From limits of quantum operations to multicopy entanglement witnesses and state-spectrum estimation

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Limits of state transformations in quantum mechanics are studied. Impossibility of physical implementation of the transformation $\varrho^{\otimes n} \rightarrow \varrho^n$ in quantum mechanics is proved. The most natural notions of structural completely positive approximation and structural physical approximations of nonphysical map are introduced. It is shown that these always exist for linear Hermitian maps and can be optimized under natural conditions. It is pointed out that it is physically possible to measure in a simple way the traces of nth power of quantum state $\mathrm{Tr}(\varrho^n)$ if only joint measurements on n copies of the system are allowed. This gives the interpretation of Tsallis entropies as values of "multicopy" observables. Following this observations, the notion of multicopy entanglement witnesses is defined and examples are provided. Finally, using notion of multicopy observable, a simple method of spectrum state estimation is discussed.

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I. INTRODUCTION

The limits of nonlinearlike operations within the quantum mechanics is an interesting question. It has been shown that [1], for example, the operation

$$\varrho \otimes \varrho \rightarrow \begin{bmatrix} \varrho_{11}^2 & \varrho_{12}^2 \\ \varrho_{21}^2 & \varrho_{22}^2 \end{bmatrix} \qquad (1)$$

can be performed with the finite probability by means of a quite simple network with two copies of ϱ as an input. On the other hand, the limits for other nonlinear operations have been shown resulting in "no-disentanglement" rule in quantum mechanics [2,3]. In this work, we want to show both further limits and advantages of nonlinear transformations in context of the quantum entanglement theory.

It can be easily seen that if the state ϱ is diagonal, then Eq. (1) provides the square of ϱ. However, to get this, we have to know the eigenvectors of the state. It is interesting that if one could physically produce square power of an unknown state, it would be possible to distill entanglement from many quantum states with little previous information about them [4]. To some extent, it would be similar to situation in compression protocol of Ref. [5] where no measurement of a source state is needed if one of its parameters (entropy) is known. We shall show that it is impossible to produce any power of the state if we do not know its eigenvectors. More precisely, it is impossible to perform the operation providing nth power of an *unknown* state from n copies of it.

However, one can consider weaker requirements: if some operation is impossible to perform exactly, one can try to perform it *approximately*. The most known examples of such approximations are cloning operation [6–8], universal NOT gate [9], and "two-qubit fidelity" map [10]. Recently more careful study of approximations of one-qubit maps has been carried out [11,12].

It is natural to expect that physical approximations of nonphysical maps could help in solving physical problems in general. To study this, we use the notion of structural completely positive approximation (SCPA) and structural physical approximation (SPA) of the nonphysical operation. These are very restrictive approximations—the key feature of such approximated maps is that these always have the direction of generalized Bloch vector of the output state the same as the output state of the original nonphysical map. Only the length of the vector is rescaled by some factor. The first example of such map has been introduced in different context as a universal NOT gate in pioneering work [9]. It is worth mentioning the first version of applications in entanglement detection [13,14]. Here, we show that SCPA and SPA always exist for linear, nonphysical Hermitian maps. We also prove that there is a natural optimization giving the best SPA.

We also consider possibility of direct measurement of trace of nth power of a state. Using generalized swap operator, we show that value $\mathrm{Tr}(\varrho^n)$ can be measured as quantum observable if joint measurements on n copies of the system are allowed. Several interesting applications of this fact are provided. In particular, one can measure what we propose to call *multicopy observables* of the system: mean value of such observables is measurable if joint measurements on several copies of the system in the same state can be performed. The mean value of such an observable can be found if such a measurement is repeated many times. Each time it consumes n copies of ϱ, but the entire procedure can provide us the interesting information about properties of single ϱ. In our analysis, we point out that all Tsallis entropies [28] with natural index correspond to some multicopy observables. Then, applying the separability conditions as the entropic inequalities theory developed first in Refs. [26,15,29] and completed in an elegant way in Refs. [16,17]), we show that each Tsallis [18] entropic separability tests (equivalent to quantum Renyi ones [15–17]) can be physically performed directly with the help of single multicopy observable. The advantage of the method is that only a finite (small) number of copies in joint measurement is required. Finally, we point

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out how to estimate the spectrum of an unknown state using the idea of multicopy observables. The above observations are in a sense complementary to the optimal procedures of Refs. [19,20], because on one hand we allow for join measurement on small number of copies and on the other hand, we focus on mean values of quantum observables directly.

