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Existence of positive solutions to second order four-point impulsive differential problems with deviating arguments

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ABSTRACT

In this paper, we discuss four-point boundary value problems for impulsive second order differential equations with deviating arguments. We investigate separately, cases when arguments are of delayed or advanced types. We formulate sufficient conditions under which our problems have at least one or two positive solutions. To obtain our results we apply the fixed point index.

1. Introduction

For $J = [0, 1]$, let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$. Put $J' = (0, 1) \setminus \{t_1, t_2, \ldots, t_m\}$. Put $\mathbb{R}_+ = [0, \infty)$ and $J_k = (t_k, t_{k+1}],\ k = 0, 1, \ldots, m-1,\ J_m = (t_m, t_{m+1})$.

Let us consider second order impulsive differential equations of type

$$\begin{cases} x''(t) + h(t)f(x(\alpha(t))) = 0, & t \in J', \\ \Delta x'(t_k) = Q_k(x(t_k)), & k = 1, 2, \ldots, m, \\ x(0) = \gamma x(\xi), & \beta x(\eta) = x(1), \end{cases} \tag{1}$$

where as usual $\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$; $x'(t_k^+)$ and $x'(t_k^-)$ denote the right and left limits of x' at t_k, respectively.

We assume that:

H_1: $f \in C(\mathbb{R}_+, \mathbb{R}_+),\ \alpha \in C(J, [0, 1]),\ t \le \alpha(t)$ for $t \in J$ and if there exists a point $\bar{t} \in J$ such that $\alpha(\bar{t}) \in \{t_1, t_2, \ldots, t_m\}$, then $\bar{t} \in \{t_1, t_2, \ldots, t_m\}$,

H_1': $f \in C(\mathbb{R}_+, \mathbb{R}_+),\ \alpha \in C(J, [0, 1]),\ \alpha(t) \le t$ for $t \in J$ and if there exists a point $\bar{t} \in J$ such that $\alpha(\bar{t}) \in \{t_1, t_2, \ldots, t_m\}$, then $\bar{t} \in \{t_1, t_2, \ldots, t_m\}$,

H_2: $h \in C(J, \mathbb{R}_+)$ and h does not vanish identically on any subinterval, $Q_k \in C(\mathbb{R}_+, (-\infty, 0])$ and are bounded for $k = 1, 2, \ldots, m$,

H_3 : $\xi, \eta \in (0, 1),\ \beta > 0,\ \gamma > 0,\ \gamma\xi(1-\beta) + (1-\beta\eta)(1-\gamma) > 0$,

H_4: $\xi, \eta \in (0, 1),\ 0 < \beta < \frac{1}{\eta},\ 0 < \gamma < \frac{1}{1-\xi},\ \gamma\xi(1-\beta) + (1-\beta\eta)(1-\gamma) > 0$.

Let us introduce the space:

$$PC^1(J, \mathbb{R}) = \left\{x \in C(J, \mathbb{R}), x|_{J_k} \in C^1(J_k, \mathbb{R}), k = 0, 1, \ldots, m \text{ and there exist } x'(t_k^+) \text{ for } k = 1, 2, \ldots, m\right\}$$

with the norm:

$$\|x\| = \sup_{t \in J} |x(t)|, \qquad \|x\|_{PC^1} = \|x\| + \|x'\|.$$

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Then $PC^1(J,\mathbb{R})$ is a Banach space. By a positive solution of problem (1) we mean a function $x \in PC^1(J,\mathbb{R}) \cap C^2(J',\mathbb{R})$ which is positive on $(0,1)$ and satisfies (1).

Throughout this paper we assume that $\alpha(t) \not\equiv t,\ t \in J$.

Impulsive differential equations have become more important in recent years; see for example books [1,2] and the references therein. In the existing literature, there is a lot of papers in which the problem of existence of positive solutions for differential equations has been investigated. The existence results are obtained by studying problems in the cone of nonnegative functions in suitable Banach spaces. Recently, we can find papers in which four-point boundary value problems for differential equations were investigated; see for example [3–8], see also papers [9–12] for three-point boundary value problems, see also [13,14]. However, only a few papers concern this problem for impulsive differential equations; see for example [15,16,9,5,6,17,18]. Usually, authors investigate differential equations without deviating arguments. It is important to indicate that it is a first paper in which the existence of positive solutions to four-point boundary value problems for impulsive differential equations is investigated by the fixed point index. Note that corresponding results for impulsive differential equations by using the Leggett–Williams theorem are obtained, for example, in papers [15,5,6].

This paper is organized as follows. In Section 2, we present some lemmas which are useful in our investigation. The main results of this paper are given in Section 4. Using the fixed point index we formulate sufficient conditions under which problem (1) has at least one or two positive solutions. Such results are formulated for cases when argument α is delayed (see Theorems 1–6), and in Theorems 7–12 if α is advanced.

2. Some lemmas

Let us consider the following problem

$$\begin{cases} u''(t) + y(t) = 0, & t \in J, \\ \Delta u'(t_k) = Q_k, & k = 1,2,\dots,m, \end{cases} \tag{2}$$

$$u(0) = \gamma u(\xi), \tag{3}$$

$$u(1) = \beta u(\eta). \tag{4}$$

We require the following:

Lemma 1 (*See [5]*). *Assume that* $\xi, \eta \in (0,1),\ \delta \equiv \gamma\xi(1-\beta) + (1-\beta\eta)(1-\gamma) \neq 0$ *and* $y \in C(J,\mathbb{R})$. *Then problem (2)–(4) has the unique solution given by the following formula*

$$\begin{aligned} u(t) = \frac{1}{\delta}\Bigg\{ &\gamma[1-\beta\eta + t(\beta-1)]\left[\sum_{i=1}^{r}{}' Q_i(\xi - t_i) - \int_0^\xi (\xi - s)y(s)\mathrm{d}s\right] \\ &+ [\gamma\xi - t(\gamma-1)]\left[\beta\sum_{i=1}^{j}{}' Q_i(\eta - t_i) - \sum_{i=1}^{m} Q_i(1-t_i) - \beta\int_0^\eta (\eta - s)y(s)\mathrm{d}s + \int_0^1 (1-s)y(s)\mathrm{d}s\right]\Bigg\} \\ &+ \sum_{i=1}^{k}{}' Q_i(t-t_i) - \int_0^t (t-s)y(s)\mathrm{d}s \end{aligned} \tag{5}$$

for $t \in J_k,\ k = 0,1,\dots,m,\ \xi \in J_r,\ \eta \in J_j$ *and* $r, j = 0,1,\dots,m$ *with* $\sum_{i=q}^{s\prime} \dots = 0$ *if* $q > s$.

Lemma 2 (*See [5]*). *Let Assumption* H_3 *hold. Assume that* $Q_i \le 0,\ i = 1,2,\dots,m$ *and* $y \in C(J,\mathbb{R}_+)$. *Then the unique solution* u *of problem (2)–(4) satisfies the condition* $u(t) \ge 0$ *on* $[0,1]$ *provided that*

(i) $\xi < \eta$ *and* $0 < \beta < \frac{1}{\eta},\ 0 < \gamma < \frac{1}{1-\xi}$, *or*
(ii) $\xi \ge \eta$.

