final Section concludes.

Research methodology

In this paper, the CIAF is defined as follows:

$$u(x)=\begin{cases}\dfrac{x^{1-\varepsilon}}{1-\varepsilon}, & for \quad 0\le\varepsilon<1, \quad (1)\\ \log x, & for \quad \varepsilon=1, \quad (2)\\ -\dfrac{x^{-(\varepsilon-1)}}{\varepsilon-1}, & for \quad \varepsilon>1 \quad (3)\end{cases}$$

where $x \geq 0$ is income, and ε is a constant parameter (Pratt, 1964). In fact, if $\varepsilon \neq 1$, the forms (1) and (2) describe the CIAF completely. However, we decided to distinguish between the variants $0 \leq \varepsilon < 1$ and $\varepsilon > 1$ of the CIAF because each of them reflects different economic properties.

The idea of inequality aversion has been borrowed from literature on making decisions under risk. If $u(x)$ is the utility function which has the first and second derivatives $u'(x)$ and $u''(x)$, respectively, the relative inequality aversion function $r(x)$ is defined as:

$$r(x)=-x\frac{u''(x)}{u'(x)} \quad (4)$$

(Pratt, 1964; Arrow, 1965). An individual whose income preferences are represented by a twice differentiable utility function $u(x)$ and $u'(x)>0$, is called a risk averter if $r(x)>0$, a risk lover if $r(x)<0$, and a risk neutral, or a risk indifferent, if $r(x)=0$.

It is easy to see that the relative risk aversion (4) for CIAF is constant. The first and second derivatives of CIAF are $u'(x)=x^{-\varepsilon}$, $u''(x)=-\varepsilon \cdot x^{-1-\varepsilon}$. Substituting these derivatives into (4) gives a risk aversion function equal to ε. We note that CIAF is concave if $\varepsilon>0$, convex if $\varepsilon<0$, and linear if $\varepsilon=0$. CIAF concavity is known as risk aversion in the context of individual choice under uncertainty (Lambert, 2001, p. 86)[3].

If $\varepsilon \leq 1$, as in (1) and (2), the CIAF is unbounded from above. On the other hand, CIAF (3), with $\varepsilon>1$, is bounded from above. An economist may chose either (1) or (3) according to his/her beliefs concerning the limits of

[3] The term 'risk aversion' is related to all increasing and concave utility functions. The CIAF is a special case of such functions.

economic welfare growth. Below we propose a simple criterion that helps to distinguish between these versions of CIAF.

In order to estimate ε parameter, we apply some well-established psychophysical methods. In a 'poikilitic' measurement, the psychophysical function is derived from the *just noticeable differences* of the stimulus variable associated with the *equal differences* of the sensation variable (see Stevens, 1975).

Let the sensation variable $u(y)$ be the household welfare (utility of income) and the current income y be the stimulus variable. A surveyed respondent is to imagine the situation where his/her household's current income increases by \$1, \$2, etc. until a *barley noticeable* difference in welfare is perceived. Next, the respondent imagines his/her current income decreases by \$1, \$2, etc. until a *barely noticeable* difference in welfare is perceived. In Kot (1998), the following questions were asked in a household survey:

– What was your household's disposable income in the last month?
– Imagine that your household's income was *higher* than the one actually earned in the last month. Please, evaluate such an income that would *just noticeably* improve your economic well-being.
– On the other hand, imagine that your household's income was *lower* than that actually earned in the last month. Please, evaluate such an income that would *barely noticeably* worsen your economic well-being.

Let y denote the household's income earned in the last month, x_1 the lower income threshold and x_2 the upper income threshold. We assume $x_1>x_2$ throughout of this paper. Fig. 1 illustrates a typical configuration of x_1, y and x_2.

In the case of a typical respondent, the y-x_1 distance is usually shorter than the x_2-y distance. This means that barely noticeable improvement welfare requires a greater increase of owned income than the decrease of owned income that barley noticeably worsened welfare would.

The calculation of inequality aversion ε of an individual who reports the quantity x_1, y and x_2, requires placing the utility of income $u(y)$ between $u(x_1)$ and $u(x_2)$. Fig. 2 illustrates this problem.

Fig. 2 presents two CIAFs, $u(x)$ and $u_1(x)$, with different parameters, ε and ε_1, respectively. For the sake of simplicity, we assume that these functions cross at x_1 and x_2 points[4].

[4] This is not a restrictive assumption. For two different CIAFs, $u(x)$ and $u^*(x)$, we can always find the numbers $a>0$, and b such that $u_1(x)=a\cdot u^*(x)+b$ satisfies the crossing conditions: $u_1(x_1)=u(x_1)$ and $u_1(x_2)=u(x_2)$. Obviously, $u_1(x)$ and $u^*(x)$ provide the same welfare ranking of income distributions because such rankings are invariant with respect to affine transformations (see, e.g. Roemer, 1996, p. 16).

The problem is to which of these utility functions the respondent's income y should be assigned. If $u(x)$ is chosen, ε will be the respondent's inequality aversion. When $u_1(x)$ is chosen, ε_1 will be the correct answer.

In general, $u(y)$ lies somewhere between $u(x_1)$ and $u(x_2)$, i.e.

$$u(y) = pu(x_1) + (1-p)u(x_2)\ ,\ 0<p<1 \tag{5}$$

If (5) holds and utility function has the form of (1) or (3), then ε will be the solution to the following nonlinear equation

$$F = px_1^{1-\varepsilon} + (1-p)x_2^{1-\varepsilon} - y^{1-\varepsilon} = 0\ ,\ \varepsilon\neq 1 \tag{6}$$

Here, F is the loss function. A numerical algorithm can be used for solving this non-linear equation. Notice that the obvious solution $\varepsilon=1$ to equation (6) should be excluded. The case of a CIAF with $\varepsilon=1$ will be discussed later.