In both approaches, collective measurements on finite (not more than dimension of Hilbert space of a single copy) number of copies are involved in one run of experiment.

The paper is organized as follows: In Sec. II, we prove the following "no-go" result: if the state ϱ is unknown, then the operation $\varrho^{\otimes n} \to \varrho^n$ is impossible. In Sec. III, we provide a general idea of the SCPA and its slight modification—the SPA. We also show that these approximations always exist for any Hermitian map. Finally, we show that the most natural SPA is optimal.

In Sec. IV, we investigate further possibility of direct measurement of nonlinear parameters. We introduce the notion of "multicopy observable" and show how Tsallis entropy with natural index can be measured with the help of such an observable. We utilize entropic separability criteria to introduce the notion of *multicopy entanglement witness*. Finally, in Sec. V, we point out how to estimate the spectrum of unknown state defined on $\mathcal{C}^m$.

II. IMPOSSIBILITY OF THE OPERATION $\varrho^{\otimes N} \to \varrho^N$

Consider an arbitrary quantum state defined on $\mathcal{C}^d$ space. We shall show that there is no quantum transformation of the kind

$$\Lambda(\varrho^{\otimes n}) \to \varrho^n, \tag{2}$$

which works for an *unknown* quantum state. Let us first note that such an operation would be *probabilistic*, i.e., it would give the required output with the probability $p = \mathrm{Tr}(\varrho^n)$ depending on the input state. Suppose that such an operation existed. Then, because of complete positivity it would be of the form of completely positive map $\Lambda(\varrho) = \sum_{i=1}^N V_i \varrho V_i^\dagger$. So, we would rewrite Eq. (2) as follows:

$$\sum_{i=1}^N V_i \varrho^{\otimes n} V_i^\dagger = \varrho^n. \tag{3}$$

Taking trace of both sides of the above equation, we get that there would exist the *positive* state independent operator $A = \sum_{i=1}^N V_i^\dagger V_i$, with the property[1] $0 \leqslant A \leqslant I$ such that

$$\mathrm{Tr}(A \varrho^{\otimes n}) = \mathrm{Tr}(\varrho^n) \tag{4}$$

for *any* state ϱ. In particular, for any pure projector $P_\phi = |\phi\rangle\langle\phi|$ corresponding to normalized vector $|\phi\rangle$, we would have $\mathrm{Tr}(A P_\phi^{\otimes n}) = \mathrm{Tr}(P_\phi) = 1$. But, because all eigenvalues of A belong to the interval $[0,1]$, this means that any vector of the form $|\Psi\rangle = |\phi\rangle^{\otimes n}$ must be an eigenvector of A. However, all $|\Psi\rangle$'s of this form span the completely symmetric subspace $\mathcal{H}_{SYM}$ of $(\mathcal{C}^d)^{\otimes n}$. Thus, the subspace is an eigensubspace of A corresponding to eigenvalue 1. On the other hand, putting into Eq. (4) full rank matrix ϱ, diagonal in some standard basis $\{|e_i\rangle\}$, one gets that positivity of A would imply $A = \sum_{i=0}^{d-1} (|e_i\rangle\langle e_i|)^{\otimes n}$. But this is in contradiction to the expected invariance of $\mathcal{H}_{SYM}$ under the action of A.

[1]If A,B are Hermitian operators, then the notation $A \leqslant B$ means that for any vector $|\psi\rangle$, one has $\langle\psi|A|\psi\rangle \leqslant \langle\psi|B|\psi\rangle$.

III. STRUCTURAL PHYSICAL APPROXIMATIONS OF NONPHYSICAL HERMITIAN MAPS

A. Definition

We introduce the following definition of the SCPA of linear Hermitian map $\Lambda: \mathcal{B}(\mathcal{C}^d) \to \mathcal{B}(\mathcal{C}^{d'})$.

Definition 1. The SCPA of linear, Hermitian map Λ is any completely positive operation of the form

$$\tilde{\Lambda}(\varrho) = \delta(\varrho)\Lambda_I + \gamma\Lambda, \tag{5}$$

with linear function $\delta \geqslant 0$, and strictly positive parameter γ. The map Λ_I is defined as $\Lambda_I(\cdot) = I\mathrm{Tr}(\cdot)$. The SPA of nonphysical map Λ is such SCPA $\tilde{\Lambda}$ that any state ϱ satisfies $\mathrm{Tr}[\tilde{\Lambda}(\varrho)] \leqslant 1$, which means that $\tilde{\Lambda}$ can be implemented experimentally.