Lemma 3 (*See Appendix, [6]*). *Let Assumption* H_4 *hold and* $\xi < \eta$. *Assume that* $y \in C(J,\mathbb{R}_+)$. *Then the unique solution* u *of problem (2)–(4) satisfies the condition*

$$\min_{[0,\xi]} u(t) \ge \Gamma_1\|u\| \quad \text{and} \quad \min_{[\eta,1]} u(t) \ge \Gamma_2\|u\|, \tag{6}$$

where

$$\Gamma_1 = \begin{cases} \min\left\{\dfrac{\xi\gamma}{1+\gamma\xi-\gamma}, \dfrac{(1-\eta)\gamma}{1-\beta\eta}\right\} & \text{if } 0 < \gamma < 1,\ 0 < \beta < 1, \\ \dfrac{\xi\gamma}{1+\gamma\xi-\gamma} & \text{if } 0 < \gamma < 1,\ 1 \le \beta < \dfrac{1}{\eta}, \\ \dfrac{1-\eta}{1-\beta\eta} & \text{if } 1 \le \gamma < \dfrac{1}{1-\xi},\ 0 < \beta < 1, \end{cases}$$

$$\Gamma_2 = \begin{cases} \min\left(\dfrac{\xi\beta}{1+\gamma\xi-\gamma}, \dfrac{(1-\eta)\beta}{1-\beta\eta}\right) & \text{if } 0<\gamma<1,\ 0<\beta<1, \\ \dfrac{\xi}{1+\gamma\xi-\gamma} \quad \text{if } 0<\gamma<1,\ 1\le\beta<\dfrac{1}{\eta}, \\ \dfrac{(1-\eta)\beta}{1-\beta\eta} \quad \text{if } 1\le\gamma<\dfrac{1}{1-\xi},\ 0<\beta<1. \end{cases}$$

Lemma 4 *(See Appendix, [6]). Let Assumption* H_3 *hold and* $\eta < \xi$. *Assume that* $y \in C(J, \mathbb{R}_+)$. *Then the unique solution* u *of problem (2)–(4) satisfies the condition*

$$\min_{[0,\eta]} u(t) \ge \Gamma_3\|u\| \quad \text{and} \quad \min_{[\xi,1]} u(t) \ge \Gamma_4\|u\|, \tag{7}$$

where

$$\Gamma_3 = \begin{cases} \gamma\min(1-\xi,\eta) & \text{if } 0<\gamma<1,\ 0<\beta<1, \\ \gamma\eta \quad \text{if } 0<\gamma<1,\ 1\le\beta, \\ 1-\xi \quad \text{if } 1\le\gamma,\ 0<\beta<1, \end{cases}$$

$$\Gamma_4 = \begin{cases} \beta\min(1-\xi,\eta) & \text{if } 0<\gamma<1,\ 0<\beta<1, \\ \eta \quad \text{if } 0<\gamma<1,\ 1\le\beta, \\ \beta(1-\xi) \quad \text{if } 1\le\gamma,\ 0<\beta<1. \end{cases}$$

Remark 1. Note that $\max(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4) < 1$.

3. The fixed point index

Let us introduce

Definition 1. Let E be a real Banach space. A nonempty convex closed set $P \subset E$ is called a cone if it satisfies the following properties:

(i) $u \in P,\ \lambda \ge 0$ implies $\lambda u \in P$,
(ii) $u \in P,\ -u \in P$ implies $u = \theta$, where θ denotes the zero element of E.

Every cone P in E defines a partial ordering given by $u \le v$ for $u, v \in P$ iff $v - u \in P$. Let us define two convex sets $P_r,\ \bar{P}_r,\ r > 0$ by relations

$$P_r = \{u \in P:\ \|u\| < r\}, \qquad \bar{P}_r = \{u \in P:\ \|u\| \le r\}.$$

We need some properties about the fixed point index of compact maps, for example see [13, Chapter 3], [19, Chapter 2], [12, p. 2015]. The index has the following properties.

Lemma 5. *Let* S *be a closed convex set in a Banach space and let* D *be a bounded open set such that* $D_S = D \cap S \ne \emptyset$. *Let* $T : \bar{D}_S \to S$ *be a compact map. Suppose that* $x \ne Tx$ *for all* $x \in \partial D_S$.

(D_1) *(Existence) If* $i(T, D_S, S) \ne 0$, *then* T *has a fixed point in* D_S.
(D_2) *(Normalization) If* $u \in D_S$, *then* $i(\tilde{u}, D_S, S) = 1$, *where* $\tilde{u}(x) = u$ *for* $x \in \bar{D}_S$.
(D_3) *(Homotopy) Let* $\zeta : J \times \bar{D}_S \to S$ *be a compact map such that* $x \ne \zeta(t, x)$ *for* $x \in \partial D_S$ *and* $t \in J$. *Then*

$$i(\zeta(0,\cdot), D_S, S) = i(\zeta(1,\cdot), D_S, S).$$

(D_4) *(Additivity) If* U_1, U_2 *are disjoint open subsets of* D_S *such that* $x \ne Tx$ *for* $x \in \bar{D}_S \setminus (U_1 \cup U_2)$, *then*

$$i(T, D_S, S) = i(T, U_1, S) + i(T, U_2, S)$$

where $i(T, U_j, S) = i(T|_{\bar{U}_j}, U_j, S), \quad j = 1, 2.$

Lemma 6. *Let* P *be a cone in a Banach space* E. *For* $\rho > 0$, *define* $\Omega_\rho = \{x \in P:\ \|x\| < \rho\}$. *Assume that* $T : \bar{\Omega}_\rho \to P$ *is a compact map such that* $x \ne Tx$ *for* $x \in \partial\Omega_\rho$.

(i) *If* $\|x\| \le \|Tx\|$ *for* $x \in \partial\Omega_\rho$, *then* $i(T, \Omega_\rho, P) = 0$.
(ii) *If* $\|x\| \ge \|Tx\|$ *for* $x \in \partial\Omega_\rho$, *then* $i(T, \Omega_\rho, P) = 1$.

4. Existence of positive solutions of problem (1) by the fixed point index

By T, we denote the operator defined by

$$\begin{aligned}(Tu)(t) &= \frac{1}{\delta}\Bigg\{\gamma[1-\beta\eta+t(\beta-1)]\left[\sum_{i=1}^{r}{}' Q_i(u(t_i))(\xi-t_i)-\int_0^{\xi}(\xi-s)h(s)f(u(\alpha(s)))\mathrm{d}s\right]\\ &\quad+[\gamma\xi-t(\gamma-1)]\Bigg[\beta\sum_{i=1}^{j}{}' Q_i(u(t_i))(\eta-t_i)-\sum_{i=1}^{m}Q_i(u(t_i))(1-t_i)\\ &\quad-\beta\int_0^{\eta}(\eta-s)h(s)f(u(\alpha(s)))\mathrm{d}s+\int_0^1(1-s)h(s)f(u(\alpha(s)))\mathrm{d}s\Bigg]\Bigg\}\\ &\quad+\sum_{i=1}^{k}{}' Q_i(u(t_i))(t-t_i)-\int_0^t(t-s)h(s)f(u(\alpha(s)))\mathrm{d}s\end{aligned}$$

for $t\in J_k,\ k=0,1,\dots,m,\ \xi\in J_r,\ \eta\in J_j$ and $r,j=0,1,\dots,m$. Indeed, $T:B\to B$, where $B=C(J,\mathbb{R}_+)$.
Theorems 1 until 6 deal with the case when $\alpha(t)\le t$ on J.

