FRACTIONAL PROBLEMS WITH RIGHT-HANDED RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVES

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(received: 30 November 2015; revised: 23 December 2015; accepted: 29 December 2015; published online: 22 February 2016)

Abstract: In this paper, we investigate the existence of solutions for advanced fractional differential equations containing the right-handed Riemann-Liouville fractional derivative both with nonlinear boundary conditions and also with initial conditions given at the end point T of interval $[0,T]$. We use both the method of successive approximations, the Banach fixed point theorem and the monotone iterative technique, as well. Linear problems are also discussed. A few examples illustrate the results.

Keywords: right-handed Riemann-Liouville fractional derivatives, nonlinear boundary problems, linear problems, existence of solutions, Mittag-Leffler functions.

1. Introduction

Put $J_0=[0,T)$, $J=[0,T]$. First, we introduce the right-handed Riemann-Liouville fractional derivative $D_T^q x$ of order q by

$$D_T^q x(t) = -\frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_t^T (s-t)^{-q}x(s)ds, \quad t\in J_0, \quad q\in(0,1) \tag{1}$$

and $D_T^1 x(t)=-x'(t)$, if $q=1$.

Similarly, we introduce the right-sided fractional integral $I_T^q x$ of order $q>0$ by

$$I_T^q x(t)=\frac{1}{\Gamma(q)}\int_t^T (s-t)^{q-1}x(s)ds, \quad t\in J_0 \tag{2}$$

The above definitions are taken from [1].

In this paper, we study the nonlinear boundary value problem of the form:

$$\begin{cases} D_T^q x(t) = f\left(t, x(t), x(\alpha(t)), \dfrac{1}{\Gamma(q_1)} \displaystyle\int_t^T (s-t)^{q_1-1} g(s)x(s)ds\right) \equiv Fx(t), \quad t \in J_0 \\ 0 = h\big(\bar{x}(T)\big) \end{cases} \tag{3}$$

where $f \in C(J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\alpha \in C(J, J)$, $g \in C(J, \mathbb{R})$, $h \in C(\mathbb{R}, \mathbb{R})$, $\bar{x}(T) = (T-t)^{1-q} x(t)|_{t=T}$ with $q \in (0,1]$, $q_1 > 0$.

We introduce the space C_{1-q} by

$$C_{1-q}(J, \mathbb{R}) = \left\{u \in C([0,T), \mathbb{R}) : \ (T-t)^{1-q} u \in C(J, \mathbb{R})\right\}, \ q \in (0,1) \tag{4}$$

and $C_0(J, \mathbb{R}) = C(J, \mathbb{R})$ if $q = 1$.

Fractional differential equations arise in many engineering and scientific disciplines. Recently, much attention has been paid to study fractional differential equations. Some authors have formulated sufficient conditions under which fractional differential equations both with initial or boundary conditions have solutions. For example, such problems have been investigated for fractional differential equations with the left-handed Riemann-Liouville fractional derivative $D_{a+}^q x$ (or shortly $D^q x$) of order q, see for example [2–8, 1, 9–20]. An interesting and fruitful technique for proving the existence results for nonlinear fractional differential problems is the monotone iterative method based on lower and upper solutions, see for example [3–6, 9–14, 16–20]. Note that fractional differential equations with the right-handed Riemann-Liouville fractional derivative $D_T^q x$ of order q have been investigated, for example in [21, 1, 15].

In our paper we use both the right-handed Riemann-Liouville fractional derivatives $D_T^q x$ and the right-sided fractional integrals $I_T^q x$ of order $q \in (0,1]$. If $g(s) = 1$, $t \in J$, then, the fractional differential equation in problem (3) takes the form

$$D_T^q x(t) = f\big(t, x(t), x(\alpha(t)), I_T^{q_1} x(t)\big), \quad t \in J_0 \tag{5}$$

If $q_1 = 1$ and $g \in C(J, J)$ then, the fractional differential equation in problem (3) takes the following form

$$D_T^q x(t) = f\left(t, x(t), x(\alpha(t)), \int_t^T g(s)x(s)ds\right), \quad t \in J_0 \tag{6}$$

First we discuss initial problems with the initial condition given at the point T for the fractional differential equations with $D_T^q x$ from (3) replacing this problem by a corresponding integral equation. Now, to find a unique solution, we apply the method of successive approximations assuming that function f appearing in the right-hand-side of problem (3) satisfies a Lipschitz condition with respect to the last three variables. We also apply the Banach fixed point theorem with the Bielecki norm for the case $q = 1$. The uniqueness of solutions is also investigated under the same Lipschitz condition. The linear fractional differential problems with initial conditions at the point T are also investigated giving their solutions in forms of Mittag-Leffler functions. Finally, to find a solution of problem (3), we use the monotone iterative method combined with lower and upper solutions. Indeed, we discuss also corresponding fractional differential inequalities. Some examples illustrate the results.

The organization of this paper is as follows. In Section 2, we discuss the nonlinear fractional differential equations of order q with advanced arguments and with initial conditions given at the end point T of interval $[0,T]$, see problem (10). We use the method of successive approximations to prove the existence and uniqueness result for problem (10) with $q\in(0,1)$, see Theorem 1. Example 1 illustrates the result of Theorem 1. Theorem 2 concerns the existence and uniqueness of solutions of problem (3) for $q=1$, by using the Banach fixed point theorem with the Bielecki norm. In the next section, we study the uniqueness of solutions of problem (10) giving sufficient conditions under which problem (10) has at most one solution, see Theorem 3. Section 4 concerns linear fractional problems with initial conditions given at the point T. Theorem 4 presents the unique solution of such problems in terms of the Mittag-Leffler function. In Section 5, some examples are given. Examples 2 and 4 concern linear fractional problems while Example 3 the system of two linear fractional equations. In Sections 6 and 7, we discuss the existence of solutions for general problems of type (3), by using the monotone iterative technique based on lower and upper solutions. The corresponding existence results are given by Theorem 5 for $q=1$, and Theorem 6 for $q\in(0,1)$. At the end of this paper, Example 5 concerns the application of Theorem 6 to a fractional differential equation with a nonlinear boundary condition.

2. Existence results for fractional problems with initial conditions

First, we cite a lemma.

Lemma 1 (see [1]). Let $0<q\le 1$, $y\in L(0,T)$. Also let $y_{1-q}(t)=I_T^{1-q}y(t)$ be the fractional integral of order $1-q$ and $y_{1-q}\in AC[0,T]$. Then,

$$I_T^qD_T^qy(t)=y(t)-\frac{y_{1-q}(T)}{\Gamma(q)}(T-t)^{q-1}\quad \text{if } 0<q<1 \tag{7}$$

and

$$I_T^1D_T^1y(t)=y(t)-y(T)\quad \text{if } q=1 \tag{8}$$

Let us introduce the following assumption:

$H_1:$ $f\in C(J\times\mathbb{R}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $\alpha\in C(J,J)$, $\alpha(t)\ge t$, $g\in C(J,\mathbb{R})$ and there exist nonnegative constants A,B,D such that

$$\left|f(t,u_1,u_2,u_3)-f(t,v_1,v_2,v_3)\right|\le A|v_1-u_1|+B|v_2-u_2|+D|u_3-v_3| \tag{9}$$

The next result concerns the problem:

$$\begin{cases} D_T^q u(t) = Fu(t), & t \in J_0 \\ \bar{u}(T) = k \in \mathbb{R} \end{cases} \tag{10}$$

where operator F is defined as in problem (3). Note that in (10) the initial point is given at the end point of interval J. Now, we formulate an existence result for problem (10).

Theorem 1. *Let Assumption H_1 hold and let $q_1 > 0$, $0 < q < 1$. Moreover, we assume that there exists a constant $M > 0$ such that*

$$\frac{1}{\Gamma(q)} \sup_{t \in J_0} \int_t^T (s-t)^{q-1} |Fu_0(s)| ds \leq M \tag{11}$$

for $u_0(t) = k(T-t)^{q-1}$. Then, problem (10) has a unique solution $u \in C_{1-q}(J, \mathbb{R})$.