Remark 1. The optimal NOT gate [9] can be treated as first example of the SCPA (and also the SPA). On the other hand, the approximated cloning machine [6–8] is not the SCPA. Indeed, if the cloning machine acts on the state ϱ, then it can be seen that the machine output is not of the form $\delta(\varrho)\Lambda_I + \gamma\varrho \otimes \varrho$.

Remark 2. The above definition could also be extended to the form $\tilde{\Lambda}(\varrho) = \delta(\varrho)\Lambda_I + \gamma(\varrho)\Lambda$. This would apparently admit nonlinear components in the SPA map. It can be shown, however, that the assumptions of (i) nontriviality of the function γ ($\gamma \not\equiv 0$) and (ii) its continuity lead to the form (5) where γ is constant and δ linear.

Let us comment on the definition. The essence of the SCPA $\tilde{\Lambda}$ of an arbitrary map Λ is that (i) it is complete positive and (ii) its output *keeps the structure of the output* of the nonphysical map Λ. In particular, for any ϱ, the "direction" of generalized Bloch vector of $\tilde{\Lambda}(\varrho)$ is the same as in $\Lambda(\varrho)$. However, the vector of $\tilde{\Lambda}(\varrho)$ is "decreased" by a factor γ [9] and the additional portion (quantified by δ) of completely random noise is admixed. The SPA is such SCPA that can be probabilistically implemented in lab (see the Appendix). Note that for finite-dimensional systems, some SPA can be obtained from nonzero SCPA by normalization,

$$\tilde{\Lambda}_{SPA} = \frac{1}{\alpha_{\tilde{\Lambda}}}\tilde{\Lambda}, \quad \alpha_{\tilde{\Lambda}} \equiv \max_{\varrho} \mathrm{Tr}[\tilde{\Lambda}(\varrho)], \tag{6}$$

where strict positivity of $\alpha_{\tilde{\Lambda}}$ is given by complete positivity of nonzero SCPA $\tilde{\Lambda}$.

Now, one can ask about possible optimality of the given SPA. Let us recall that question of optimality was frequently posed in context of approximate cloning machines. Here, we define optimality as follows.

Definition 2. The optimal SPA of Λ is such SPA map $\tilde{\Lambda}_{opt}$ that for any fixed ϱ (i) the ratio $\delta(\varrho)/\gamma$ in Eq. (5) is minimal and (ii) the probability of implementation $p(\Lambda_{opt};\varrho) \equiv \mathrm{Tr}[\tilde{\Lambda}_{opt}(\varrho)]$ is maximal if compared to any other SPA $\tilde{\Lambda}$.

Sometimes, instead of the term "the optimal SPA," we shall use name "the best SPA."

Summarizing, for the best SPA, one requires as much less "noise" in Eq. (5) as possible under the condition of maximal probability of implementation (see the Appendix).

B. Natural construction

In what follows, we shall briefly prove the following.

Proposition 1. For any Hermitian linear map $\Lambda:\mathcal{B}(\mathcal{C}^d)\to\mathcal{B}(\mathcal{C}^{d'})$, there exists a SPA $\tilde{\Lambda}$ defined by

$$\tilde{\Lambda}_a = \beta_a^{-1}(a\Lambda_I + \Lambda), \tag{7}$$

with the parameter $a \geqslant \lambda d$, where $\lambda = \max[0,-\lambda']$ and λ' is the minimal eigenvalue of the operator $[\mathbb{I}\otimes\Lambda](P_+)$. Here, $P_+ = |\Psi_+\rangle\langle\Psi_+|$ corresponds to the "isotropic" maximally entangled $d\otimes d$ state $|\Psi_+\rangle = (1/d)\Sigma_{i=1}^d |i\rangle|i\rangle$, $\beta_a \equiv \max_\varrho \mathrm{Tr}([a\Lambda_I + \Lambda](\varrho))$.

The above approximation follows immediately from the well-known fact that linear, Hermitian operation Λ is completely positive if and only if $[\mathbb{I}\otimes\Lambda](P_+)$ has non-negative spectrum.

C. Remarks on structural physical approximations of nonlinear maps

One might expect (as it was conjectured in first version of this paper) that there is no physical approximation of the map $\varrho^{\otimes n}\to\varrho^n$. However, quite recently, it has been shown that such operations are possible. Namely, in Ref. [21], the explicit construction of such a map for two qubits ($n=2$) has been provided. It happens that generalization of this result to many copies of multilevel system is possible. This issue will be considered elsewhere [22].