Theorem 1. *Let Assumptions H_1', H_2, H_4 hold and $\xi<\eta$. We assume that there exist constants b,c,Q such that $0<b\le\min[\frac{1}{\mu},\Gamma_1^2]c$, $Q\ge 0$, where Γ_1 is defined as in Lemma 3 and*

$$\mu\ge\frac{1}{\delta}\left[\Delta\int_0^1(1-s)h(s)\mathrm{d}s+Q\right],\qquad \Delta=\max_{t\in J}[\gamma\xi-t(\gamma-1)]$$

with δ defined as in Lemma 1.

In addition, we assume that:

(A_1) $-[\gamma\xi-t(\gamma-1)]\sum_{i=1}^{m}Q_i(u(t_i))(1-t_i)\le\frac{Q}{\mu}c,\ f(u(t))\le\frac{1}{\mu}c$ for $0\le u(t)\le c,\ t\in J$,
(A_2) $f(u(t))\ge\frac{b}{l_1}$ for $b\le u(t)\le\frac{b}{\Gamma_1^2},\ 0\le t\le\xi$ with

$$l_1=\min(\gamma,1)\frac{1-\xi+\beta(\xi-\eta)}{\delta}\int_0^{\xi}sh(s)\mathrm{d}s.$$

Then problem (1) has at least one positive solution.

Proof. We see that problem (1) can be written as the fixed point problem $Tu=u$. It is easy to show that operator T is completely continuous. Indeed T is compact. This results from the Ascoli–Arzela theorem.
Let

$$P=\{u\in B:\ u(t)\ge 0,\ t\in J \text{ and } \min_{[0,\xi]}u(t)\ge\Gamma_1\|u\|\}.$$

Take $u\in P$. Then Lemma 3 and

$$\min_{[0,\xi]}(Tu)(t)=\min_{[0,\xi]}u(t)\ge\Gamma_1\|u\|=\Gamma_1\|Tu\|$$

show that $TP\subset P$.
First we want to show that

$$\min_{[0,\xi]}(Tu)(t)\ge\Gamma_1\|Tu\|,\quad u\in\bar{P}_c. \tag{8}$$

Take $u\in\bar{P}_c$. Then $0\le u(t)\le c$ on J; so $\|u\|\le c$. Because $\alpha\in C(J,[0,1])$, then $0\le u(\alpha(t))\le c$ on J. Indeed $(Tu)(t)\ge 0$, by Lemma 2. Moreover, in view of Assumption (A_1), we have

$$\begin{aligned}\|Tu\| &= \sup_{t\in J}|(Tu)(t)|=\sup_{t\in J}(Tu)(t)\\ &=\sup_{t\in J}\Bigg\{\frac{1}{\delta}\Bigg[\gamma[1-\beta\eta+t(\beta-1)]\left(\sum_{i=1}^{r}{}' Q_i(u(t_i))(\xi-t_i)-\int_0^{\xi}(\xi-s)h(s)f(u(\alpha(s)))\mathrm{d}s\right)\\ &\quad+[\gamma\xi-t(\gamma-1)]\Bigg(\beta\sum_{i=1}^{j}{}' Q_i(u(t_i))(\eta-t_i)-\sum_{i=1}^{m}Q_i(u(t_i))(1-t_i)\end{aligned}$$

$$
\begin{aligned}
&- \beta \int_0^\eta (\eta - s)h(s)f(u(\alpha(s)))\mathrm{d}s + \int_0^1 (1-s)h(s)f(u(\alpha(s)))\mathrm{d}s\Bigg) \\
&+ \delta\left(\sum_{i=1}^{k}{}' Q_i(u(t_i))(t-t_i) - \int_0^t (t-s)h(s)f(u(\alpha(s)))\mathrm{d}s\right)\Bigg]\Bigg\} \\
\le{}& \sup_{t\in J}\left\{-\frac{1}{\delta}[\gamma\xi - t(\gamma-1)]\left[\sum_{i=1}^{m} Q_i(u(t_i))(1-t_i) - \int_0^1 (1-s)h(s)f(u(\alpha(s)))\mathrm{d}s\right]\right\} \\
\le{}& \frac{1}{\delta}\left[\frac{Q}{\mu} + \frac{\Delta}{\mu}\int_0^1 (1-s)h(s)\mathrm{d}s\right]c \le c.
\end{aligned}
$$

It proves that $T : \bar{P}_c \to \bar{P}_c$. Hence, in view of Lemma 3, we have relation (8).

Now we want to show that

$$\min_{[0,\xi]}(Tu)(t) > b \quad \text{for } u \in \bar{P}_c \text{ with } b \le u(t) \le \frac{b}{\Gamma_1^2},\ \ t \in [0,\xi]. \tag{9}$$

Let $b \le u(t) \le \frac{b}{\Gamma_1^2}$ for $0 \le t \le \xi$. Then $\alpha(t) \le t \le \xi$ for $t \in [0,\xi]$ and also

$$b \le u(\alpha(t)) \le \frac{b}{\Gamma_1^2} \le \frac{c}{\Gamma_1^2}\min\left(\frac{1}{\mu}, \Gamma_1^2\right) \le c \quad \text{for } 0 \le t \le \xi.$$

Moreover

$$(Tu)''(t) = -h(t)f(u(\alpha(t))) \le 0, \quad t \in J'.$$

Now we consider two cases.

Case (1) Let $0 < \gamma < 1,\ 0 < \beta < \frac{1}{\eta}$. Then

$$\min_{[0,\xi]}(Tu)(t) = \min_{[0,\xi]} u(t) = \min[u(0), u(\xi)] = \min[\gamma u(\xi), u(\xi)] = \gamma u(\xi).$$

We see that

$$
\begin{aligned}
\min_{[0,\xi]}(Tu)(t) &= \gamma u(\xi) \\
&= \frac{\gamma}{\delta}\Bigg\{[1-\xi+\beta(\xi-\eta)]\left[-\sum_{i=1}^{r}{}' Q_i(u(t_i))t_i + \int_0^\xi sh(s)f(u(\alpha(s)))\mathrm{d}s\right] \\
&\quad + \xi\Bigg[-\sum_{i=r+1}^{j}{}' Q_i(u(t_i))(1-\beta\eta+t_i(\beta-1)) - \sum_{i=j+1}^{m}{}' Q_i(u(t_i))(1-t_i) \\
&\quad + \int_\xi^\eta (1-\beta\eta+s(\beta-1))h(s)f(u(\alpha(s)))\mathrm{d}s + \int_\eta^1 (1-s)h(s)f(u(\alpha(s)))\mathrm{d}s\Bigg]\Bigg\} \\
&> \frac{\gamma}{\delta}[1-\xi+\beta(\xi-\eta)]\int_0^\xi sh(s)f(u(\alpha(s)))\mathrm{d}s \\
&\ge \frac{\gamma}{\delta}\frac{b}{l_1}[1-\xi+\beta(\xi-\eta)]\int_0^\xi sh(s)\mathrm{d}s \ge b,
\end{aligned}
$$

by Assumption (A_2).