It is also worth to notice that the solution to equation (6) is 'scale invariant'. For instance, if income y and thresholds x_1 and x_2 are deflated by the same constant, e.g., household size or any equivalence scale, equation (6) will remain unchanged.

The most important advantage of Eq. (6) is that it enables estimating inequality aversion and then the utility function for individual households as well as for individual household members. The latter possibility follows from the commonly accepted assumption that household income and welfare is evenly divided among household members (see, e.g., Moyes, 2012).

Some of the consequences of (5) and (6) are important for applications of the proposed method.

When $\varepsilon=1$, i.e. when the utility function has form (2), the equality (5) implies

$$y = x_1^p x_2^{1-p} = G \tag{7}$$

where G is the (generalised) geometric mean of the thresholds x_1 and x_2.

If $\varepsilon\neq 1$, i.e., when the utility function has either the form (1) or (3), then income y, calculated from eq. (6), is

$$y = \left[px_1^{1-\varepsilon} + (1-p)x_2^{1-\varepsilon}\right]^{1/(1-\varepsilon)} \tag{8}$$

In general, the following two theorems provide criteria for identification of the CIAF versions.

Theorem 1.

Let $M=px_1+(1-p)x_2$ and $G = x_1^p x_2^{1-p}$. If $x_1<y<x_2$ and (5) holds, then for all $p\epsilon(0,1)$

$$y>M, \text{ if and only if } \varepsilon<0 \quad (9a)$$

$$y=M, \text{ if and only if } \varepsilon=0 \quad (9b)$$

$$G<y<M, \text{ if and only if } 0<\varepsilon<1 \quad (9c)$$

$$y=G \text{ if and only if } \varepsilon=1 \quad (9d)$$

$$y<G, \text{ if and only if } \varepsilon>1 \quad (9e)$$

Theorem 2.

If (5) holds and $y=x_1$ or $y=x_2$, then $\varepsilon=1$. (for proofs see: Appendix).

Notice that M and G are the arithmetic mean and the geometric mean of the thresholds x_1 and x_2. If n households are surveyed, we use the symbols x_{1i}, y_i, x_{2i}, M_i and G_i, $i=1,\ldots,n$. The individual statistics M_i and G_i should not be confused with the arithmetic and geometric means of household incomes y_i.

From the above theorems, it follows that we need to calculate inequality aversion only for the cases (9a), (9c) and (9e). In other cases, ε is set either to 0 or to 1.

The 'location' parameter p should be set for a unique solution of equation (6). The lack of knowledge about p means the state of ignorance, i.e. the *state of maximal entropy* where all functions $u(y)$ belonging to open interval $(u(x_1),u(x_2))$ are equally probable. The entropy will be maximal, if $p=1/2$, i.e., if $u(y)=[u(x_1)+u(x_2)]/2$. In other words, we assign the mean of utilities $u(x_1)$ and $u(x_2)$ to income y. This approach is in accordance with Lerner's (1944) advice (see Lambert, 2001, p. 92)[5]. Hereafter, $p=0.5$ will be applied.

[5] Lerner (1944, p. 9) was the first to propose the mean value solution to the problem of assigning a utility function to a person, assuming a state of ignorance. Also see Thistle

It is worth to notice that the identification of the CIAF's version is independent of the choice of *p*. This is due to that Theorems 1 and 2 are valid for all $p\epsilon(0,1)$.

Fig. 3 illustrates the calculation of ε for a household which provides the following data: x_1=400, y=510, x_2=600. Here, income y is greater than the arithmetic mean M=500 of the thresholds x_1 and x_2, so $\varepsilon<0$ (the case (9a) of Theorem 1).

The calculation gives ε= -1.01. Negative ε means that the utility function is convex.

In Economics, however, concave utility functions are generally preferred. Such functions have a declining marginal utility of income, which is a fundamental property to all approaches (Lambert, 2001, p.94).

Figures 4 illustrates the loss function *F* for the case (9c) of Theorem 1, i.e. for $0<\varepsilon<1$. Fig. 5 illustrates the loss function *F* for the case (9e) of Theorem 1, i.e. for $\varepsilon>1$.

Special attention should be paid to the precision of calculating $\varepsilon>1$. As Fig. 5 shows, loss function *F* crosses zero in the extremely narrow interval of $\pm\ 2\cdot 10^{-15}$. Therefore, the solution to eq. (6) might be overlooked unless calculations are performed with double precision.

Empirical results of the research

We used — with permission — archival statistical data from the survey conducted among Polish households by The Public Opinion Research Center (CBOS) in October, 1999. The details are presented in Kot (2000, Ch. IV). In this paper, the same data are used for quite a different purpose, i.e., to check the validity of the proposed method of estimating individual inequality aversion. A new survey is currently being planned.

Table 1 presents the structure of the sample of 812 households with regard to inequality aversion. The selection criterion bases on Theorems 1 and 2. Additionally, the mean and coefficient of the variation of the per capita disposable income are shown.

The analysis of the results presented in Table 1 shows that in the year 1999 the Polish society was predominantly (80%) inequality averse. Utility function (3) with $\varepsilon>1$ is the dominant form of the CIAF. Only 2 per cent of surveyed households exhibited ε in the (0,1) interval, and yet this model is

(1997, pp. 2–3).

most usually assumed by economists. It is also worth noting that utility functions with $\varepsilon>1$ are negative.[6]

Our empirical analysis reveals the group of households with convex utility functions. The results presented in Table 1 show that the null or negative inequality aversion characterises rich households. On the other hand, positive inequality aversion is typical of relatively poor households.

Table 2 presents mean values of thresholds x_1, x_2 and household income y in the same groups as in Table 1.