Proof. Using Lemma 1, it is easy to show that problem (10) is equivalent to the integral equation:

$$u(t) = k(T-t)^{q-1} + \frac{1}{\Gamma(q)} \int_t^T (s-t)^{q-1} Fu(s) ds \tag{12}$$

To find the solution of (12) we use the method of successive approximations. Let

$$\begin{cases} u_0(t) = k(T-t)^{q-1} \\ u_n(t) = k(T-t)^{q-1} + I_T^q Fu_{n-1}(t), & n = 1, 2, \cdots \end{cases} \tag{13}$$

Put

$$w_n(t) = |u_n(t) - u_{n-1}(t)|, \ n = 1, 2, \cdots, \quad L = A + B + \frac{DG}{\Gamma(q_1+1)} T^{q_1} \tag{14}$$

with $G = \max_{t \in J} |g(t)|$. Then,

$$\begin{aligned} w_1(t) &\leq \frac{1}{\Gamma(q)} \int_t^T (s-t)^{q-1} \big|Fu_0(s)\big| ds \leq M, \\ w_2(t) &\leq \frac{1}{\Gamma(q)} \int_t^T (s-t)^{q-1} \big|Fu_1(s) - Fu_0(s)\big| ds \\ &\leq \frac{1}{\Gamma(q)} \int_t^T (s-t)^{q-1} \left[Aw_1(s) + Bw_1\big(\alpha(s)\big) + \frac{DG}{\Gamma(q_1)} \int_s^T (\tau - s)^{q_1 - 1} w_1(\tau) d\tau \right] ds \\ &\leq \frac{M}{\Gamma(q)} \int_t^T (s-t)^{q-1} \left[A + B + \frac{DG}{\Gamma(q_1)} \int_s^T (\tau - s)^{q_1 - 1} d\tau \right] ds \\ &\leq \frac{ML}{\Gamma(q)} \int_t^T (s-t)^{q-1} ds = \frac{ML}{\Gamma(q+1)} (T-t)^q \end{aligned} \tag{15}$$

Now, we have to prove that

$$w_n(t) \leq \frac{ML^{n-1}}{\Gamma(q(n-1)+1)} (T-t)^{q(n-1)} \equiv z_n(t), \quad n = 1, 2, \cdots. \tag{16}$$

Assume that (16) holds for some integer $m>1$. As $\alpha(s)\geq s$, so $w_m(\alpha(s))\leq z_m(s)$, and $w_m(\tau)\leq z_m(s)$, $\tau\in[s,T]$. Using Assumption H_1 and relation (16) for $n=m$, we obtain

$$\begin{aligned} w_{m+1}(t) &\leq \frac{1}{\Gamma(q)}\int_t^T (s-t)^{q-1}\big|Fu_m(s)-Fu_{m-1}(s)\big|ds \\ &\leq \frac{1}{\Gamma(q)}\int_t^T (s-t)^{q-1}\Big[Aw_m(s)+Bw_m(\alpha(s))+\frac{DG}{\Gamma(q_1)}\int_s^T(\tau-s)^{q_1-1}w_m(\tau)d\tau\Big]ds \\ &\leq \frac{1}{\Gamma(q)}\int_t^T (s-t)^{q-1}z_m(s)\Big[A+B+\frac{DG}{\Gamma(q_1)}\int_s^T(\tau-s)^{q_1-1}d\tau\Big]ds \\ &\leq \frac{L}{\Gamma(q)}\int_t^T (s-t)^{q-1}z_m(s)ds=\frac{ML^m}{\Gamma(qm+1)}(T-t)^{qm} \end{aligned} \tag{17}$$

This and the mathematical induction show that (16) holds.

Now, we have to show that the sequence $\{u_n\}$ is convergent. First, we note that

$$u_n(t)=u_0(t)+\sum_{j=1}^{n}\big[u_j(t)-u_{j-1}(t)\big], \quad n=1,2,\cdots \tag{18}$$

In view of (16), we see that

$$\begin{aligned} \sum_{j=1}^{\infty}w_j(t) &\leq \sum_{j=1}^{\infty}\frac{ML^{j-1}}{\Gamma((j-1)q+1)}(T-t)^{(j-1)q}=M\sum_{j=0}^{\infty}\frac{L^j}{\Gamma(jq+1)}(T-t)^{jq} \\ &\leq M\sum_{j=0}^{\infty}\frac{L^j}{\Gamma(jq+1)}T^{jq}=ME_{q,1}(LT^q) \end{aligned} \tag{19}$$

where $E_{q,1}$ is the Mittag-Leffler function defined by

$$E_{q,1}(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(jq+1)} \tag{20}$$

It proves that $\lim_{n\to\infty}u_n(t)$ exists, so $u(t)=\lim_{n\to\infty}u_n(t)$. Indeed, $u-u_0$ is a continuous function on J and u is a continuous function on J_0. Taking the limit $n\to\infty$ in (13), we see that $u\in C_{1-q}(J,\mathbb{R})$ is a solution of problem (12).

Now we have to prove that u is a unique solution of (12). Suppose that v is another solution distinct from u and such that $D_0=\sup_{t\in J_0}V(t)$ with $V(t)=|u(t)-v(t)|$. Then,

$$\begin{aligned} V(t) &= \frac{1}{\Gamma(q)}\left|\int_t^T (s-t)^{q-1}\big[Fu(s)-Fv(s)\big]ds\right| \\ &\leq \frac{1}{\Gamma(q)}\int_t^T (s-t)^{q-1}\Big[AV(s)+BV(\alpha(s))+\frac{DG}{\Gamma(q_1)}\int_s^T(\tau-s)^{q_1-1}V(\tau)d\tau\Big]ds \end{aligned} \tag{21}$$

by Assumption H_1. Then,

$$V(t)\leq \frac{D_0L}{\Gamma(q)}\int_t^T (s-t)^{q-1}ds=\frac{D_0L}{\Gamma(q+1)}(T-t)^q\leq \frac{D_0L}{\Gamma(q+1)}T^q\equiv D_1 \tag{22}$$

This and the previous relation on V give

$$\begin{aligned} V(t) &\leq \frac{1}{\Gamma(q)} \int_t^T (s-t)^{q-1} \left[AV(s) + BV\big(\alpha(s)\big) + \frac{DG}{\Gamma(q_1)} \int_s^T (\tau - s)^{q-1} V(\tau) d\tau \right] ds \\ &\leq \frac{D_1 L}{\Gamma(q)} \int_t^T (s-t)^{q-1} ds = \frac{D_1 L}{\Gamma(q+1)} (T-t)^q \end{aligned} \tag{23}$$

Repeating it, we can show, by induction, that

$$V(t) \leq \frac{D_1 L^n}{\Gamma(nq+1)} (T-t)^{nq}, \quad n = 0, 1, \cdots \tag{24}$$

so

$$V(t) \leq \frac{D_1 L^n}{\Gamma(nq+1)} T^{nq}, \quad n = 0, 1, \cdots \tag{25}$$

Indeed,

$$\lim_{n\to\infty} \frac{L^n}{\Gamma(nq+1)} T^{nq} = 0 \tag{26}$$

This shows that u is the unique solution of (12). This also proves that u is the unique solution of (10). This ends the proof. ■

Remark 1. Put $Z_n(t) = |u_n(t) - u(t)|$, where u is the unique solution of problem (10) and u_n is defined as in the proof of Theorem 1. Indeed, $Z_0(t) \leq \max_{t\in J} Z_0(t) \equiv K$. Moreover,

$$\begin{aligned} Z_n(t) &\leq \frac{1}{\Gamma(q)} \int_t^T (s-t)^{q-1} \big| Fu_{n-1}(s) - Fu(s) \big| ds \\ &\leq \frac{1}{\Gamma(q)} \int_t^T (s-t)^{q-1} \Big[AZ_{n-1}(s) + BZ_{n-1}(\alpha(s)) \\ &\quad + \frac{DG}{\Gamma(q_1)} \int_s^T (\tau - s)^{q_1 - 1} Z_{n-1}(\tau) d\tau \Big] ds \end{aligned} \tag{27}$$

for $n = 1, 2, \cdots$.

Similarly as in the proof of Theorem 1, we see that

$$\big| u_n(t) - u(t) \big| \leq \frac{KL^n}{\Gamma(nq+1)} (T-t)^{nq}, \quad n = 0, 1, \cdots \tag{28}$$

The above relation gives the estimation between the approximate solution u_n of problem (10) and the unique solution u of problem (10).

Lemma 2. *Assume that there exists a nonnegative constant M_1 such that*

$$\begin{aligned} &(i) \quad \sup_{t\in J_0} |Fu_0(t)| \leq M_1, \quad 0 < q \leq \frac{1}{2} \\ &(ii) \quad \sup_{t\in J_0} (T-t)^{1-q} |Fu_0(t)| \leq M_1, \quad \frac{1}{2} < q < 1 \end{aligned} \tag{29}$$

Then, condition (11) holds.