Case (2) Let $1 \le \gamma < \frac{1}{1-\xi},\ 0 < \beta < 1$. Then

$$\min_{[0,\xi]}(Tu)(t) = u(\xi).$$

Indeed, by Assumption (A_2), we obtain

$$
\begin{aligned}
\min_{[0,\xi]}(Tu)(t) &= u(\xi) \\
&> \frac{1}{\delta}[1-\xi+\beta(\xi-\eta)]\int_0^\xi sh(s)f(u(\alpha(s)))\mathrm{d}s \\
&\ge \frac{1}{\delta}\frac{b}{l_1}[1-\xi+\beta(\xi-\eta)]\int_0^\xi sh(s)\mathrm{d}s \ge b;
\end{aligned}
$$

see also the proof in Case (1).

This shows that (9) holds.

To show the assertion of Theorem 1 we apply Lemma 5. As a closed convex set S (from Lemma 5) we take the set $\bar{P}_c$. Put

$$U = \{u \in \bar{P}_c : \min_{[0,\xi]} u(t) > b\}.$$

We want to show that $Tu \neq u$ for $u \in \partial U$. Assume that this relation is not true. It means that there exists $u_0 \in \partial U$ such that $Tu_0 = u_0$ and $\min_{[0,\xi]} u_0(t) = b$. We consider two cases.

Case (a) Let $u_0 \in \{u \in \bar{P}_c : \|u\| \leq \frac{b}{\Gamma_1^2}, \min_{[0,\xi]} u(t) = b\}$. It means that

$$b = \min_{[0,\xi]} u(t) \leq \|u\| \leq \frac{b}{\Gamma_1^2}.$$

This and (9) give

$$b = \min_{[0,\xi]} u_0(t) = \min_{[0,\xi]} (Tu_0)(t) > b.$$

This is a contradiction.

Case (b) Let $\|u_0\| > \frac{b}{\Gamma_1^2}$. In view of (8), we obtain

$$b = \min_{[0,\xi]} u_0(t) = \min_{[0,\xi]} (Tu_0)(t) \geq \Gamma_1 \|Tu_0\| = \Gamma_1 \|u_0\| > \Gamma_1 \frac{b}{\Gamma_1^2} = \frac{b}{\Gamma_1} > b.$$

This is a contradiction too. It proves that $Tu \neq u$ for $u \in \partial U$.

To prove that T has a fixed point in U we need to show that $i(T, U, \bar{P}_c) \neq 0$; see condition (D_1) of Lemma 5. To do it we use conditions (D_3) and (D_2). Take $u_0 \in P$ such that $\min_{[0,\xi]} u_0(t) > b$, $\|u_0\| \leq \frac{b}{\Gamma_1^2}$ and define the map ζ by relation $\zeta(\lambda, u) = \lambda u_0 + (1 - \lambda) Tu$ for $\lambda \in [0, 1]$. Note that

$$\begin{aligned} \|\zeta(\lambda, u)\| &\leq \lambda \|u_0\| + (1 - \lambda)\|Tu\| \leq \lambda \frac{b}{\Gamma_1^2} + (1 - \lambda)c \\ &\leq \lambda c + (1 - \lambda)c = c. \end{aligned}$$

We see that $\zeta : [0, 1] \times \bar{U} \to \bar{P}_c$. Indeed, ζ is compact. Now, we need to show that $u \neq \zeta(\lambda, u)$ on $[0, 1] \times \partial U$. Assume that it is not true; so there exist $(\lambda_1, u_1) \in [0, 1] \times \partial U$ such that $\zeta(\lambda_1, u_1) = u_1$. We have two cases.

Case (i) Let $\|Tu_1\| > \frac{b}{\Gamma_1^2}$. Then,

$$\min_{[0,\xi]} (Tu_1)(t) \geq \Gamma_1 \|Tu_1\| > \Gamma_1 \frac{b}{\Gamma_1^2} = \frac{b}{\Gamma_1} > b,$$

by relation (8). Moreover,

$$\begin{aligned} b &= \min_{[0,\xi]} u_1(t) = \min_{[0,\xi]} [\lambda_1 u_0(t) + (1 - \lambda_1)(Tu_1)(t)] \\ &\geq \min_{[0,\xi]} \lambda_1 u_0(t) + \min_{[0,\xi]} (1 - \lambda_1)(Tu_1)(t) > \lambda_1 b + (1 - \lambda_1) b = b. \end{aligned}$$

This contradicts.

Case (ii) Let $\|Tu_1\| \leq \frac{b}{\Gamma_1^2}$. Then

$$\begin{aligned} \|u_1\| &= \|\lambda_1 u_0 + (1 - \lambda_1) Tu_1\| \leq \lambda_1 \|u_0\| + (1 - \lambda_1)\|Tu_1\| \\ &\leq \lambda_1 \frac{b}{\Gamma_1^2} + (1 - \lambda_1) \frac{b}{\Gamma_1^2} = \frac{b}{\Gamma_1^2}. \end{aligned}$$

Note that

$$b = \min_{[0,\xi]} u_1(t) \leq \|u_1\| \leq \frac{b}{\Gamma_1^2}.$$

This and relation (9) give

$$\begin{aligned} b &= \min_{[0,\xi]} u_1(t) = \min_{[0,\xi]} [\lambda_1 u_0(t) + (1 - \lambda_1)(Tu_1)(t)] \\ &\geq \min_{[0,\xi]} \lambda_1 u_0(t) + \min_{[0,\xi]} (1 - \lambda_1)(Tu_1)(t) > \lambda_1 b + (1 - \lambda_1) b = b. \end{aligned}$$

This is a contradiction.

It proves that $\zeta(\lambda, u) \neq u$ for $(\lambda, u) \in [0, 1] \times \partial U$. Hence, in view of (D_3), we have

$$i(T, U, \bar{P}_c) = i(u_0, U, \bar{P}_c).$$

This and (D_2) show that $i(u_0, U, \bar{P}_c) = 1$, so $i(T, U, \bar{P}_c) = 1$. In view of condition (D_1), it proves that T has a fixed point $x^* \in U$. Because,

$$\min_{[0,\xi]} x^*(t) > b, \quad \|x^*\| \leq c,$$

x^* is a nonzero fixed point of T in U. This ends the proof. ∎

In the next two theorems we use constants Γ_1 and l_1 defined earlier.