IV. OPTIMIZATION

Here, we shall give the optimal form of the SPA of given linear, Hermitian map as follows.

Proposition 2. The best SPA of linear, Hermitian map Λ is the approximation (7) with minimal a (i.e., $a=\lambda d$). It can be written in the form

$$\tilde{\Lambda}_{opt} = \frac{\lambda d}{\lambda d d' + \alpha_\Lambda}\Lambda_I + \frac{1}{\lambda d d' + \alpha_\Lambda}\Lambda, \tag{8}$$

with $\alpha_\Lambda = \max_\varrho \mathrm{Tr}[\Lambda(\varrho)]$.

Proof. Recall that, following Riesz theorem applied to Hilbert-Schmidt space, the linear function of quantum state denoted by δ is uniquely determined by some Hermitian operator D in the following way: $\delta(\varrho)=\mathrm{Tr}(D\varrho)$. Now, the complete positivity of $\tilde{\Lambda}$ results in condition $\delta(\varrho)\geqslant\gamma a$, with parameter $a\geqslant\lambda d$ defined as in proposition 1. where d is a dimension of the Hilbert space and $a\geqslant\lambda$ is defined as in proposition 2. Minimization of the rate δ/γ according to the condition (i) of the Definition 2 leads to constant δ that amounts to

$$\delta = \gamma\lambda d. \tag{9}$$

In this way, one gets the family of SPA maps $\tilde{\Lambda}_\gamma = \gamma(\lambda d\Lambda_I + \Lambda)$ depending on single parameter γ. On the other hand, according to the condition (ii) of Definition 2, to get the best SPA, we have to maximize the value $\mathrm{Tr}[\tilde{\Lambda}_\gamma(\varrho)]$ under the condition $\mathrm{Tr}[\tilde{\Lambda}_\gamma(\varrho)]\leqslant 1$. Since the latter must be satisfied for any ϱ, one gets $\gamma\leqslant 1/(\lambda dd' + \alpha_\Lambda)$ with $\alpha_\Lambda = \max_\varrho \mathrm{Tr}[\Lambda(\varrho)]$. This after maximization leads to optimal parameters: $\gamma = 1/(\lambda dd' + \alpha_\Lambda)$, $\delta = \lambda d/(\lambda dd' + \alpha_\Lambda)$. In this way, we have obtained the formula (8). To conclude the proof of the Proposition 2, it is enough to check that the map (8) coincides with Eq. (7) for $a=\lambda d$.

Interpretation and example

There is a simple interpretation of the optimal formula (8) if $\alpha_\Lambda > 0$. To get the best SPA $\tilde{\Lambda}$, one needs the following two operations: (i) rescale the given linear, Hermitian Λ by taking $\Lambda' = \alpha_\Lambda^{-1}\Lambda$, which already satisfies $\mathrm{Tr}[\Lambda'(\varrho)]\leqslant 1$, (ii) take the following convex combination:

$$\tilde{\Lambda}_{opt} = p_* \Lambda_{dep} + (1-p_*)\Lambda', \tag{10}$$

with probability $p_* = \lambda dd'\alpha_\Lambda^{-1}/(\lambda dd'\alpha_\Lambda^{-1}+1)$ and $\lambda = \max[0,-\lambda']$, where λ' is the minimal eigenvalue of the operator $[\mathbb{I}\otimes\Lambda](P_+)$. In the above formula, we have also used the usual depolarizing channel

$$\Lambda_{dep}(\varrho) = I/d' \tag{11}$$

that turns any state on $\mathcal{C}^d$ into a maximally mixed state on $\mathcal{C}^{d'}$. Note that $\Lambda_{dep} = \Lambda_I/d'$.

In the case of trace-preserving maps (or, in general, for all maps that satisfy $\alpha_\Lambda = 1$), the above protocol gives $\Lambda' = \Lambda$ so the only step (ii) above is important. This means that in these cases, to get the optimal SPA, one has to construct a probabilistic mixture of Λ with depolarizing channel. For very efficient applications of such maps in domain of physical detection of quantum entanglement, see Refs. [13,14].