Proof. Case 1. Assume that $0<q\le \frac{1}{2}$. Indeed,

$$\frac{1}{\Gamma(q)}\int_t^T (s-t)^{q-1}|Fu_0(s)|ds \le \frac{M_1}{\Gamma(q)}\int_t^T (s-t)^{q-1}ds \le \frac{M_1}{\Gamma(q+1)}T^q \equiv M \tag{30}$$

Case 2. Let $\frac{1}{2}<q<1$. Using the Schwartz inequality for integrals, we have

$$\begin{aligned}\int_t^T (s-t)^{q-1}(T-s)^{q-1}ds &\le \sqrt{\int_t^T (s-t)^{2(q-1)}ds}\sqrt{\int_t^T (T-s)^{2(q-1)}ds} \\ &= \frac{\Gamma(2q-1)}{\Gamma(2q)}(T-t)^{2q-1} \le \frac{\Gamma(2q-1)}{\Gamma(2q)}T\end{aligned} \tag{31}$$

Hence,

$$\begin{aligned}\frac{1}{\Gamma(q)}\int_t^T (s-t)^{q-1}\big|Fu_0(s)\big|ds &= \frac{1}{\Gamma(q)}\int_t^T (s-t)^{q-1}(T-s)^{q-1}(T-s)^{1-q}\big|Fu_0(s)\big|ds \\ &\le \frac{M_1}{\Gamma(q)}\frac{\Gamma(2q-1)}{\Gamma(2q)}T \equiv M\end{aligned} \tag{32}$$

This ends the proof. ∎

Remark 2. For example, in papers [2, 15] the assumption

$$\sup_{t\in J_1}\big|f(t,y)\big| \le M \tag{33}$$

has been used for in initial value problem:

$$\begin{cases} D^q x(t) = f\big(t,x(t)\big), & t\in J_1=(0,T],\ q\in(0,1] \\ \bar{x}(0)=k, \quad \bar{x}(0)=t^{1-q}x(t)|_{t=0} \end{cases} \tag{34}$$

Example 1. Consider the following nonlinear fractional differential problem:

$$\begin{cases} D_T^q x(t) = \lambda \sin x(t) + \sigma(t), \quad t\in J_0=[0,T) \\ \bar{x}(T)=k \end{cases} \tag{35}$$

where $\lambda,k\in \mathbb{R}$, $\sigma\in C(J,\mathbb{R})$. Note that all assumptions of Theorem 1 hold with

$$A=|\lambda|, \quad B=D=0, \quad M=\frac{T^q}{\Gamma(q+1)}\left[\max_{t\in J}|\sigma(t)|+|\lambda|\right] \tag{36}$$

In view of Theorem 1, problem (35) has a unique solution x being the limit of the sequence $\{x_n\}$ defined by

$$\begin{cases} x_0(t) = k(T-t)^{q-1} \\ x_{n+1}(t) = k(T-t)^{q-1} + \dfrac{1}{\Gamma(q)}\displaystyle\int_t^T (s-t)^{q-1}\big[\lambda \sin x_n(s)+\sigma(s)\big]ds \end{cases} \tag{37}$$

for $n=0,1,\cdots$. Moreover

$$\big|x(t)-x_n(t)\big| \le \frac{KL^n}{\Gamma(nq+1)}(T-t)^{nq}, \quad n=0,1,\cdots \tag{38}$$

by Remark 1. Here $L=|\lambda|$, $K=M$. ∎

Now, we consider the case $q=1$, so problem (10) takes the form

$$\begin{cases} u'(t)=-Fu(t), & t\in J \\ u(T)=k\in\mathbb{R} \end{cases} \tag{39}$$

Theorem 2. *Let $q=1$, $q_1>0$. Suppose that Assumption H_1 holds. Then, problem (39) has a unique solution $u\in C^1(J,\mathbb{R})$.*

Proof. Note that problem (39) is equivalent to the following one

$$u(t)=k+\int_t^T Fu(s)ds\equiv Au(t), \quad t\in J \tag{40}$$

Put

$$\|u\|_*=\max_{t\in J} e^{\lambda(t-T)}|u(t)| \text{ for } \lambda\geq L,\ \lambda>0, \text{ and } Q=\left(1-e^{-\lambda T}\right)<1 \tag{41}$$

where L is defined as in the proof of Theorem 1. We show that operator A is a contraction with the Bielecki norm $\|\cdot\|_*$. Let $u,v\in C(J,\mathbb{R})$. Then, in view of Assumption H_1, we obtain

$$\begin{aligned}
\|Au-Av\|_* &\leq \max_{t\in J} e^{\lambda(t-T)}\int_t^T \big|Fu(s)-Fv(s)\big|ds \\
&\leq \max_{t\in J} e^{\lambda(t-T)}\int_t^T \Bigg[A\big|u(s)-v(s)\big|+B\big|u\big(\alpha(s)\big)-v\big(\alpha(s)\big)\big| \\
&\quad +\frac{DG}{\Gamma(q_1)}\int_s^T (\tau-s)^{q_1-1}|u(\tau)-v(\tau)|d\tau\Bigg]ds \\
&\leq \|u-v\|_*\max_{t\in J} e^{\lambda t}\int_t^T \left[Ae^{-\lambda s}+Be^{-\lambda\alpha(s)}+\frac{DG}{\Gamma(q_1)}\int_s^T (\tau-s)^{q_1-1}e^{-\lambda\tau}d\tau\right]ds \\
&\leq \|u-v\|_* L\max_{t\in J} e^{\lambda t}\int_t^T e^{-\lambda s}ds \\
&=\frac{L}{\lambda}\, Q\|u-v\|_*\leq Q\|u-v\|_*
\end{aligned} \tag{42}$$

Then, problem (39) has a unique solution, by the Banach fixed point theorem. This ends the proof. ∎

Remark 3. To show that problem (10) with $q\in(0,1)$ has a unique solution we can also use the Banach fixed point theorem with a corresponding norm using the Hölder inequality for integrals.

3. Uniqueness of solutions of problem (10)

Basing on the proof of Theorem 1, we can formulate some sufficient conditions for the uniqueness of the solution of problem (10) but it does not guarantee the existence of this solution.

Theorem 3. *Let Assumption H_1 hold and let $q_1>0$, $0<q<1$. Then, problem (10) has at most one solution in the space $C_{1-q}(J,\mathbb{R})$.*

Proof. Note that u is a solution of (10) if and only if

$$u(t) = k(T-t)^{q-1} + I_T^q Fu(t) \tag{43}$$

Assume that the above problem has two distinct solutions $U, V \in C_{1-q}(J, \mathbb{R})$ and put $P(t) = |U(t) - V(t)|$, $P_0 = \sup_{t \in J_0} P(t)$. Then, using Assumption H_1, we obtain

$$\begin{aligned}
P(t) &\le I_T^q \big|FU(t) - FV(t)\big| \\
&\le \frac{1}{\Gamma(q)} \int_t^T (s-t)^{q-1} \left[AP(s) + BP(\alpha(s)) + \frac{DG}{\Gamma(q_1)} \int_s^T (\tau - s)^{q_1 - 1} P(\tau) d\tau \right] ds \\
&\le P_0 \frac{1}{\Gamma(q)} \int_t^T (s-t)^{q-1} \left[A + B + \frac{DG}{\Gamma(q_1)} \int_s^T (\tau - s)^{q_1 - 1} d\tau \right] ds \\
&\le \frac{P_0 L}{\Gamma(q)} \int_t^T (s-t)^{q-1} ds = \frac{P_0 L}{\Gamma(q+1)} (T-t)^q \le \frac{P_0 L}{\Gamma(q+1)} T^q \equiv P_1
\end{aligned} \tag{44}$$

where L is defined as in Theorem 1. Now, similarly as in the proof of Theorem 1, we can show

$$P(t) \le \frac{P_1 L^n}{\Gamma(nq+1)} T^{nq}, \quad n = 0, 1, \cdots \tag{45}$$

Hence, the assertion holds because

$$\lim_{n \to \infty} \frac{P_1 L^n}{\Gamma(nq+1)} T^{nq} = 0 \tag{46}$$

■

4. Linear fractional differential equations

Let us consider the following linear problem

$$D_T^q x(t) = \lambda I_T^{q_1} x(t) + \sigma(t), \quad t \in J_0, \ \bar{x}(T) = k, \tag{47}$$

where $\lambda, k \in \mathbb{R}$, $\sigma \in C_{1-q}(J, \mathbb{R})$.