Theorem 2. *Let Assumptions* H_1', H_2, H_4 *hold and* $\xi < \eta$. *In addition, we assume that:*

(A_3) $f_0 = f_\infty = 0$ *and there exists constants* δ_1, $\bar{Q}_i \geq 0$, $i = 1, 2, \ldots, m$ *such that*

$$\begin{cases} -Q_i(u(t_i)) \leq \bar{Q}_i u(t_i) & \text{if } u(t_i) \geq 0, \\ \dfrac{1}{\delta} \max(\gamma\xi, \gamma\xi - \gamma + 1) \displaystyle\sum_{i=1}^{m} \bar{Q}_i (1 - t_i) \leq \delta_1 < 1, \end{cases} \tag{10}$$

where

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u},$$

(A_4) *there exists a constant* $\rho > 0$ *such that*

$$f(u) \geq \frac{1}{l_1}\rho \quad \text{for } u \in [\Gamma_1 \rho, \rho].$$

Then problem (1) has at least two positive solutions x_1^* *and* x_2^* *such that*

$$0 < \|x_1^*\| < \rho < \|x_2^*\|.$$

Proof. Indeed, problem (1) has a solution u if and only if u is a solution of the operator equation $u = Tu$, where T is defined earlier. Put

$$\mu_0 > \frac{1}{\delta} \max(\gamma\xi, \gamma\xi - \gamma + 1) \int_0^1 (1 - s)h(s)\mathrm{d}s.$$

If $f_0 = 0$, then we may choose a constant $r \in (0, \rho)$ such that $f(u) \leq \delta_2 u$ for $0 < u \leq r$, where $0 < \delta_2 < \frac{1-\delta_1}{\mu_0}$. Take $u \in P$ such that $\|u\| = r$. In view of assumptions H_1', H_2,

$$\begin{aligned} (Tu)(t) &\leq \frac{1}{\delta}[\gamma\xi - t(\gamma - 1)] \left[-\sum_{i=1}^{m} Q_i(u(t_i))(1 - t_i) + \int_0^1 (1 - s)h(s)f(u(\alpha(s)))\mathrm{d}s \right] \\ &\leq \frac{1}{\delta} \max(\gamma\xi, \gamma\xi - \gamma + 1) \left[\sum_{i=1}^{m} \bar{Q}_i u(t_i)(1 - t_i) + \delta_2 \int_0^1 (1 - s)h(s)u(\alpha(s))\mathrm{d}s \right] \\ &\leq (\delta_1 + \delta_2\mu_0)\|u\| < \|u\|. \end{aligned}$$

Then $\|Tu\| < \|u\|$ for $u \in \partial\Omega$, where $\Omega = \{u \in P : \|u\| < r\}$. It means that $u \neq Tu$ for $u \in \partial\Omega$. It shows that $i(T, \Omega, P) = 1$, by Lemma 6(ii).

Now, we consider the case when $f_\infty = 0$. It means that we may choose $\omega > \rho$ such that $f(u) \leq \kappa u$ for $u \geq \omega$ with $0 < \kappa < \frac{1-\delta_1}{\mu_0}$. We consider two cases.

Case (C_1) We assume that f is bounded; so there exists a constant $M > 0$ such that $f(u) \leq M$ for $u \in [0, \infty)$. Put $\max_i u(t_i) = N$. Let us choose $u \in P$ such that $\|u\| = \bar{r}$, where $\bar{r} > \max(\delta_1 N + \mu_0 M, \omega)$. Then

$$\begin{aligned} (Tu)(t) &\leq \frac{1}{\delta}[\gamma\xi - t(\gamma - 1)] \left[-\sum_{i=1}^{m} Q_i(u(t_i))(1 - t_i) + \int_0^1 (1 - s)h(s)f(u(\alpha(s)))\mathrm{d}s \right] \\ &\leq \frac{1}{\delta} \max(\gamma\xi, \gamma\xi - \gamma + 1) \left[\sum_{i=1}^{m} \bar{Q}_i u(t_i)(1 - t_i) + M \int_0^1 (1 - s)h(s)\mathrm{d}s \right] \\ &\leq \delta_1 N + M\mu_0 < \bar{r} = \|u\|; \end{aligned}$$

so $\|Tu\| < \|u\|$, $u \in \partial\Omega_0$, where $\Omega_0 = \{u \in P : \|u\| < \bar{r}\}$.

Case (C_2) We assume that f is unbounded. Note that $f \in C([0,\infty),[0,\infty))$; so there exists $\bar{r} > \max(\omega, \frac{r}{\Gamma_1})$ such that $f(u) \leq f(\bar{r})$ for $0 < u \leq \bar{r}$. Take $u \in P$ such that $\|u\| = \bar{r}$. Then

$$\begin{aligned}(Tu)(t) &\leq \frac{1}{\delta}[\gamma\xi - t(\gamma-1)]\left[-\sum_{i=1}^{m} Q_i(u(t_i))(1-t_i) + \int_0^1 (1-s)h(s)f(u(\alpha(s)))\mathrm{d}s\right] \\ &\leq \frac{1}{\delta}\max(\gamma\xi, \gamma\xi-\gamma+1)\left[\sum_{i=1}^{m}\bar{Q}_i u(t_i)(1-t_i) + f(\bar{r})\int_0^1(1-s)h(s)\mathrm{d}s\right] \\ &\leq (\delta_1 + \kappa\mu_0)\bar{r} < \bar{r} = \|u\|.\end{aligned}$$

Hence, in view of (C_1) and (C_2), we have $\|Tu\| < \|u\|$ for $u \in \partial\Omega_0$. It shows that $i(T,\Omega_0,P) = 1$, by Lemma 6(ii).

Let $\Lambda = \{u \in P : \|u\| < \rho\}$. Note that $\partial\Lambda \subset P$; so $\min_{[0,\xi]} u(\alpha(t)) \geq \Gamma_1\|u\| = \Gamma_1\rho$. For $u \in \partial\Lambda$, we have,

$$\begin{aligned}\|Tu\| \geq u(\xi) &> \frac{1}{\delta}[1-\xi+\beta(\xi-\eta)]\int_0^\xi sh(s)f(u(\alpha(s)))\mathrm{d}s \\ &> \frac{\rho}{l_1}\frac{1}{\delta}[1-\xi+\beta(\xi-\eta)]\int_0^\xi sh(s)\mathrm{d}s \geq \rho = \|u\|,\end{aligned}$$

by Assumption (A_4). It shows that $\|Tu\| > \|u\|$ for $u \in \partial\Lambda$. Hence, $i(T,\Lambda,P) = 0$, by Lemma 6(ii). Because $r < \rho < \bar{r}$, we see that $i(T,\Lambda\setminus\bar{\Omega},P) = i(T,\Lambda,P) - i(T,\Omega,P) = -1$ and $i(T,\Omega_0\setminus\bar{\Lambda},P) = i(T,\Omega_0,P) - i(T,\Lambda,P) = 1$. It proves that problem (1) has two positive solutions x_1^* and x_2^* such that $x_1^* \in \Lambda\setminus\bar{\Omega}$, $x_2^* \in \Omega_0\setminus\bar{\Lambda}$ and $0 < \|x_1^*\| < \rho < \|x_2^*\|$. This ends the proof. ■

Theorem 3. *Let Assumptions* H_1', H_2, H_4 *hold and* $\xi < \eta$. *In addition, we assume that condition* (10) *holds and moreover*