Table 3 presents the estimates of inequality aversion and 95% confidence intervals. The distributions of ε is presented in Fig. 6, 7 and 8.

The distribution of $\varepsilon<0$ (Fig.6) is skewed to the left. The mode of this distribution is about -2.2. The distribution of $0<\varepsilon<1$ (Fig. 7) is also skewed to the left and has two modes: 0.52 and 0.96. The distribution of $\varepsilon>1$ (Fig.8) is skewed to the right and has the mode 1.99.

The shapes of the utility functions are presented in Fig. 9, 10, and 11. The mean values of ε from Table 3 were applied.

Finally, we analyse the impact of household welfare on inequality aversion. In Fig. 12, the fitted line suggests diminishing inequality aversion when income increases.

In order to test the statistical significance of the relationship presented in Fig. 12, we estimated the parameters of linear regression function $\varepsilon=\alpha_0+\alpha_1\cdot x$ The results are presented in Table 4.

This shows that parameter α_1 is less than zero at 0.05 significance level. This means that inequality aversion is a diminishing function of household income per capita.

Conclusions

The results of our research allow us to draw the following general conclusions:

- The proposed method of estimating individual inequality aversion proves to be quite accurate in confrontation with empirical data. This means that the individual utility of income can be treated as empirically observable.
- A priori identification of the form of utility function is possible when the arithmetic and geometric means of the thresholds are available.

[6] In mathematics, negative numbers are as good as positive ones. However, economists prefer positive rather than negative quantities. Atkinson (1970) uses $u^*(x)=a\cdot u(x)+b$, $a, b>0$, which makes some values of $u^*(x)$ positive, but negative utilities still remain.

– In reality, not only concave utility functions appear, but also convex ones. Economic theories cannot ignore this fact.
– In 1999, the Polish society was predominantly inequality averse. However, further investigations are necessary in order to ascertain whether or not this aversion is still so prevalent.
– Inequality aversion diminishes when individual income increases.

Although there are convincing arguments to support the assumption of a mid-point location of utility of income, these need to be verified by further investigations.

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Annex

Proofs of theorems 1 and 2

For the sake of convenience, we recall symbols used in Section 2:

$$u(x) = \begin{cases} \dfrac{x^{1-\varepsilon}}{1-\varepsilon}, & for \quad 0 \le \varepsilon < 1, \quad (1) \\ \log x, & for \quad \varepsilon = 1, \quad (2) \\ -\dfrac{x^{-(\varepsilon-1)}}{\varepsilon - 1}, & for \quad \varepsilon > 1 \quad (3) \end{cases}$$

For every $p\epsilon(0,1)$, $0<x_1<y<x_2$,

$$u(y) = p \cdot u(x_1) + (1-p) \cdot u(x_2) \qquad (5)$$

$$px_1^{1-\varepsilon} + (1-p)x_2^{1-\varepsilon} - y^{1-\varepsilon} = 0 \qquad (6)$$

$$y = x_1^p x_2^{1-p} = G \text{, for } \varepsilon=1 \qquad (7)$$

$$y = \left[px_1^{1-\varepsilon} + (1-p)x_2^{1-\varepsilon}\right]^{1/(1-\varepsilon)} \text{, for } \varepsilon \neq 1 \qquad (8)$$

$M=px_1+(1\text{-}p)x_2$, Obviously, $M>G$..

Theorem 1.

Let us assume that $x_1<y<x_2$ and (5) holds. Then for every $p\epsilon(0,1)$

$y>M$, if and only if $\varepsilon<0$ (9a)

$y=M$, if and only if $\varepsilon=0$ (9b)

$G<y<M$, if and only if $0<\varepsilon<1$ (9c)

If $\varepsilon=1$ if and only if $y=G$, (9d)

$y<G$, if and only if $\varepsilon>1$ (9e)

We apply Jensen's inequality when proving the Theorem 1.

If $f(x)$ is a real continuous function that is strictly concave and x is a non-degenerate random variable then

$$\mathrm{E}[f(x)] < f(\mathrm{E}[\mathrm{x}]) \tag{a1}$$

If $f(x)$ is strictly convex then

$$\mathrm{E}[f(x)] > f(\mathrm{E}[\mathrm{x}]) \tag{a2}$$

where $\mathrm{E}[\cdot]$ is the expectation's operator (Lambert, 2001, p. 11).

<u>Proof of thesis (9a).</u>

First we prove the implication: $\varepsilon<0 \rightarrow y>M$. For $\varepsilon<0$ $u(x)$ is strictly convex. Then (5) and (a2) imply

$$pu(x_1)+(1-p)u(x_2) > u[px_1+(1-p)x_2] \tag{a3}$$

Hence

$$\left[p\frac{x_1^{1-\varepsilon}}{1-\varepsilon}+(1-p)\frac{x_2^{1-\varepsilon}}{1-\varepsilon}\right] > \frac{1}{1-\varepsilon}\left[px_1+(1-p)x_2\right]^{1-\varepsilon}$$

and then

$$[px_1^{1-\varepsilon}+(1-p)x_2^{1-\varepsilon}]^{1/(1-\varepsilon)} > \left[px_1+(1-p)x_2\right]=M \tag{a4}$$

The left hand size of the above inequality is equal to y (8). Finally we have $y \geq M$.

Now we prove the implication: $y>M \rightarrow \varepsilon<0$. Assume $y>M$. i.e., (a4) holds. Then (a3) also holds. Observe that (a3) defines a strict convex function (Rudin (1976, p. 101). But $u(x)$ (1) or (3) is convex if and only if $\varepsilon<0$ that completes proof of the thesis (9a).

Proof of thesis (9b).