Theorem 4. *Let $q \in (0,1]$, $q_1 > 0$, $\lambda, k \in \mathbb{R}$, $\sigma \in C_{1-q}(J, \mathbb{R})$. Then, problem (47) has a unique solution given by the formula*

$$\begin{aligned}
x(t) &= k\Gamma(q)(T-t)^{q-1} E_{q+q_1, q}\big(\lambda (T-t)^{q+q_1}\big) \\
&\quad + \int_t^T (s-t)^{q-1} E_{q+q_1, q}\big(\lambda (s-t)^{q+q_1}\big) \sigma(s) ds
\end{aligned} \tag{48}$$

where $E_{\nu,\beta}(\zeta) = \sum_{r=0}^{\infty} \frac{\zeta^r}{\Gamma(\nu r + \beta)}$ is the Mittag-Leffler function.

Proof. Indeed, problem (47) is equivalent in the space $C_{1-q}(J, \mathbb{R})$ to the following fractional integral equation

$$x(t) = x_0(t) + \lambda I_T^{q+q_1} x(t) + I_T^q \sigma(t), \quad t \in J_0 \tag{49}$$

where $x_0(t) = k(T-t)^{q-1}$.

We apply the method of successive approximations to find the solution of problem (49), so for $n=0,1,\cdots$, we have

$$x_{n+1}(t)=x_0(t)+\lambda I_T^{q+q_1}x_n(t)+I_T^q\sigma(t) \tag{50}$$

Hence,

$$\begin{aligned}
x_1(t)&=x_0(t)+\lambda I_T^{q+q_1}x_0(t)+I_T^q\sigma(t)\\
x_2(t)&=x_0(t)+\lambda I_T^{q+q_1}x_1(t)+I_T^q\sigma(t)\\
&=x_0(t)+\lambda I_T^{q+q_1}[x_0(t)+\lambda I_T^{q+q_1}x_0(t)+I_T^q\sigma(t)]+I_T^q\sigma(t)\\
&=x_0(t)+\lambda I_T^{q+q_1}x_0(t)+\lambda^2 I_T^{2(q+q_1)}x_0(t)+\lambda I_T^{2q+q_1}\sigma(t)+I_T^q\sigma(t)
\end{aligned} \tag{51}$$

using the relation $I_T^r I_T^m x(t)=I_T^{r+m}x(t)$, $r,m>0$.

Thus, in general, we get by induction x_n as follows

$$x_n(t)=x_0(t)+\sum_{i=1}^{n}\lambda^i I_T^{i(q+q_1)}x_0(t)+\sum_{i=1}^{n}\lambda^{i-1}I_T^{(i-1)(q+q_1)+q}\sigma(t), \quad n=1,2,\cdots \tag{52}$$

Using the following formula

$$I_T^\delta x_0(t)=x_0(t)\frac{\Gamma(q)}{\Gamma(\delta+q)}(T-t)^\delta, \quad \delta>0 \tag{53}$$

to (52), we obtain

$$\begin{aligned}
x_n(t)&=x_0(t)\left[1+\Gamma(q)\sum_{i=1}^{n}\lambda^i\frac{1}{\Gamma(i(q+q_1)+q)}(T-t)^{i(q+q_1)}\right]\\
&\quad+\sum_{i=1}^{n}\lambda^{i-1}\frac{1}{\Gamma((i-1)(q+q_1)+q)}\int_t^T(s-t)^{(i-1)(q+q_1)+q-1}\sigma(s)ds\\
&=x_0(t)\Gamma(q)\sum_{i=0}^{n}\lambda^i\frac{1}{\Gamma(i(q+q_1)+q)}(T-t)^{i(q+q_1)}\\
&\quad+\int_t^T(s-t)^{q-1}\left[\sum_{i=0}^{n-1}\lambda^i\frac{1}{\Gamma(i(q+q_1)+q)}(s-t)^{i(q+q_1)}\right]\sigma(s)ds
\end{aligned} \tag{54}$$

for $n=0,1,\cdots$. Taking the limit as $n\to\infty$, we obtain the unique solution x in terms of the Mittag-Lefler function given by formula (48). ■

Remark 4. Put $q=1$, then, problem (47) takes the form

$$-x'(t)=\lambda I_T^{q_1}x(t)+\sigma(t), \quad t\in J_0,\ x(T)=k \tag{55}$$

Let $q_1=1$, $\lambda=1$. Then, $E_{2,1}(t^2)=\cosh(t)$, so, in view of (48), the solution has the form

$$x(t)=k\cosh(T-t)+\int_t^T\cosh(s-t)\sigma(s)ds \tag{56}$$

5. Examples

In this section, some examples are given.

Example 2. For $q \in (0,1]$, $q_1 > 0$, let us consider the following problem

$$\begin{cases} D_T^q x(t) = I_T^{q_1} x(t) - \dfrac{\Gamma(q)}{\Gamma(q+q_1)}(T-t)^{q+q_1-1}, & t \in J_0 = [0,T) \\ \bar{x}(T) = 1 \end{cases} \tag{57}$$

Comparing this problem with (47) we see that

$$\lambda = 1, \quad \sigma(t) = -\frac{\Gamma(q)}{\Gamma(q+q_1)}(T-t)^{q+q_1-1}, \quad k = 1 \tag{58}$$

In view of Theorem 4, problem (57) has a unique solution given by

$$\begin{aligned} x(t) &= \Gamma(q)(T-t)^{q-1} E_{q+q_1,q}\big((T-t)^{q+q_1}\big) + \int_t^T (s-t)^{q-1} E_{q+q_1,q}\big((s-t)^{q+q_1}\big)\sigma(s)ds \\ &= \Gamma(q)(T-t)^{q-1} E_{q+q_1,q}\big((T-t)^{q+q_1}\big) \\ &\quad - \frac{\Gamma(q)}{\Gamma(q+q_1)} \sum_{n=0}^{\infty} \frac{1}{\Gamma(n(q+q_1)+q)} \int_t^T (s-t)^{n(q+q_1)+q-1}(T-s)^{q+q_1-1} ds \\ &= \Gamma(q)(T-t)^{q-1} E_{q+q_1,q}\big((T-t)^{q+q_1}\big) \\ &\quad - \Gamma(q) \sum_{n=0}^{\infty} \frac{1}{\Gamma((n+1)(q+q_1)+q)} (T-t)^{(n+1)(q+q_1)+q-1} \\ &= \Gamma(q)(T-t)^{q-1} \left[E_{q+q_1,q}\big((T-t)^{q+q_1}\big) - \sum_{n=1}^{\infty} \frac{1}{\Gamma(n(q+q_1)+q)}(T-t)^{n(q+q_1)} \right] \\ &= \Gamma(q)(T-t)^{q-1} \left[E_{q+q_1,q}\big((T-t)^{q+q_1}\big) - E_{q+q_1,q}((T-t)^{q+q_1}) + \frac{1}{\Gamma(q)} \right] \\ &= (T-t)^{q-1} \end{aligned} \tag{59}$$

It proves that $x(t) = (T-t)^{q-1}$ is the unique solution of problem (57).