Example. Consider the best SPA of the transposition map $T:\mathcal{B}(\mathcal{C}^d)\to\mathcal{B}(\mathcal{C}^d)$ that transposes the matrix $[T(A)]_{mn} = A_{nm}$. Applying the prescription above for $\Lambda = T$, we get $[\mathbb{I}\otimes T](P_+) = 1/dV$ where V is a "flip" or "swap" operator [23]. As V has spectrum ± 1, this gives $\lambda = 1/d$, $\alpha_\Lambda = 1$ that results in the optimal parameter $p_* = d/(d+1)$. In this way, we get trace-preserving SPA, which is the following quantum channel:

$$\tilde{T}(\varrho) = \frac{d}{d+1}\frac{I}{d} + \frac{1}{d+1}\varrho^T \tag{12}$$

or $\widetilde{T}=[d/(d+1)](I/d)+[1/(d+1)]T$. It has already been introduced as a byproduct of optimal quantum cloning machines [6–8] and is equivalent to one-qubit universal quantum NOT gate [9].

Another important example of the SPA is the approximation of partial transposition $\mathbb{I}\otimes T$ that found application in detection [13] and estimation [14] of quantum entanglement.

V. TSALLIS ENTROPY AS "MULTICOPY" OBSERVABLE AND MULTICOPY ENTANGLEMENT WITNESSES

A. Multicopy observables

We propose to extend the notion of quantum observable A to *n-copy observable* $A^{(n)}$. Suppose that the system state is defined on the Hilbert space $\mathcal{H}$. Then, measurement of A performed on ϱ leads to the mean value $\langle A\rangle\equiv\mathrm{Tr}(A\varrho)$.

Definition 3. Let $A^{(n)}$ be the Hermitian operator on $\mathcal{H}^{\otimes n}$. We interpret it as n-copy observable with respect to the single system defined on a single Hilbert space $\mathcal{H}$ by defining "mean value" of $A^{(n)}$ on ϱ as

$$\langle\langle A^{(n)}\rangle\rangle_{\varrho}=\langle A^{(n)}\rangle_{\varrho\otimes\cdots\otimes\varrho}\equiv\mathrm{Tr}(A^{(n)}\varrho^{\otimes n}). \qquad (13)$$

.

Remark. The above concept of multicopy observables involves an interaction between many copies of the system. One can explain the concept with the help of the following physical example. Suppose that we are given an ensemble of many copies of the state ϱ of single spin-$\frac{1}{2}$ particle. One can consider an observable that measures z coordinate of global spin of each three particles of the ensemble *treated as a new joint spin-*$\frac{3}{2}$ *system*. Such an observable is just multicopy (three-copy) observable. It involves an interaction since evidently it is not a product of three spin-$\frac{1}{2}$ observables. Hence, after measurement of the observable, the three-particle system will, in general, remain in entangled state.

Below, we shall consider examples of multicopy observables that are important for further analysis.

Example 1. "Swap" observable. Consider the swap operator [8] on two-system space, which has the property $V|\Phi\rangle\otimes|\Psi\rangle=|\Psi\rangle\otimes|\Phi\rangle$ for any Φ, $\Psi\in\mathcal{C}^d$. It is known [8] that V is hermitian and satisfies $\mathrm{Tr}(VA\otimes B)=\mathrm{Tr}(AB)$. In particular, its mean value on the state $\varrho\otimes\varrho$ is

$$\mathrm{Tr}(V\varrho\otimes\varrho)=\mathrm{Tr}(\varrho^2). \qquad (14)$$

Thus, swap V operator can be viewed as 2-*copy* observable: the above formula (14) leads to the conclusion that the value $\mathrm{Tr}(\varrho^2)$ is measurable if joint measurements on two copies of the state are allowed. From that, we get immediately that the Tsallis entropy $S_q^T(\varrho)=[1-\mathrm{Tr}(\varrho^q)]/(q-1)$ is measurable for $q=2$. Indeed, we take the observable $W\equiv I-V$ and $S_2^T(\varrho)=\langle W\rangle_{\varrho\otimes\varrho}=\mathrm{Tr}(W\varrho\otimes\varrho)$.

Example 2. "Shift" operation. Tsallis entropies as induced multicopy observables. Consider the well-known natural generalization of the swap. This is a shift operation V_n, which can be interpreted as some cyclic permutation. It is defined as $V^{(n)}u_1\otimes u_2\otimes\cdots\otimes u_n=u_2\otimes\cdots\otimes u_n\otimes u_1$. It is known that (see Ref. [24]) $\mathrm{Tr}(V^{(n)}A_1\otimes\cdots\otimes A_n)=\mathrm{Tr}(A_1\cdots A_n)$. Unfortunately, it is, in general, not Hermitian (which can be checked directly looking at its decomposition into swaps $V=V^{(2)}$). However, note that value $\mathrm{Tr}(X\sigma)$ for *any* operator X and state σ can be experimentally checked by measuring Hermitian [defined as $X_h\equiv\frac{1}{2}(X+X^\dagger)$] and anti-Hermitian [defined as $X_a\equiv(-i/2)(X-X^\dagger)$] part of X because of elementary observation $\langle X\rangle=\mathrm{Tr}(X\sigma)=\mathrm{Tr}(X_h\sigma)+i\mathrm{Tr}(X_a\sigma)=\langle X_h\rangle+i\langle X_a\rangle$. Thus, one can determine mean value $\mathrm{Tr}(X\sigma)$ of non-Hermitian operator X by an experimental measurement of only *two* Hermitian observables.