(A_5) $f_0 = f_\infty = \infty$ *where* f_0 *and* f_∞ *are defined as in Theorem 2,*
(A_6) *there exists a positive constant* ρ *such that*

$$f(u) < \frac{1-\delta_1}{\mu_0}\rho \quad \text{for } u \in [0,\rho],$$

where μ_0 *is defined as in Theorem 2.*

Then problem (1) *has at least two positive solutions* x_1^* *and* x_2^* *such that*

$$0 < \|x_1^*\| < \rho < \|x_2^*\|.$$

Proof. Case (1) Let $f_0 = \infty$. Then there exists $r \in (0,\rho)$ such that $f(u) \geq \delta_3 u$ for $0 < u \leq r$ with $\delta_3 > \frac{1}{\Gamma_1 l_1}$. Let $\Omega = \{u \in P : \|u\| < r\}$. Take $u \in P$ such that $\|u\| = r$; so $u \in \partial\Omega$. Then, as in the proof of Theorem 2, we have

$$\begin{aligned}\|Tu\| \geq u(\xi) &> \frac{1}{\delta}[1-\xi+\beta(\xi-\eta)]\int_0^\xi sh(s)f(u(\alpha(s)))\mathrm{d}s \\ &\geq \frac{1}{\delta}[1-\xi+\beta(\xi-\eta)]\,\delta_3\int_0^\xi sh(s)u(\alpha(s))\mathrm{d}s \\ &\geq \frac{1}{\delta}[1-\xi+\beta(\xi-\eta)]\,\delta_3\Gamma_1\int_0^\xi sh(s)\|u\|\mathrm{d}s \geq \delta_3\Gamma_1 l_1\|u\| > \|u\|.\end{aligned}$$

It shows that $\|Tu\| > \|u\|$ for $u \in \partial\Omega$. Hence, $i(T,\Omega,P) = 0$, by Lemma 6(i).

Case (2) Let $f_\infty = \infty$. It means that there exists $\nu > \rho$ such that $f(u) \geq \kappa u$ for $u \geq \nu$ and $\kappa > \frac{1}{\Gamma_1 l_1}$. Put $\Omega_0 = \{u \in P : \|u\| < \omega\}$, where $\omega > \max\left(\frac{\nu}{\Gamma_1},\rho\right)$. Then for $u \in \partial\Omega_0$, we have

$$\min_{[0,\xi]} u(\alpha(t)) \geq \Gamma_1\|u\| = \Gamma_1\omega > \Gamma_1\frac{\nu}{\Gamma_1} = \nu.$$

Hence, as before, we have

$$\|Tu\| \geq u(\xi) > \kappa\Gamma_1 l_1\|u\| > \|u\|.$$

This shows that $i(T,\Omega_0,P) = 0$, by Lemma 6(i).

Now we apply condition (A_6). Put $\Omega_1 = \{u \in P : \|u\| < \rho\}$. As in Theorem 2, for $u \in \partial\Omega_1$, we have

$$
\begin{aligned}
(Tu)(t) &\le \frac{1}{\delta}[\gamma\xi - t(\gamma-1)]\left[-\sum_{i=1}^{m} Q_i(u(t_i))(1-t_i) + \int_0^1 (1-s)h(s)f(u(\alpha(s)))\mathrm{d}s\right] \\
&\le \left[\delta_1 + \frac{1-\delta_1}{\mu_0}\frac{1}{\delta}\max(\gamma\xi, \gamma\xi-\gamma+1)\int_0^1 (1-s)h(s)\mathrm{d}s\right]\rho \\
&< \rho = \|u\|.
\end{aligned}
$$

Then $\|Tu\| < \|u\|$ for $u \in \partial\Omega_1$. It shows that $i(T, \Omega_1, P) = 1$.

Note that $r < \rho < \omega$. Hence $i(T, \Omega_1 \setminus \bar{\Omega}, P) = i(T, \Omega_1, P) - i(T, \Omega, P) = 1$ and $i(T, \Omega_0 \setminus \bar{\Omega}_1, P) = i(T, \Omega_0, P) - i(T, \Omega_1, P) = -1$. It shows that operator T has two positive fixed points $x_1^* \in \Omega_1 \setminus \bar{\Omega}$, $x_2^* \in \Omega_0 \setminus \bar{\Omega}_1$ such that $0 < \|x_1^*\| < \|x_2^*\|$. This ends the proof. ■

We formulate the next three theorems without proofs since they are respectively similar to those of Theorems 1–3. Throughout this paper we use the constant μ defined as in Theorem 1.

Theorem 4. *Let Assumptions* H_1', H_2, H_3 *hold and* $\eta < \xi$. *We assume that there exist constants* b, c, Q *such that* $0 < b \le \min[\frac{1}{\mu}, \Gamma_3^2]c$, $Q \ge 0$, *where* Γ_3 *is defined as in Lemma 4. Let Assumption* $(A1)$ *hold. In addition, we assume that:*

(A_2') $f(u(t)) \ge \frac{b}{l_3}$ *for* $b \le u(t) \le \frac{b}{\Gamma_3^2}$, $0 \le t \le \eta$ *with*

$$
l_3 = \min\left(\frac{\gamma}{\delta}[1-\beta\eta+\xi(\beta-1)], \frac{1-\eta}{\delta}\right)\int_0^\eta sh(s)\mathrm{d}s.
$$

Then problem (1) *has at least one positive solution.*

Theorem 5. *Let Assumptions* H_1', H_2, H_3, (A_3) *hold and* $\eta < \xi$. *In addition, we assume that:*

(A_4') *there exists a constant* $\rho > 0$ *such that*

$$
f(u) \ge \frac{1}{l_3}\rho \quad \text{for } u \in [\Gamma_3\rho, \rho].
$$

Then problem (1) *has at least two positive solutions* x_1^* *and* x_2^* *such that*

$$
0 < \|x_1^*\| < \rho < \|x_2^*\|.
$$

Theorem 6. *Let Assumptions* H_1', H_2, H_3, (A_3), (A_5), (A_6) *hold and* $\eta < \xi$. *Then problem* (1) *has at least two positive solutions* x_1^* *and* x_2^* *such that*

$$
0 < \|x_1^*\| < \rho < \|x_2^*\|.
$$

The next six theorems concern the case when $\alpha(t) \ge t$ on J. The next results we also formulate without proofs (they are similar to the previous ones).

Theorem 7. *Let Assumptions* H_1, H_2, H_3 *hold and* $\xi < \eta$. *We assume that there exist constants* b, c, Q *such that* $0 < b \le \min[\frac{1}{\mu}, \Gamma_2^2]c$, $Q \ge 0$, *where* Γ_2 *is defined as in Lemma 4. Let Assumption* (A_1) *hold. In addition, we assume that:*

(A_2'') $f(u(t)) \ge \frac{b}{l_2}$ *for* $b \le u(t) \le \frac{b}{\Gamma_2^2}$, $\eta \le t \le 1$ *with*

$$
l_2 = \min(\beta, 1)\frac{\gamma\xi - \gamma\eta + \eta}{\delta}\int_\eta^1 (1-s)h(s)\mathrm{d}s.
$$

Then problem (1) *has at least one positive solution.*

Theorem 8. *Let Assumptions* H_1, H_2, H_3, (A_3) *hold and* $\xi < \eta$. *In addition, we assume that:*