The implication $\varepsilon=0 \rightarrow y=M$ is obvious if we substitute $\varepsilon=0$ into equation (8). To prove the reverse implication: $y=M \rightarrow \varepsilon=0$, we again use (8) and set $y=M$, i.e.

$$\left[px_1^{1-\varepsilon}+(1-p)x_2^{1-\varepsilon}\right]^{1/(1-\varepsilon)} = px_1+(1-p)x_2$$

Then

$$\left[px_1^{1-\varepsilon}+(1-p)x_2^{1-\varepsilon}\right]=[px_1+(1-p)x_2]^{1-\varepsilon}$$

The preceding equality holds when either $\varepsilon=0$ or $p=0$ or $p=1$ or $x_1=x_2$. The latter three cases are excluded by the assumptions $0<p<1$ and $x_1>x_2$. When $\varepsilon\neq 0$, inequalities '<' or '>' hold instead of equality '=' because of Jensen's inequality. Therefore $\varepsilon=0$.

Proof of thesis 9c.

First we prove the implication: $0<\varepsilon<1 \rightarrow G<y<M$. When $\varepsilon>0$, $u(x)$ (1) is strictly concave. Then $u(x_1)^p \cdot u(x_2)^{1-p} < pu(x_1)+(1-p)u(x_2)$, i.e. the arithmetic mean is greater than the geometric mean. According to (a1), the right hand side of this inequality is less than $u[px_1+(1-p)x_2]$, i.e.,

$$u(x_1)^p \cdot u(x_2)^{1-p} < pu(x_1)+(1-p)u(x_2) < u[px_1+(1-p)x_2] \quad \text{(a5)}$$

The components in the preceding double inequality can be expressed respectively as

$$u(x_1)^p \cdot u(x_2)^{1-p} = \left(\frac{x_1^{1-\varepsilon}}{1-\varepsilon}\right)^p \left(\frac{x_2^{1-\varepsilon}}{1-\varepsilon}\right)^{1-p} =$$

$$\frac{1}{1-\varepsilon}\left(x_1^p x_2^{1-p}\right)^{1-\varepsilon} = \frac{1}{1-\varepsilon} G^{1-\varepsilon}$$

$$pu(x_1)+(1-p)u(x_2) = \frac{1}{1-\varepsilon}[px_1^{1-\varepsilon}+(1-p)x_2^{1-\varepsilon}]$$

$$u[px_1 + (1-p)x_2] = \frac{1}{1-\varepsilon}[px_1 + (1-p)x_2]^{1-\varepsilon} = \frac{1}{1-\varepsilon}M^{1-\varepsilon}$$

Then Jensens's inequality (a3) can be expressed as

$$\frac{1}{1-\varepsilon}G^{1-\varepsilon} < \frac{1}{1-\varepsilon}[px_1^{1-\varepsilon} + (1-p)x_2^{1-\varepsilon}] < \frac{1}{1-\varepsilon}M^{1-\varepsilon}$$

Multiplying all sides of this double inequality by a positive number 1-ε and raising to the power 1/(1-ε) give

$$G < \left[px_1^{1-\varepsilon} + (1-p)x_2^{1-\varepsilon}\right]^{1/(1-\varepsilon)} < M \tag{a6}$$

The middle component in (a4) is y (8). Therefore $G<y<M$. The implication $0<\varepsilon<1 \rightarrow G<y<M$ is proved.

Now we prove the reverse implication: $G<y<M \rightarrow 0<\varepsilon<1$. The 'if and only if' nature of the theses (9a) and (9b) excludes $\varepsilon \leq 0$. Therefore, condition $\varepsilon \neq 1$ means that ε must belong either to (0,1) or to (1,∞). Condition $G<y<M$ (a6) can be written as

$$G^{1-\varepsilon} < px_1^{1-\varepsilon} + (1-p)x_2^{1-\varepsilon} < M^{1-\varepsilon}$$

The indirect proof of the implication $G<y<M \rightarrow 0<\varepsilon<1$ runs as follows. We assume $\varepsilon>1$. Then the right hand side of preceding inequality can be expressed as

$$px_1^{-(\varepsilon-1)} + (1-p)x_2^{-(\varepsilon-1)} < M^{-(\varepsilon-1)} \tag{a7}$$

Notice that now function $f(x)=x^{-(\varepsilon-1)}$ is convex because its second derivative $f''(x)=\varepsilon(\varepsilon-1)$ is positive. Applying Jensen's inequality (a2) gives

$$px_1^{-(\varepsilon-1)} + (1-p)x_2^{-(\varepsilon-1)} > [px_1 + (1-p)x_2]^{-(\varepsilon-1)} = M^{-(\varepsilon-1)}$$

which contradicts to (a7). Therefore $\varepsilon \in (0,1)$ which completes the proof.

Proof of thesis (9d).

First, we prove the implication ε=1→ y=G. When ε=1, equations (2) and (5) imply

$$\log y = p\log x_1 + (1-p)\log x_2$$

Therefore $y = x_1^p x_2^{1-p} = G$.

The indirect proof of the implication y=G → ε=1 runs as follows. Assume that y=G but ε≠1. Then using (8) we get

$$y^{1-\varepsilon} = px_1^{1-\varepsilon} + (1-p)x_2^{1-\varepsilon} > (x_1^{1-\varepsilon})^p (x_2^{1-\varepsilon})^{1-p} = (x_1^p x_2^p)^{1-\varepsilon}$$

Therefore

$$y > x_1^p x_2^{1-p} = G$$

which contradicts to the assumption y=G. This completes the proof of the thesis (9d).

Proof of thesis (9e).