Example 3. Consider the system of fractional linear equations:

$$\begin{cases} D_T^q x(t) = 2I_T^{q_1} x(t) - 2I_T^{q_1} y(t) + \sigma_1(t), & t \in J_0 = [0,T) \\ D_T^q y(t) = -2I_T^{q_1} x(t) + 2I_T^{q_1} y(t) + \sigma_2(t), & t \in J_0 \\ \bar{x}(T) = 1, \quad \bar{y}(T) = 0 \end{cases} \tag{60}$$

with $q \in (0,1)$, $q_1 > 0$ and

$$\begin{aligned} \sigma_1(t) &= -\frac{2\Gamma(q)}{\Gamma(q+q_1)}(T-t)^{q+q_1-1} + \frac{10}{\Gamma(2+q_1)}(T-t)^{1+q_1} + \frac{4}{\Gamma(3-q)}(T-t)^{2-q}, \\ \sigma_2(t) &= \frac{2\Gamma(q)}{\Gamma(q+q_1)}(T-t)^{q+q_1-1} - \frac{10}{\Gamma(2+q_1)}(T-t)^{1+q_1} + \frac{4}{\Gamma(3-q)}(T-t)^{2-q} \\ &\quad + \frac{5}{\Gamma(2-q)}(T-t)^{1-q} \end{aligned} \tag{61}$$

Put $P = x+y$, $Q = x-y$. Then, in view of (60), we obtain

$$\begin{cases} D_T^q P(t) = \sigma_1(t) + \sigma_2(t), & t \in J_0 \\ \bar{P}(T) = 1 \end{cases} \tag{62}$$

$$\begin{cases} D_T^q Q(t) = 4I_T^{q_1} Q(t) + \sigma_1(t) - \sigma_2(t) \\ \bar{Q}(T) = 1 \end{cases} \tag{63}$$

In view of (48), the solution P of problem (62) is given by

$$\begin{aligned} P(t) &= (T-t)^{q-1} + \frac{1}{\Gamma(q)} \int_t^T (s-t)^{q-1} \big[\sigma_1(s) + \sigma_2(s)\big] ds \\ &= (T-t)^{q-1} + \frac{1}{\Gamma(q)} \int_t^T (s-t)^{q-1} \left[\frac{8}{\Gamma(3-q)}(T-s)^{2-q} + \frac{5}{\Gamma(2-q)}(T-s)^{1-q}\right] ds \\ &= (T-t)^{q-1} + 4(T-t)^2 + 5(T-t) \end{aligned} \tag{64}$$

Similarly, for Q we have

$$\begin{aligned} Q(t) &= \Gamma(q)(T-t)^{q-1} E_{q+q_1,q}\big(4(T-t)^{q+q_1}\big) \\ &\quad + \int_t^T (s-t)^{q-1} E_{q+q_1,q}\big(4(s-t)^{q+q_1}\big) \big[\sigma_1(s) - \sigma_2(s)\big] ds \\ &= \Gamma(q)(T-t)^{q-1} E_{q+q_1,q}\big(4(T-t)^{q+q_1}\big) \\ &\quad + \sum_{r=0}^{\infty} \frac{4^r}{\Gamma(r(q+q_1)+q)} \int_t^T (s-t)^{r(q+q_1)+q-1} \left[-\frac{4\Gamma(q)}{\Gamma(q+q_1)}(T-s)^{q+q_1-1}\right. \\ &\quad \left. + \frac{20}{\Gamma(2+q_1)}(T-s)^{1+q_1} - \frac{5}{\Gamma(2-q)}(T-s)^{1-q}\right] \\ &= \Gamma(q)(T-t)^{q-1} E_{q+q_1,q}\big(4(T-t)^{q+q_1}\big) \\ &\quad - \Gamma(q) \sum_{r=0}^{\infty} \frac{4^{r+1}}{\Gamma((r+1)(q+q_1)+q)} (T-t)^{(r+1)(q+q_1)+q-1} \\ &\quad + 5 \sum_{r=0}^{\infty} \frac{4^{r+1}}{\Gamma((r+1)(q+q_1)+2)} (T-t)^{(r+1)(q+q_1)+1} \\ &\quad - 5 \sum_{r=0}^{\infty} \frac{4^r}{\Gamma(r(q+q_1)+2)} (T-t)^{r(q+q_1)+1} \\ &= \Gamma(q)(T-t)^{q-1} E_{q+q_1,q}\big(4(T-t)^{q+q_1}\big) \\ &\quad - \Gamma(q)(T-t)^{q-1} \left[E_{q+q_1,q}\big(4(T-t)^{q+q_1}\big) - \frac{1}{\Gamma(q)}\right] \\ &\quad + 5(T-t)\big[E_{q+q_1,2}\big(4(T-t)^{q+q_1}\big) - 1\big] - 5(T-t) E_{q+q_1,2}\big(4(T-t)^{q+q_1}\big) \\ &= (T-t)^{q-1} - 5(T-t) \end{aligned} \tag{65}$$

Now, solving the system:

$$\begin{cases} x(t) + y(t) = (T-t)^{q-1} + 4(T-t)^2 + 5(T-t) \\ x(t) - y(t) = (T-t)^{q-1} - 5(T-t) \end{cases} \tag{66}$$

we see that the solution (x,y) of (60) is given by

$$\begin{cases} x(t) = (T-t)^{q-1} + 2(T-t)^2 \\ y(t) = 2(T-t)^2 + 5(T-t) \end{cases} \tag{67}$$

Example 4. Consider the problem:

$$\begin{cases} D_T^q x(t) = \lambda \dfrac{1}{\Gamma(q_1)} \displaystyle\int_t^T (s-t)^{q_1-1}(T-s)^r x(s)ds, \quad t \in J_0 = [0,T) \\ \bar{x}(T) = k \end{cases} \tag{68}$$

where $\lambda, k \in \mathbb{R}$, $r > -q$, $q_1 > 0$; so $g(s) = (T-s)^r$ in comparing with the operator F from problem (3). Problem (68) is equivalent to the integral equation

$$x(t) = k(T-t)^{q-1} + \frac{\lambda}{\Gamma(q)\Gamma(q_1)} \int_t^T (s-t)^{q-1} \left[\int_s^T (\tau-s)^{q_1-1}(T-\tau)^r x(\tau)d\tau \right] ds. \tag{69}$$

To find a solution of (69) we use the method of successive approximation, so

$$\begin{cases} x_0(t) = k(T-t)^{q-1} \\ x_{n+1}(t) = x_0(t) + \dfrac{\lambda}{\Gamma(q)\Gamma(q_1)} \displaystyle\int_t^T (s-t)^{q-1} \left[\int_s^T (\tau-s)^{q_1-1}(T-\tau)^r x_n(\tau)d\tau \right] ds \end{cases} \tag{70}$$

for $n = 0, 1, \cdots$.

Indeed,

$$\begin{aligned} x_1(t) &= x_0(t) + \frac{\lambda k}{\Gamma(q)\Gamma(q_1)} \int_t^T (s-t)^{q-1} \left[\int_s^T (\tau-s)^{q_1-1}(T-\tau)^{r+q-1} d\tau \right] ds \\ &= x_0(t) + \frac{\lambda k}{\Gamma(q)} \frac{\Gamma(r+q)}{\Gamma(r+q+q_1)} \int_t^T (s-t)^{q-1}(T-s)^{r+q+q_1-1} ds \\ &= x_0(t) \left[1 + \lambda \frac{\Gamma(r+q)}{\Gamma(2q+r+q_1)} (T-t)^{q+r+q_1} \right] \end{aligned} \tag{71}$$

and

$$\begin{aligned} x_2(t) &= x_0(t) + \frac{\lambda}{\Gamma(q)\Gamma(q_1)} \int_t^T (s-t)^{q-1} \left[\int_s^T (\tau-s)^{q_1-1}(T-\tau)^r x_1(\tau)d\tau \right] ds \\ &= x_0(t) + \frac{\lambda k}{\Gamma(q)\Gamma(q_1)} \int_t^T (s-t)^{q-1} \left[\int_s^T (\tau-s)^{q_1-1}(T-\tau)^{r+q-1} d\tau \right. \\ &\quad \left. + \lambda \frac{\Gamma(r+q)}{\Gamma(2q+r+q_1)} \int_s^T (\tau-s)^{q_1-1}(T-\tau)^{2r+2q+q_1-1} d\tau \right] ds \\ &= x_0(t) + \frac{\lambda k}{\Gamma(q)} \frac{\Gamma(r+q)}{\Gamma(q+r+q_1)} \int_t^T (s-t)^{q-1}(T-s)^{r+q+q_1-1} ds \\ &\quad + \frac{\lambda^2 k}{\Gamma(q)} \frac{\Gamma(r+q)\Gamma(2r+2q+q_1)}{\Gamma(2q+r+q_1)\Gamma\big(2(r+q+q_1)\big)} \int_t^T (s-t)^{q-1}(T-s)^{2q+2q_1+2r-1} ds \\ &= x_0(t) \left[1 + \lambda \frac{\Gamma(q+r)}{\Gamma(2q+r+q_1)} (T-t)^{q+r+q_1} \right. \\ &\quad \left. + \Lambda^2 \frac{\Gamma(q+r)\Gamma(2(q+r)+q_1)}{\Gamma(2q+r+q_1)\Gamma(3q+2(r+q_1)} (T-t)^{2(q+r+q_1)} \right] \end{aligned} \tag{72}$$