(A_4'') *there exists a constant* $\rho > 0$ *such that*

$$
f(u) \ge \frac{1}{l_2}\rho \quad \text{for } u \in [\Gamma_2\rho, \rho].
$$

Then problem (1) has at least two positive solutions x_1^* and x_2^* such that

$$0 < \|x_1^*\| < \rho < \|x_2^*\|.$$

Theorem 9. *Let Assumptions* $\mathrm{H}_1, \mathrm{H}_2, \mathrm{H}_3$, (A_3), (A_5), (A_6) *hold and* $\xi < \eta$. *Then problem (1) has at least two positive solutions x_1^* and x_2^* such that*

$$0 < \|x_1^*\| < \rho < \|x_2^*\|.$$

Theorem 10. *Let Assumptions* $\mathrm{H}_1, \mathrm{H}_2, \mathrm{H}_3$ *hold and* $\eta < \xi$. *We assume that there exist constants* b, c, Q *such that* $0 < b \le \min[\frac{1}{\mu}, \Gamma_4^2]c$, $Q \geq 0$, *where* Γ_4 *is defined as in Lemma 4. Let Assumption* (A_1) *hold. In addition, we assume that:*

(A_2''') $f(u(t)) \geq \frac{b}{l_4}$ *for* $b \leq u(t) \leq \frac{b}{\Gamma_4^2}$, $\xi \leq t \leq 1$ *with*

$$l_4 = \min\left(\frac{\beta}{\delta}[\gamma\xi - \eta\gamma + \eta], \frac{\xi}{\delta}\right)\int_\xi^1 (1-s)h(s)\mathrm{d}s.$$

Then problem (1) has at least one positive solution.

Theorem 11. *Let Assumptions* $\mathrm{H}_1, \mathrm{H}_2, \mathrm{H}_3$, (A_3) *hold and* $\eta < \xi$. *In addition, we assume that:*

(A_4''') *there exists a constant* $\rho > 0$ *such that*

$$f(u) \geq \frac{1}{l_4}\rho \quad \text{for } u \in [\Gamma_4\rho, \rho].$$

Then problem (1) has at least two positive solutions x_1^* and x_2^* such that

$$0 < \|x_1^*\| < \rho < \|x_2^*\|.$$

Theorem 12. *Let Assumptions* $\mathrm{H}_1, \mathrm{H}_2, \mathrm{H}_3$, (A_3), (A_5), (A_6) *hold and* $\eta < \xi$. *Then problem (1) has at least two positive solutions x_1^* and x_2^* such that*

$$0 < \|x_1^*\| < \rho < \|x_2^*\|.$$

Appendix

In this part, we provide the proofs of Lemmas 3 and 4 from paper [6].

Proof of Lemma 3. Put $u(t^*) = \|u\|$. We divide the proof into four cases.

Case (1) Let $0 < \gamma < 1$, $0 < \beta < 1$. In this case

$$u(0) = \gamma u(\xi) \leq u(\xi), \qquad u(1) = \beta u(\eta) \leq u(\eta).$$

Subcase (1a) Let $u(\xi) \leq u(\eta)$. Note that $t^* \in (\xi, \eta)$ or $t^* \in (\eta, 1)$ and

$$\min_{[0,\xi]} u(t) = u(0) \quad \text{and} \quad \min_{[\eta,1]} u(t) = u(1).$$

Then

$$\begin{aligned} u(t^*) &\leq u(\xi) + \frac{[u(\xi) - u(0)]}{\xi - 0}(t^* - \xi) = \frac{t^*(1-\gamma) + \gamma\xi}{\xi}u(\xi) \leq \frac{1 + \gamma\xi - \gamma}{\xi}u(\xi) \\ &= \frac{1 + \xi\gamma - \gamma}{\gamma\xi}u(0). \end{aligned}$$

Hence

$$\min_{[0,\xi]} u(t) \geq \frac{\xi\gamma}{1 + \gamma\xi - \gamma}\|u\| \quad \text{and} \quad \min_{[\eta,1]} u(t) \geq \frac{\xi\beta}{1 + \gamma\xi - \gamma}\|u\|.$$

Subcase (1b) Let $u(\xi) > u(\eta)$. In this case $t^* \in (0, \xi)$ or $t^* \in (\xi, \eta)$ and

$$\min_{[0,\xi]} u(t) = u(0) \quad \text{and} \quad \min_{[\eta,1]} u(t) = u(1).$$

Then

$$u(t^*) \le u(\eta) + \frac{u(\eta) - u(1)}{\eta - 1}(t^* - \eta) = \frac{u(\eta)}{1-\eta}[1 - \beta\eta + t^*(\beta - 1)] \le \frac{1-\beta\eta}{1-\eta}u(\eta)$$
$$\le \frac{1-\beta\eta}{(1-\eta)\gamma}u(0).$$

Hence

$$\min_{[0,\xi]} u(t) \ge \frac{(1-\eta)\gamma}{1-\beta\eta}\|u\| \quad \text{and} \quad \min_{[\eta,1]} u(t) \ge \frac{(1-\eta)\beta}{1-\beta\eta}\|u\|.$$

Case (2) Let $0 < \gamma < 1,\ 1 \le \beta < \frac{1}{\eta}$. In this case

$$u(0) = \gamma u(\xi) \le u(\xi), \qquad u(1) = \beta u(\eta) \ge u(\eta).$$

Let $u(\xi) \le u(\eta)$. Note that $t^* \in (\eta, 1)$,

$$\min_{[0,\xi]} u(t) = u(0) \quad \text{and} \quad \min_{[\eta,1]} u(t) = u(\eta).$$

Then

$$u(t^*) \le u(\xi) + \frac{u(\xi) - u(0)}{\xi - 0}(t^* - \xi) = \frac{u(\xi)}{\xi}[t^*(1-\gamma) + \gamma\xi] \le \frac{1-\gamma+\gamma\xi}{\xi}u(\xi)$$
$$= \frac{1-\gamma+\gamma\xi}{\xi\gamma}u(0).$$

It yields

$$\min_{[0,\xi]} u(t) \ge \frac{\xi\gamma}{1-\gamma+\gamma\xi}\|u\| \quad \text{and} \quad \min_{[\eta,1]} u(t) \ge \frac{\xi}{1-\gamma+\gamma\xi}\|u\|.$$

The case when $u(\xi) > u(\eta)$ contradicts with the concavity of u.

Case (3) Let $1 \le \gamma < \frac{1}{1-\xi},\ 0 < \beta < 1$. In this case

$$u(0) = \gamma u(\xi) \ge u(\xi), \qquad u(1) = \beta u(\eta) \le u(\eta).$$

If $u(\xi) \ge u(\eta)$, then $t^* \in (0, \xi)$ and

$$\min_{[0,\xi]} u(t) = u(\xi) \quad \text{and} \quad \min_{[\eta,1]} u(t) = u(1).$$

Moreover,

$$u(t^*) \le u(\eta) + \frac{u(\eta) - u(1)}{\eta - 1}(t^* - \eta) = \frac{u(\eta)}{1-\eta}[1 - \beta\eta + t^*(\beta - 1)] \le \frac{1-\beta\eta}{1-\eta}u(\eta)$$
$$\le \frac{1-\beta\eta}{1-\eta}u(\xi).$$

It yields

$$\min_{[0,\xi]} u(t) \ge \frac{1-\eta}{1-\beta\eta}\|u\| \quad \text{and} \quad \min_{[\eta,1]} u(t) \ge \frac{(1-\eta)\beta}{1-\beta\eta}\|u\|.$$

The case when $u(\xi) < u(\eta)$ contradicts with the concavity of u.