First, we prove the implication: ε>1 → G<y. When ε>0 the utility function has the form (3). Then equation (5) has the form

$$-\frac{1}{\varepsilon-1}y^{-(\varepsilon-1)} = -\left[p\frac{x_1^{-(\varepsilon-1)}}{\varepsilon-1} + (1-p)\frac{x_2^{-(\varepsilon-1)}}{\varepsilon-1}\right]$$

or

$$y^{-(\varepsilon-1)} = px_1^{-(\varepsilon-1)} + (1-p)x_2^{-(\varepsilon-1)} \qquad \text{(a7)}$$

Notice that the right hand side of the above equation is the mean of positive and different factors $x_1^{-(\varepsilon-1)}$ and $x_2^{-(\varepsilon-1)}$. Therefore the arithmetic mean of these factors is grater than their geometric mean, i.e.

$$px_1^{-(\varepsilon-1)} + (1-p)x_2^{-(\varepsilon-1)} > (x_1^{-(\varepsilon-1)})^p (x_2^{-(\varepsilon-1)})^{1-p} = G^{-(\varepsilon-1)} \qquad \text{(a8)}$$

Combining (a7) and (a8) we get

$$y^{-(\varepsilon-1)} > G^{-(\varepsilon-1)}$$

Rising both sides of this inequality to negative power -1/(ε-1) (and changing inequality direction) we get $y \leq G$.
The indirect proof of the implication: $G<y \rightarrow \varepsilon>1$ is trivial. Assume that $G<y$ and $\varepsilon<1$. If $\varepsilon<0$ then $y>M>G$ due to (9a). If $\varepsilon=0$ the $y=M>G$ due to (9b). And finally, if $\varepsilon \in (0,1)$ then (9c) implies $y>G$. In other words, assumption $\varepsilon<1$ leads to contradiction with $y<G$. It follows that $\varepsilon>1$ which completes the proof of the (9e).

The Theorem 2 concerns situations where household income y is equal to either of thresholds, i.e. either to x_1 or to x_2. In Theorem 1, we excluded such situations assuming $x_1<y<x_2$.

Theorem 2.

If (5) holds and $y=x_1$ or $y=x_2$, and $x_1 \neq x_2$, then $\varepsilon=1$.

Proof. If we set $y=x_1$ Eq. (6) gives

$$px_1^{1-\varepsilon} + (1-p)x_2^{1-\varepsilon} - x_1^{1-\varepsilon} = 0 \text{ , } \varepsilon \neq 1$$

Then

$$x_1^{1-\varepsilon} = x_2^{1-\varepsilon} \text{ ,}$$

which is true for $\varepsilon=1$, because $x_1 \neq x_2$. Obviously, when $y=x_2$, we will get the same result, i.e. $\varepsilon=1$.

Table 1. Versions of utility functions and household disposable income per capita

Inequality aversion	Selection criterion	Households		Income/ capita	
		N	%N	Mean	V.
$\varepsilon<0$	$y>M$	62	7.64	614	70.5
$\varepsilon=0$	$y=M$	100	12.32	595	69.4
$0<\varepsilon<1$	$G<y<M$	15	1.85	592	46.9
$\varepsilon=1$	$y=x_1$	81	10.00	282	53.3
	$y=x_2$	6	0.71	471	70.8
	$y=G$	5	0.62	380	22.4
$\varepsilon>1$	$x_1<y<G$	543	66.87	402	64.6
Total		**812**	**100.0**	**432**	**70.6**

Note: *M* –arithmetic mean of the thresholds x_1 and x_2 , *G*-geometric mean of the thresholds, *N*- the number of households in the sample, *%N* -percentage of households, *V*- coefficient of variation (standard deviation as the percentage of the arithmetic mean).

Source: own elaboration using data from Kot (2000), with kind permission.

Table 2. Average threshold and household income

Inequality aversion	Selection criterion	Lower Threshold x1	Household Income y	Upper Threshold x2
$\varepsilon<0$	$y>M$	1385	2099	2402
$\varepsilon=0$	$y=M$	1569	1983	2396
$0<\varepsilon<1$	$M>y>G$	1073	1893	3180
$\varepsilon=1$	$y=x_1$	1016	1016	1704
	$y=x_2$	1200	1650	1650
	$y=G$	1320	1900	2800
$\varepsilon>1$	$x_1<y<G$	1177	1367	2076
Total		**1224**	**1479**	**2125**

Source: own elaboration using data from Kot (2000), with kind permission.

Table 3. Estimates of inequality aversion

Version	Mean	Standard error	95% Confidence		Min.	Max.
			Lower	Upper		
$\varepsilon<0$	-3.68399	2.5207	-4.32412	-3.04386	-11.6697	-0.3066
$\varepsilon=0$	0.0					
$0<\varepsilon<1$	0.74449	0.2568	0.60229	0.88670	0.2992	1.0000
$\varepsilon=1$	1.0					
$\varepsilon>1$	2.27269	0.7250	2.21158	2.33381	1.0721	9.8693
Total	**1.36556**	**1.89282**	**1.23517**	**1.49594**	**-11.6697**	**9.8693**

Source: own elaboration using data from Kot (2000), with kind permission.