By induction, we can show

$$x_n(t) = x_0(t)\sum_{j=0}^{n}\lambda^j c_j (T-t)^{j(q+r+q_1)}, \quad n=1,2,\cdots \tag{73}$$

where

$$c_0 = 1, \quad c_j = \prod_{i=0}^{j-1}\frac{\Gamma(i(q+r+q_1)+q+r)}{\Gamma((i+1)(q+r+q_1)+q)}, \quad j=1,2,\cdots \tag{74}$$

Now, taking the limit as $n\to\infty$, we obtain a solution of (68), by formula:

$$x(t) = k(T-t)^{q-1}\sum_{j=0}^{\infty}c_j\left[\lambda(T-t)^{(q+r+q_1)}\right]^j \tag{75}$$

This solution of (68) can be written in the form

$$x(t) = k(T-t)^{q-1}E_{q+q_1,1+\frac{r}{q+q_1},1+\frac{r-1-q_1}{q+q_1}}\left[\lambda(T-t)^{(q+r+q_1)}\right] \tag{76}$$

where $E_{\nu,m,n}$ is the Mittag-Leffler function given by

$$E_{\nu,m,n}(z) = \sum_{j=0}^{\infty}c_j^* z^j \tag{77}$$

with

$$c_0^* = 1, \quad c_j^* = \prod_{i=0}^{j-1}\frac{\Gamma(\nu(im+n)+1)}{\Gamma(\nu(im+n+1)+1)}, \quad j=1,2,\cdots \tag{78}$$

see p. 48 of [1]. Indeed, $c_j = c_j^*$ for $\nu = q+q_1$, $m = 1+\frac{r}{q+q_1}$, $n = 1+\frac{r-1-q_1}{q+q_1}$. If $r\ge 0$, then, x given by (76) is the unique solution of (68), by Theorem 1. Note that if $r=0$, then, $c_j = \frac{\Gamma(q)}{\Gamma(j(q+q_1)+q)}$ and $E_{q+q_1,1,\frac{q-1}{q+q_1}}(z) = E_{q+q_1,q}(z)$. ■

6. Existence results for fractional problems of type (3) with $q=1$

In this section, we consider the existence of extremal solutions of problem (3) in the case $q=1$. To obtain it, we apply the monotone iterative technique, therefore we first formulate a comparison result which will play a very important role in our research.

Lemma 3. *Let $\alpha\in C(J,J)$, $t\le\alpha(t)\le T$ on J. Suppose that $M\in C(J,\mathbb{R})$, $p\in C^1(J,\mathbb{R})$ and*

$$\begin{cases} p'(t) \ge M(t)p(t) + \mathcal{G}p(t), & t\in J \\ p(T)\le 0 \end{cases} \tag{79}$$

where operator $\mathcal{G}$ is defined by

$$\mathcal{G}p(t) = N(t)p(\alpha(t)) + P(t)I_T^{q_1}p(t) \tag{80}$$

with nonnegative functions N,P integrable on J and the right-sided fractional integral $I_T^{q_1}p$ of order $q_1>0$

In addition, we assume that

H_2: *$r \le 1$ with*

$$r = \int_0^T \left[N(t)\exp\left(\int_t^{\alpha(t)} M(s)ds \right) + \frac{P(t)}{\Gamma(q_1)} \int_t^T (s-t)^{q_1-1} \exp\left(\int_t^s M(\tau)d\tau \right) ds \right] dt \quad (81)$$

Then, $p(t) \le 0$ on J.

Proof. Put

$$q(t) = \exp\left(\int_t^T M(s)ds \right) p(t), \quad t \in J \quad (82)$$

This and (79) give $q(T) = p(T) \le 0$, and

$$\begin{aligned} q'(t) &= \exp\left(\int_t^T M(s)ds \right) [-M(t)p(t) + p'(t)] \ge \exp\left(\int_t^T M(s)ds \right) \mathcal{G}p(t) \\ &= N(t)\exp\left(\int_t^{\alpha(t)} M(s)ds \right) q(\alpha(t)) + \frac{P(t)}{\Gamma(q_1)} \int_t^T (s-t)^{q_1-1} \exp\left(\int_t^s M(\tau)d\tau \right) q(s)ds \end{aligned} \quad (83)$$

so

$$\begin{cases} -q'(t) \le -N(t)\exp\left(\int_t^{\alpha(t)} M(s)ds \right) q\big(\alpha(t)\big) \\ \qquad\qquad - \dfrac{P(t)}{\Gamma(q_1)} \displaystyle\int_t^T (s-t)^{q_1-1} \exp\left(\int_t^s M(\tau)d\tau \right) q(s)ds \\ q(T) \le 0 \end{cases} \quad (84)$$

We need to prove that $q(t) \le 0$, $t \in J$. Suppose that the inequality $q(t) \le 0$, $t \in J$ is not true. Then, we can find $t_0 \in [0,T)$ such that $q(t_0) > 0$. Put

$$q(t_1) = \min_{[t_0,T]} q(t) \le 0 \quad (85)$$

Integrating the differential inequality in (84) from t_0 to t_1, we obtain

$$\begin{aligned} q(t_0) - q(t_1) &\le -\int_{t_0}^{t_1} \left[N(t)\exp\left(\int_t^{\alpha(t)} M(s)ds \right) q\big(\alpha(t)\big) \right. \\ &\qquad \left. + \frac{P(t)}{\Gamma(q_1)} \int_t^T (s-t)^{q_1-1} \exp\left(\int_t^s M(\tau)d\tau \right) q(s)ds \right] dt \\ &\le -rq(t_1) \le -q(t_1) \end{aligned} \quad (86)$$

It contradicts the assumption that $q(t_0) > 0$. This proves that $q(t) \le 0$ on J. This also proves that $p(t) \le 0$ on J and the proof is complete. ■

Remark 5. Assume $M(t) \ge 0$ on J and

$$\int_0^T \left[N(t)\exp\left(\int_t^T M(s)ds \right) + \frac{P(t)}{\Gamma(q_1)} \int_t^T (s-t)^{q_1-1} \exp\left(\int_t^s M(\tau)d\tau \right) ds \right] dt \le 1 \quad (87)$$

Note that the above condition does not depend on α and moreover Assumption H_2 holds.

Remark 6. Assume that $1 \le A_0 \exp\left(\int\limits_s^T M(\tau)d\tau\right)(T-s)^a$, $A_0>0$, $a\ge 0$. Then,

$$r \le \int_0^T \left[N(t)\exp\left(\int\limits_t^{\alpha(t)} M(s)ds\right) + \frac{A_0P(t)}{\Gamma(q_1)}\exp\left(\int\limits_t^T M(\tau)d\tau\right)\int_t^T (s-t)^{q_1-1}(T-s)^a ds\right]dt$$
$$= \int_0^T \left[N(t)\exp\left(\int\limits_t^{\alpha(t)} M(s)ds\right) + \frac{A_0P(t)\Gamma(a+1)}{\Gamma(a+q_1+1)}\exp\left(\int\limits_t^T M(\tau)d\tau\right)(T-t)^{a+q_1}\right]dt \equiv r_1 \tag{88}$$

Indeed, Assumption H_2 holds if $r_1 \le 1$.

We can also obtain another condition for r from Assumption H_2, namely using the following estimation

$$\exp\left(\int\limits_0^T |M(\tau)|d\tau\right) \le P_0 \tag{89}$$

Then, it is easy to see that Assumption H_2 holds, if we assume that

$$\int_0^T \left[N(t)\exp\left(\int\limits_t^{\alpha(t)} M(s)ds\right) + \frac{P(t)P_0}{\Gamma(q_1+1)}(T-t)^{q_1}\right]dt \le 1 \tag{90}$$

Now, we are going to use the monotone iterative technique to find a solution of (3) for $q=1$. Let us introduce the following definition.

Let $q=1$ and $q_1>0$. We say that $u\in C^1(J,\mathbb{R})$ is a lower solution of (3) if

$$u'(t) \le -Fu(t), \quad t\in J, \quad h\big(u(T)\big) \le 0 \tag{91}$$

and it is an upper solution of (3) if the above inequalities are reversed.

A solution $y\in C^1(J,\mathbb{R})$ of problem (3) is called maximal if $x(t)\le y(t)$, $t\in J$ for each solution x of (3), and minimal, if the reverse inequality holds. If both minimal and maximal solutions exist, we call them extremal solutions of (3).

If we know the existence of lower and upper solutions y_0, z_0 of problem (3) such that $z_0(t)\le y_0(t)$, $t\in J$, then, under corresponding conditions, we can prove the existence of the extremal solutions of (3) in the sector

$$[z_0,y_0]_* = \big\{ w\in C^1(J,\mathbb{R}): \quad z_0(t)\le w(t)\le y_0(t), \quad t\in J\big\} \tag{92}$$

It is the content of the next result.