Case (4) Let $1 \le \gamma < \frac{1}{1-\xi},\ 1 \le \beta < \frac{1}{\eta}$. Then $u(0) \ge u(\xi),\ u(1) \ge u(\eta)$. This case contradicts with the concavity of u. This ends the proof. ■

Proof of Lemma 4. Put $u(t^*) = \|u\|$. First we prove the first inequality of (7).

Case (1) Let $0 < \gamma < 1,\ 0 < \beta < 1$. In this case

$$u(0) = \gamma u(\xi) \le u(\xi), \qquad u(1) = \beta u(\eta) \le u(\eta).$$

Subcase (1a) Let $u(\xi) \le u(\eta)$ and $u(1) \le u(\xi)$. Note that $t^* \in (0, \eta)$ or $t^* \in (\eta, \xi)$ and

$$\min_{[0,\eta]} u(t) = u(0) \quad \text{and} \quad \min_{[\xi,1]} u(t) = u(1).$$

Then

$$u(t^*) \le u(\xi) + \frac{[u(\xi) - u(1)]}{\xi - 1}(t^* - \xi) = \frac{u(\xi)}{1-\xi}(1-t^*) - \frac{u(1)}{1-\xi}(\xi - t^*) \le \frac{u(\xi)}{1-\xi}$$
$$= \frac{u(0)}{(1-\xi)\gamma}.$$

Hence

$$\min_{[0,\eta]} u(t) \ge \gamma(1-\xi)\|u\| \quad \text{and} \quad \min_{[\xi,1]} u(t) \ge \beta(1-\xi)\|u\|.$$

Subcase (1ab) Let $u(1) > u(\xi)$. This case contradicts with the concavity of u.
Subcase (1b) Let $u(\xi) > u(\eta)$.
Subcase (1ba) Let $u(0) \le u(\eta)$. Then $t^* \in (\eta, \xi)$ or $t^* \in (\xi, 1)$ and

$$\min_{[0,\eta]} u(t) = u(0) \quad \text{and} \quad \min_{[\xi,1]} u(t) = u(1).$$

Then

$$u(t^*) \le u(\eta) + \frac{u(\eta) - u(0)}{\eta}(t^* - \eta) = \frac{u(\eta)}{\eta}\, t^* - \frac{u(0)}{\eta}(t^* - \eta) \le \frac{u(\eta)}{\eta}$$
$$\le \frac{u(\xi)}{\eta} = \frac{u(0)}{\gamma\eta}.$$

It yields

$$\min_{[0,\eta]} u(t) \ge \gamma\eta\|u\| \quad \text{and} \quad \min_{[\xi,1]} u(t) \ge \beta\eta\|u\|.$$

Subcase (1bb) Let $u(0) > u(\eta)$. This case contradicts with the concavity of u.
Case (2) Let $0 < \gamma < 1,\ 1 \le \beta$. In this case

$$u(0) = \gamma u(\xi) \le u(\xi), \qquad u(1) = \beta u(\eta) \ge u(\eta).$$

Subcase (2a) Let $u(\xi) \le u(\eta)$. This case contradicts with the concavity of u.
Subcase (2b) Let $u(\xi) > u(\eta)$.
Subcase (2ba) Let $u(0) < u(\eta)$ and $u(1) \le u(\xi)$. It yields $t^* \in (\eta, \xi)$ or $t^* \in (\xi, 1)$ and

$$\min_{[0,\eta]} u(t) = u(0) \quad \text{and} \quad \min_{[\xi,1]} u(t) = u(1).$$

Then

$$u(t^*) \le u(\eta) + \frac{u(\eta) - u(0)}{\eta}(t^* - \eta) \le \frac{1}{\eta}u(\eta) \le \frac{u(\xi)}{\eta}$$
$$= \frac{u(0)}{\gamma\eta}.$$

Hence

$$\min_{[0,\eta]} u(t) \ge \gamma\eta\|u\| \quad \text{and} \quad \min_{[\xi,1]} u(t) \ge \beta\eta\|u\|.$$

Subcase (2bb) Let $u(0) < u(\eta)$ and $u(1) > u(\xi)$. Then $t^* \in (\eta, \xi)$ or $t^* \in (\xi, 1)$ and

$$\min_{[0,\eta]} u(t) = u(0) \quad \text{and} \quad \min_{[\xi,1]} u(t) = u(\xi).$$

Moreover

$$u(t^*) \le u(\eta) + \frac{u(\eta) - u(0)}{\eta}(t^* - \eta) \le \frac{1}{\eta}u(\eta)$$
$$\le \frac{u(\xi)}{\eta} = \frac{u(0)}{\gamma\eta}.$$

Hence

$$\min_{[0,\eta]} u(t) \ge \gamma\eta\|u\| \quad \text{and} \quad \min_{[\xi,1]} u(t) \ge \eta\|u\|.$$

Subcase (2bc) Let $u(0) > u(\eta)$. This case contradicts with the concavity of u.

Case (3) Let $1 \le \gamma,\ 0 < \beta < 1$. In this case

$$u(0) = \gamma u(\xi) \ge u(\xi), \qquad u(1) = \beta u(\eta) \le u(\eta).$$

Subcase (3a) Let $u(\xi) \le u(\eta)$.

Subcase (3aa) Let $u(1) < u(\xi)$ and $u(0) \le u(\eta)$. It yields $t^* \in (0, \eta)$ or $t^* \in (\eta, \xi)$ and

$$\min_{[0,\eta]} u(t) = u(0) \quad \text{and} \quad \min_{[\xi,1]} u(t) = u(1).$$

Moreover,

$$u(t^*) \le u(\xi) + \frac{u(\xi) - u(1)}{\xi - 1}(t^* - \xi) \le \frac{u(\xi)}{1-\xi} = \frac{u(0)}{\gamma(1-\xi)}.$$

It yields

$$\min_{[0,\eta]} u(t) \ge \gamma(1-\xi)\|u\| \quad \text{and} \quad \min_{[\xi,1]} u(t) \ge \beta(1-\xi)\|u\|.$$

Subcase (3ab) Let $u(1) < u(\xi)$ and $u(\eta) \le u(0)$. It yields $t^* \in (0, \eta)$ and

$$\min_{[0,\eta]} u(t) = u(\eta) \quad \text{and} \quad \min_{[\xi,1]} u(t) = u(1).$$

Moreover

$$u(t^*) \le u(\xi) + \frac{u(\xi) - u(1)}{\xi - 1}(t^* - \xi) \le \frac{u(\xi)}{1-\xi} \le \frac{u(\eta)}{1-\xi}.$$

Hence

$$\min_{[0,\eta]} u(t) \ge (1-\xi)\|u\| \quad \text{and} \quad \min_{[\xi,1]} u(t) \ge \beta(1-\xi)\|u\|.$$

Subcase (3ac) Let $u(1) > u(\xi)$. This contradicts with the concavity of u.

Subcase (3b) Let $u(\xi) > u(\eta)$. This contradicts with the concavity of u too.

Case (4) Let $1 \le \gamma,\ 1 \le \beta$. Then $u(0) \ge u(\xi),\ u(1) \ge u(\eta)$. This case contradicts with the concavity of u. This ends the proof. ■

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