Table 4. Estimates of the regression function $\varepsilon = \alpha_0 + \alpha_1 \cdot x$

Parameter	Estimate	Standard error	t(810)	p
α_0	2.106519	0.118023	17.84830	0.000000
α_1	-0.001453	0.000194	-7.48573	0.000000

Figure 1. Income y and income thresholds x_1 and x_2 in a concave utility function

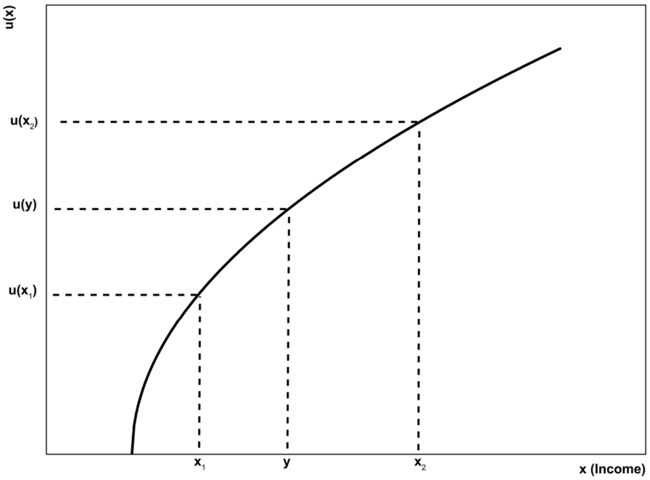

Figure 2. The position of $u(y)$ against $u(x_1)$ and $u(x_2)$ for two utility functions with different parameters of inequality aversion

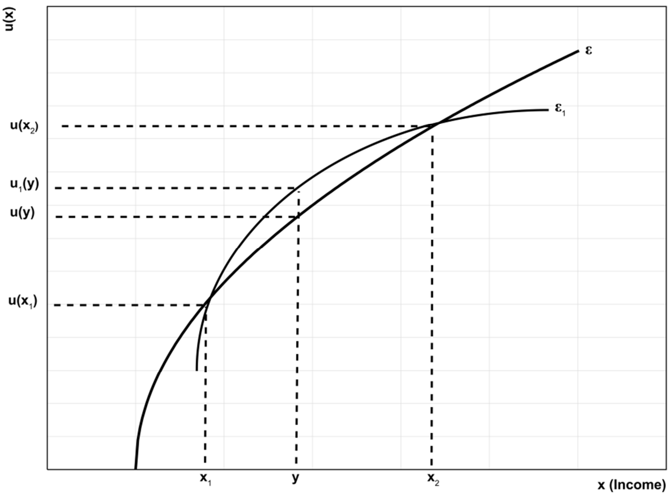

Figure 3. The loss function *F* for $y>M$ (x_1=400, y=510, x_2=600, ε= -1.01)

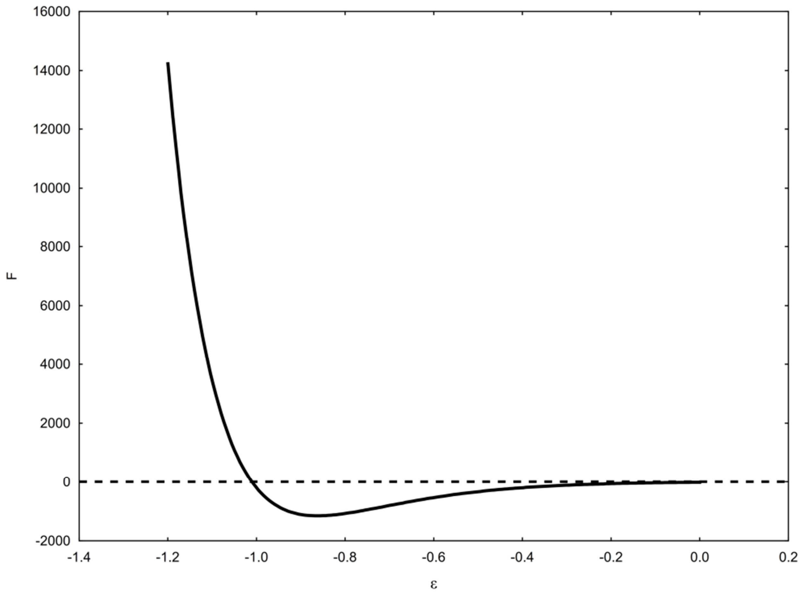

Figure 4. The loss function *F* for G<y<M (x_1=500, y=720, x_2=1000, ε=0.6982)

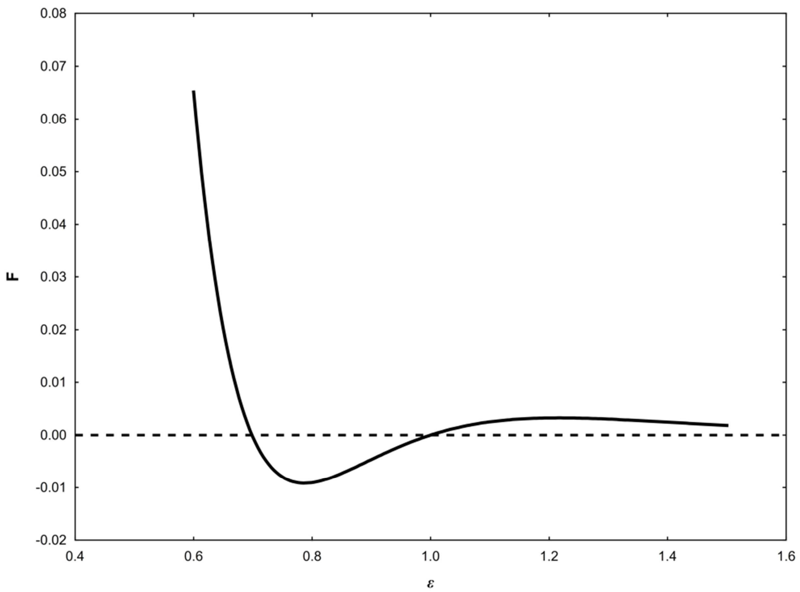

Figure 5. The loss function F for $y<G$ (x_1=350, y=400, x_2=700 ε=5.9525)