Theorem 5. *Let $q=1$, and $q_1>0$. Let Assumption H_1 hold (with $g(t)=1$, $t\in J$) and $h\in C(\mathbb{R},\mathbb{R})$. Let $y_0,z_0\in C^1(J,\mathbb{R})$ be lower and upper solutions of (3), respectively and $z_0(t)\le y_0(t)$, $t\in J$. In addition, we assume that*

H_3: *there exist functions $M\in C(J,\mathbb{R})$, $N,P\in C(J,\mathbb{R}_+)$ such that Assumption H_2 holds and*

$$f(t,v_1,v_2,v_3) - f(t,u_1,u_2,u_3) \ge -M(t)[v_1-u_1] - N(t)[v_2-u_2] - P(t)[v_3-u_3] \tag{93}$$

if $z_0(t)\le u_1\le v_1\le y_0(t)$, $z_0(\alpha(t))\le u_2\le v_2\le y_0(\alpha(t))$, $I_T^{q_1}z_0(t)\le u_3\le v_3\le I_T^{q_1}y_0(t)$

H_4: *there exists a constant* $\mu > 0$ *such that*

$$h(u) - h(u_0) \le \mu(u_0 - u) \text{ if } z_0(T) \le u \le u_0 \le y_0(T) \tag{94}$$

Then, problem (3) has extremal solutions in the sector $[z_0, y_0]_*$.

Proof. Put $(z_0, y_0)_* = \{w \in C(J, \mathbb{R}) : \ z_0(t) \le w(t) \le y_0(t), \ t \in J\}$. Let $\eta, \xi \in (z_0, y_0)_*$ and let $\varphi(t) = \min[\eta(t), \xi(t)]$, $\Phi(t) = \max[\eta(t), \xi(t)]$.

Consider the boundary value problems

$$\begin{cases} v'(t) = M(t)[v(t) - \Phi(t)] + \mathcal{G}v(t) - \mathcal{G}\Phi(t) - F\Phi(t), \quad t \in J \\ v(T) = \dfrac{1}{\mu} h\big(\Phi(T)\big) + \Phi(T) \end{cases} \tag{95}$$

$$\begin{cases} w'(t) = M(t)[w(t) - \varphi(t)] + \mathcal{G}w(t) - \mathcal{G}\varphi(t) - F\varphi(t), \quad t \in J \\ w(T) = \dfrac{1}{\mu} h\big(\varphi(T)\big) + \varphi(T) \end{cases} \tag{96}$$

where operator $\mathcal{G}$ is defined as in Lemma 3. By Theorem 2, problems (95), (96) have a unique solution. Therefore, we can define the operator

$$B: \ \bar{\Omega} \to C(J, \mathbb{R}) \times C(J, \mathbb{R}), \quad (z_0, y_0)_* \subset C(J, \mathbb{R}), \ B(\eta, \xi) = (v, w) \tag{97}$$

where v, w are solutions of (95) and (96), respectively with $\bar{\Omega} = (z_0, y_0)_* \times (z_0, y_0)_*$.

Now, we want to show that

$$z_0(t) \le w(t) \le v(t) \le y_0(t), \quad t \in J \tag{98}$$

Put $p = z_0 - w$. Then, in view of Assumption H_3, we have

$$\begin{aligned} p'(t) &\ge -Fz_0(t) - M(t)\big[w(t) - \varphi(t)\big] - \mathcal{G}w(t) + \mathcal{G}\varphi(t) + F\varphi(t) \\ &\ge -M(t)\big[\varphi(t) - z_0(t)\big] - \mathcal{G}\varphi(t) + \mathcal{G}z_0(t) - \mathcal{G}w(t) + \mathcal{G}\varphi(t) - M(t)\big[w(t) - \varphi(t)\big] \\ &= M(t)p(t) + \mathcal{G}p(t) \end{aligned} \tag{99}$$

Moreover, in view of Assumption H_4,

$$\begin{aligned} p(T) &= z_0(T) - \frac{1}{\mu}\big[h\big(\varphi(T)\big) - h\big(z_0(T)\big) + h\big(z_0(T)\big)\big] - \varphi(T) \\ &\le z_0(T) - \varphi(T) + \varphi(T) - z_0(T) = 0 \end{aligned} \tag{100}$$

This and Lemma 3 show that $z_0(t) \le w(t)$, $t \in J$. Similarly we can show that $v(t) \le y_0(t)$, $t \in J$. To show that $w(t) \le v(t)$, $t \in J$, we put $p = w - v$. Then,

$$\begin{aligned} p'(t) &= M(t)\big[w(t) - \varphi(t) - v(t) + \Phi(t)\big] + \mathcal{G}w(t) - \mathcal{G}\varphi(t) - F\varphi(t) - \mathcal{G}v(t) \\ &\quad + \mathcal{G}\Phi(t) + F\Phi(t) \\ &\ge -M(t)\big[\Phi(t) - \varphi(t)\big] - \mathcal{G}\Phi(t) + \mathcal{G}\varphi(t) + M(t)\big[w(t) - \varphi(t) - v(t) + \Phi(t)\big] \\ &\quad + \mathcal{G}w(t) - \mathcal{G}\varphi(t) - \mathcal{G}v(t) + \mathcal{G}\Phi(t) \\ &= M(t)p(t) + \mathcal{G}p(t) \end{aligned} \tag{101}$$

Moreover

$$p(T) = \frac{1}{\mu} h\big(\varphi(T)\big) + \varphi(T) - \frac{1}{\mu} h\big(\Phi(T)\big) - \Phi(T) \le 0 \tag{102}$$

Hence $B : \bar{\Omega} \to \bar{\Omega}$.

Note that operator $B:\bar{\Omega}\to\bar{\Omega}$ is compact by direct application of Arzeli-Ascoli theorem. Hence, by Schauder's fixed point theorem, the operator B has a fixed point, i.e. there exist $(v,w)\in\bar{\Omega}$ such that $B(v,w)=(v,w)$ and $w\le v$.

Now, by (95) and (96), we see that v,w satisfy the following relations

$$\begin{cases} v'(t)=M(t)\big[v(t)-v(t)\big]+\mathcal{G}v(t)-\mathcal{G}v(t)-Fv(t), & t\in J\\ v(T)=\dfrac{1}{\mu}h\big(v(T)\big)+v(T) \end{cases} \tag{103}$$

$$\begin{cases} w'(t)=M(t)\big[w(t)-w(t)\big]+\mathcal{G}w(t)-\mathcal{G}w(t)-Fw(t), & t\in J\\ w(T)=\dfrac{1}{\mu}h\big(w(T)\big)+w(T) \end{cases} \tag{104}$$

It shows that $v,w\in C^1(J)$ are solutions of problem (3). This ends the proof. ■

7. Existence results for fractional problems of type (3) with $q\in(0,1)$

In this Section, we will use the monotone iterative method to show that problem (3) with $q_1>0$, $0<q<1$ has a solution. First, we cite some comparison results.

Lemma 4 (see [21]). *Let $q\in(0,1)$, $M\in C(J,[0,\mathbb{R}_+)$. Suppose that $p\in C_{1-q}(J,\mathbb{R})$ satisfies the problem:*

$$\begin{cases} D_T^q p(t)\le -M(t)p(t), & t\in J_0\\ \bar{p}(T)\le 0 \end{cases} \tag{105}$$

Then, $p(t)\le 0$ on J.

Lemma 5 (see [21]). *Let $q\in(0,1)$, $M\in\mathbb{R}$. Suppose that $p\in C_{1-q}(J,\mathbb{R})$ satisfies the problem:*

$$\begin{cases} D_T^q p(t)\le -Mp(t), & t\in J_0\\ \bar{p}(T)\le 0 \end{cases} \tag{106}$$

Then, $p(t)\le 0$ on J.

Now, we introduce the following definition.

Let $q_1>0$, $0<q<1$. We say that $u\in C_{1-q}(J,\mathbb{R})$ is a lower solution of (3) if

$$D_T^q u(t)\le Fu(t),\quad t\in J_0,\quad h\big(\bar{u}(T)\big)\le 0 \tag{107}$$

and it is an upper solution of (3), if the above inequalities are reversed.