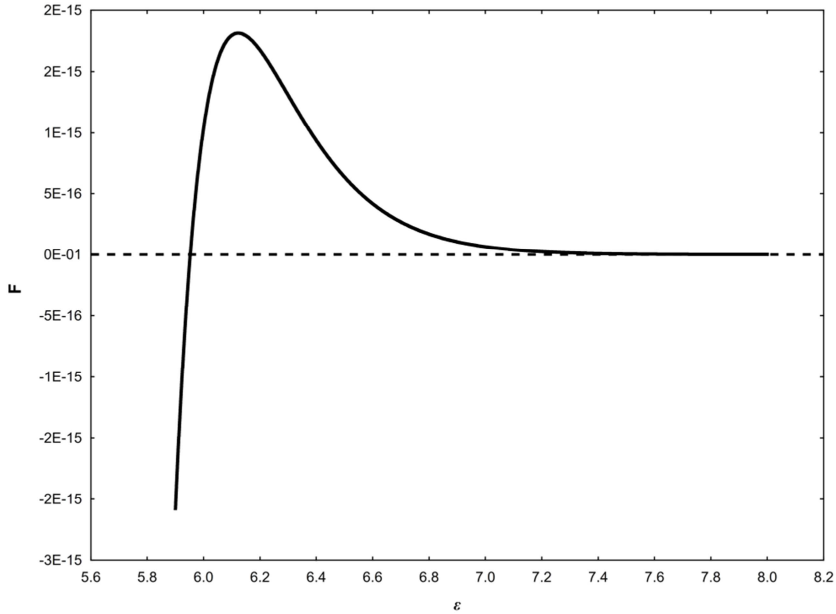

Figure 6. Kernel density function of inequality aversion (ε<0)

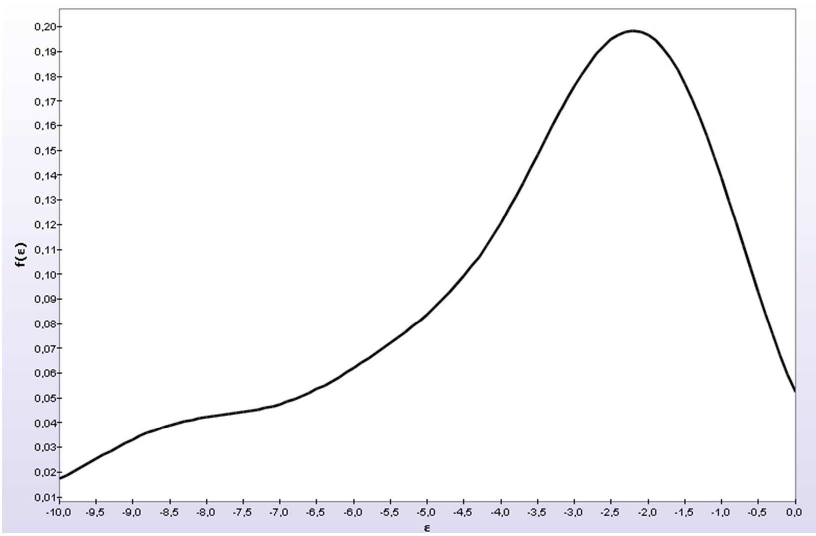

Figure 7. Kernel density function of inequality aversions (0<*ε*<1)

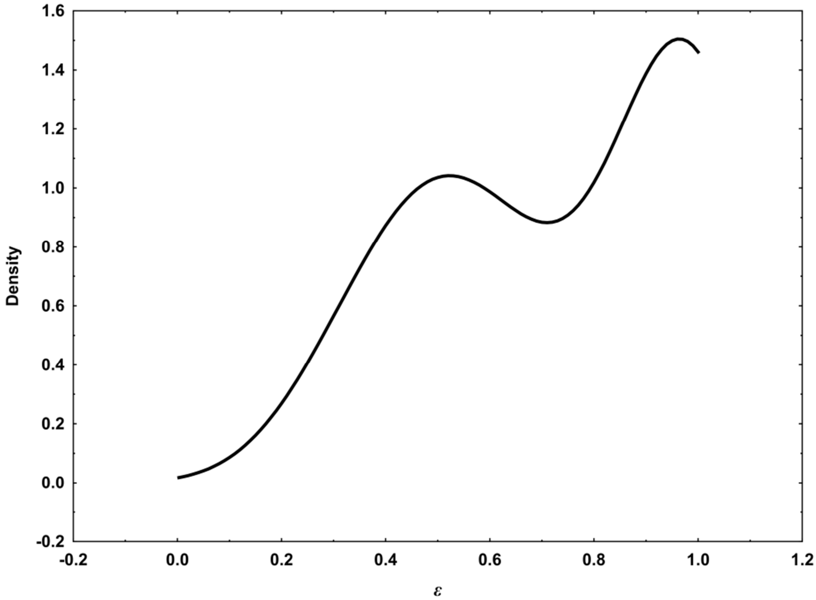

Figure 8. Kernel density function of inequality aversion (*ε*>1)