Theorem 6. *Let $q_1>0$, $0<q<1$. Let Assumption H_1 hold (with $g(t)=1$, $t\in J$) and $h\in C(\mathbb{R},\mathbb{R})$. Let $y_0,z_0\in C_{1-q}(J,\mathbb{R})$ be lower and upper solutions of (3), respectively and $y_0(t)\le z_0(t)$, $t\in J$. In addition, we assume that*

H_5: *there exist a function $M\in C(J,\mathbb{R}_+)$ such that*

$$f(t,u_1,u_2,u_2)-f(t,v_1,v_2,v_3)\le M(t)[v_1-u_1] \tag{108}$$

if $y_0(t)\le u_1\le v_1\le z_0(t)$, $y_0(\alpha(t))\le u_2\le v_2\le z_0(\alpha(t))$, $I_T^{q_1}y_0(t)\le u_3\le v_3\le I_T^{q_1}z_0(t)$,

H_6: *there exists a constant* $\mu > 0$ *such that*

$$h(u_0) - h(u) \leq \mu(u_0 - u) \text{ if } \bar{y}_0(T) \leq u \leq u_0 \leq \bar{z}_0(T) \tag{109}$$

Then, problem (3) has extremal solutions in the sector

$$[y_0, z_0] = \{w \in C_{1-q}(J, \mathbb{R}): \ y_0(t) \leq w(t) \leq z_0(t), \ t \in J_0, \ \bar{y}_0(T) \leq \bar{w}(T) \leq \bar{z}_0(T)\} \tag{110}$$

Proof. Let

$$\begin{cases} D_T^q y_{n+1}(t) = F y_n(t) - M(t)[y_{n+1}(t) - y_n(t)], \quad t \in J_0 \\ \bar{y}_{n+1}(T) = -\dfrac{1}{\mu} h\big(\bar{y}_n(T)\big) + \bar{y}_n(T) \end{cases} \tag{111}$$

$$\begin{cases} D_T^q z_{n+1}(t) = F z_n(t) - M(t)\big[z_{n+1}(t) - z_n(t)\big], \quad t \in J_0 \\ \bar{z}_{n+1}(T) = -\dfrac{1}{\mu} h\big(\bar{z}_n(T)\big) + \bar{z}_n(T) \end{cases} \tag{112}$$

for $n = 0, 1, \cdots$. Note that problems (111) and (112) have a unique solution, in view of Theorem 1.

Put $p = y_0 - y_1$. Then,

$$\begin{aligned} D_T^q p(t) &\leq F y_0(t) - F y_0(t) + M(t)\big[y_1(t) - y_0(t)\big] = -M(t)p(t) \\ \bar{p}(T) &= \bar{y}_0(T) + \frac{1}{\mu} h\big(\bar{y}_0(T)\big) - \bar{y}_0(T) \leq 0 \end{aligned} \tag{113}$$

Hence, $y_0(t) \leq y_1(t)$, in view of Lemma 4. Similarly, $z_1(t) \leq z_0(t)$. Put $p = y_1 - z_1$. Then,

$$\begin{aligned} D_T^q p(t) &= F y_0(t) - M(t)\big[y_1(t) - y_0(t)\big] - F z_0(t) + M(t)\big[z_1(t) - z_0(t)\big] \\ &\leq M(t)\big[z_0(t) - y_0(t)\big] - M(t)\big[y_1(t) - y_0(t) - z_1(t) + z_0(t)\big] = -M(t)p(t) \\ \bar{p}(T) &= -\frac{1}{\mu} h\big(\bar{y}_0(T)\big) + \bar{y}_0(T) + \frac{1}{\mu} h\big(\bar{z}_0(T)\big) - \bar{z}_0(T) \\ &\leq \bar{z}_0(T) - \bar{y}_0(T) + \bar{y}_0(T) - \bar{z}_0(T) = 0 \end{aligned} \tag{114}$$

by Assumptions H_5, H_6. This proves that

$$y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t), \quad t \in J \tag{115}$$

Now, we prove that y_1 is a lower solution of problem (3). Indeed,

$$\begin{aligned} D_T^q y_1(t) &= F y_0(t) - M(t)\big[y_1(t) - y_0(t)\big] - F y_1(t) + F y_1(t) \\ &\leq M(t)\big[y_1(t) - y_0(t)\big] - M(t)\big[y_1(t) - y_0(t)\big] + F y_1(t) = F y_1(t) \\ \bar{y}_1(T) &= -\frac{1}{\mu}\big[h\big(\bar{y}_0(T)\big) - h\big(\bar{y}_1(T)\big) + h\big(\bar{y}_1(T)\big)\big] + \bar{y}_0(T) \\ &\leq \bar{y}_1(T) - \bar{y}_0(T) + \bar{y}_0(T) - h\big(\bar{y}_1(T)\big) \end{aligned} \tag{116}$$

so $h(\bar{y}_1(T)) \leq 0$. This proves that y_1 is a lower solution of problem (3). Similarly, we can show that z_1 is an upper solution of (3).

By induction, we can prove that

$$y_0(t)\le y_1(t)\le\cdots\le y_n(t)\le z_n(t)\le\cdots\le z_1(t)\le z_0(t),\quad t\in J \tag{117}$$

Sequences $\{y_n\},\{z_n\}$ are monotone. It is easy to show that they converge to y and z, respectively, and $y\le z$. Indeed, there is no problem to prove that problem (3) has minimal and maximal solutions in $[y_0,z_0]$. This ends the proof. ■

Remark 7. If we extra assume that $M(t)=0$, $t\in J$, then, f is nondecreasing with respect to the last three variables.

Remark 8. If condition (108) holds for $\bar{M}\in C(J,\mathbb{R})$, then, it is also satisfied for some $M\in C(J,\mathbb{R}_+)$.

Example 5. Consider the problem:

$$\begin{cases} D_T^q x(t)=Ae^{-x(t)}+Bx\big(\alpha(t)\big)+CI_T^{q_1}x(t)+\dfrac{D}{\sqrt{\pi(1-t)}}\equiv Fx(t), & t\in J_0\\ 0=\bar{x}(1)\big[1-\bar{x}(1)\big] \end{cases} \tag{118}$$

where $J_0=[0,1)$, $q=\frac{1}{2}$, $q_1>0$, $\alpha\in C\big([0,1],[0,1]\big)$, $\alpha(t)\ge t$. Moreover, we assume that $A,B,C,D\ge 0$ and such that

$$\begin{aligned} Ae^{-1}+2B+C(1-t)^{q_1}\left[\frac{1}{\Gamma(q_1+1)}+\frac{1-t}{\Gamma(q_1+2)}\right]+\frac{D}{\sqrt{\pi(1-t)}}&\\ \le\frac{1}{\sqrt{\pi(1-t)}}+\frac{2}{\sqrt{\pi}}\sqrt{1-t},&\quad t\in[0,1) \end{aligned} \tag{119}$$

Put $y_0(t)=0$, $z_0(t)=2-t$, so $\bar{y}_0(1)=\bar{z}_0(1)=0$. Note that $M(t)=Ae^2$, $\mu=1$, from Theorem 6. Indeed, y_0 is a lower solution of problem (118). Moreover,

$$\begin{aligned} Fz_0(t)&=Ae^{-(2-t)}+B(2-\alpha(t))+\frac{C}{\Gamma(q_1)}\int_t^T(s-t)^{q_1-1}(1+1-s)ds+\frac{D}{\sqrt{\pi(1-t)}}\\ &\le Ae^{-1}+2B+C(1-t)^{q_1}\left[\frac{1}{\Gamma(q_1+1)}+\frac{1-t}{\Gamma(q_1+2)}\right]+\frac{D}{\sqrt{\pi(1-t)}}\\ &\le\frac{1}{\sqrt{\pi(1-t)}}+\frac{2}{\sqrt{\pi}}\sqrt{1-t}=D_T^q z_0(t) \end{aligned} \tag{120}$$

in view of (119). This proves that z_0 is an upper solution of problem (118). Hence, problem (118) has extremal solutions in the region $[y_0,z_0]$, by Theorem 6.

Remark 9. We can also discuss the problem with more right-sided fractional integrals, namely

$$\begin{cases} D_T^q x(t)=f\big(t,x(t),x(\alpha(t)),I_T^{q_1}x(t),I_T^{q_2}x(t),\cdots,I_T^{q_r}x(t)\big), & t\in J_0\\ 0=h\big(\bar{x}(T)\big) \end{cases} \tag{121}$$

with $q_1,q_2,\cdots,q_r>0$.

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