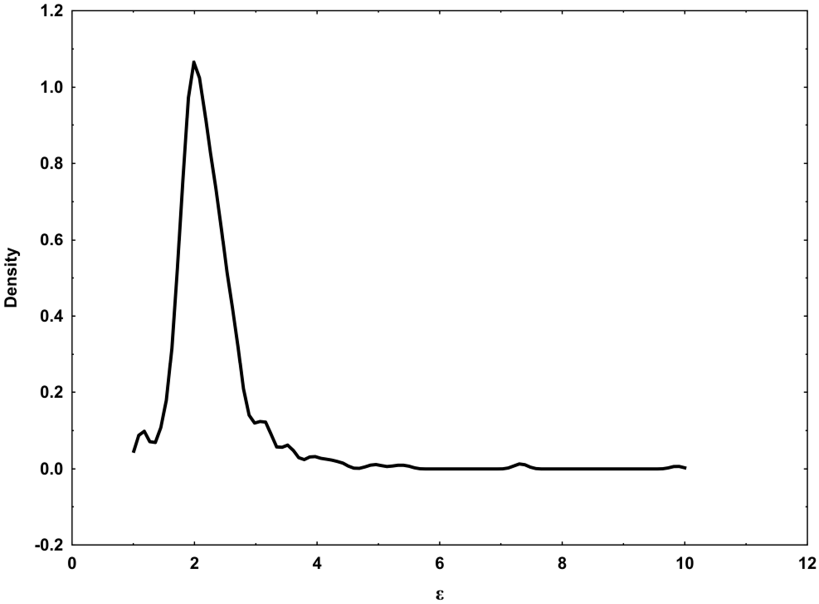

Figure 9. The utility function with ε= -3.68399

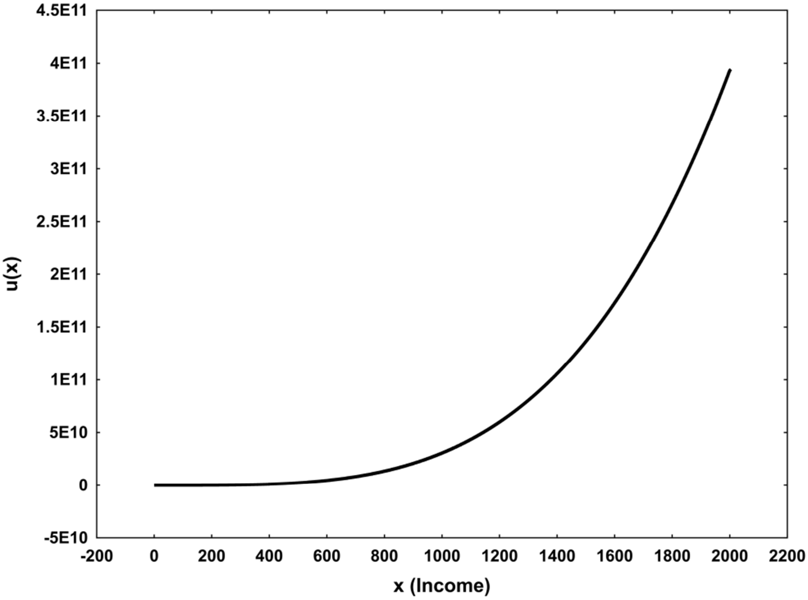

Figure 10. The utility functions with ε= 0.74449 (solid line) and ε=1 (doted line)

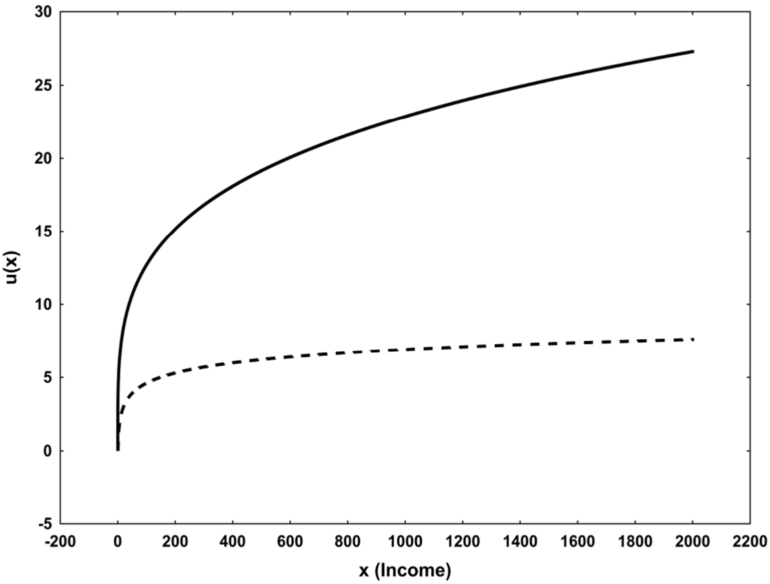

Figure 11. The utility function with ε= 2.27269

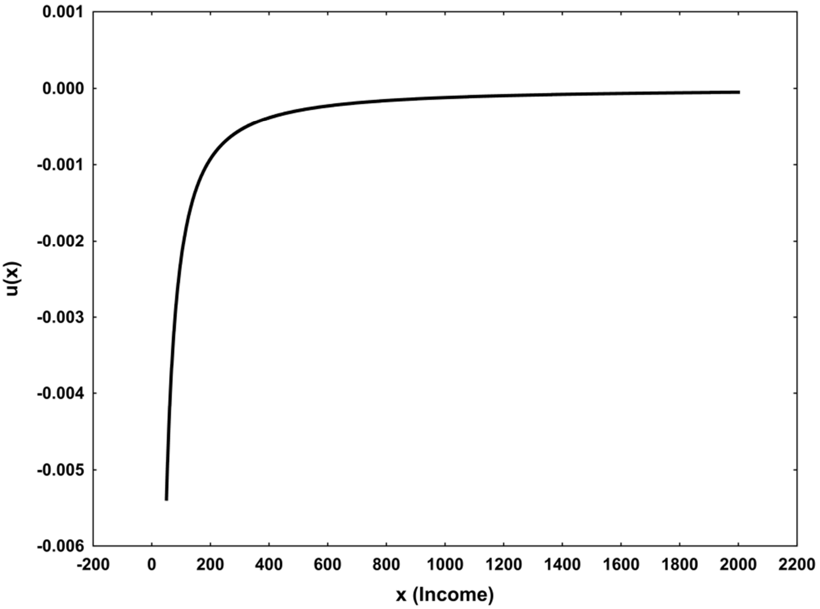

Figure 12. The relationship between inequality aversion and income