The interesting case is when we know *a priori* that the resulting value $\mathrm{Tr}(V^{(n)}\sigma)$ is real. Then, there is the following simple observation (cf. Refs. [13,20]).

Observation. Consider an arbitrary (may be non-Hermitian) operator X. If the value $\langle X\rangle=\mathrm{Tr}(X\sigma)$ is real, then it coincides with the usual mean value of observable defined by Hermitian part of X.

This leads to the interesting consequences. Namely, application of the above observation to the moments $\mathrm{Tr}(\varrho^n)=\mathrm{Tr}(V^{(n)}\varrho^{\otimes n})$, which are *real*, leads to an immediate conclusion that each of them can be measured as the mean value of n-copy observable

$$W^{(n)}=[V^{(n)}+(V^{(n)})^\dagger]/2. \qquad (15)$$

Immediately, all Tsallis entropies $\widetilde{S}_q(\varrho)=[1-\mathrm{Tr}(\varrho^q)]/(q-1)$ with positive integer q can be measured as the mean value of *single* multicopy observable

$$\widetilde{W}^{(q)}=(I-W^{(q)})/(1-q). \qquad (16)$$

For another implications, see remarks on spectrum state estimation (Sec. VI).

Estimation of mean values of multicopy observables. The above observables have, in general, complicated structure. However, their mean values can be estimated via measurement on single ancilla coupled to the system. This may be done either in a way proposed recently in Ref. [20] with the help of the encoding observable into the ancilla (this requires ancilla that is bigger than the system, but allows for elegant interferometric implementation) or via binary measurement on a single spinlike ancilla coupled to the system [25]. The latter method works also for any bounded continuous variable observables. It is interesting that in the case of multicopy observable (15), the corresponding mean value on $\varrho^{\otimes n}$ can be measured interferometrically directly using controlled unitary operation $V^{(n)}$ [20].

B. Multicopy entanglement witnesses

Separability inequalities in terms of entropic inequalities were first investigated in Ref. [26]. Following the Renyi entropy analysis, we know that any separable state satisfies the entropy inequality (see Refs. [26,15,29,16,17])

$$S_\alpha(\varrho_{AB})-S_\alpha(\varrho_X)\geqslant 0, X=A,B. \qquad (17)$$

This is equivalent to the Tsallis entropy inequalities

$$\widetilde{S}_q(\varrho_{AB})-\widetilde{S}_q(\varrho_X)\geqslant 0, X=A,B. \qquad (18)$$

All these inequalities have been shown to be satisfied by bipartite separable states (see Ref. [17]). As such these form a necessary condition for separability of quantum states. Consider the bipartite state ϱ defined on $\mathcal{H}_{AB}=\mathcal{H}_A\otimes\mathcal{H}_B$. We shall define the q -observable on the system of q copies of the state: $\varrho_{AB}^{\otimes q}$. Let $\widetilde{W}^{(q)}$ be just the observable (15) acting on $\mathcal{H}_{AB}^{\otimes q}$. Let us recall that this is the multicopy observable that corresponds to entropy $\widetilde{S}_q$. Let also $\widetilde{W}_X^{(q)}$ stand for counterpart of $\widetilde{W}^{(q)}$ but acting on $\mathcal{H}_X^{\otimes q}$, $X=A,B$. Finally, let

$$R_A^{(q)}=\widetilde{W}^{(q)}-I_A\otimes\widetilde{W}_B^{(q)},$$

$$R_B^{(q)}=\widetilde{W}^{(q)}-\widetilde{W}_A^{(q)}\otimes I_B, \qquad (19)$$

where I_X corresponds to identity matrix on $\mathcal{H}_X^{\otimes q}$, $X=A,B$.

Evidently, mean value $\langle\langle R_X^{(q)}\rangle\rangle_\varrho\equiv\langle R_X^{(q)}\rangle_{\varrho^{\otimes q}}$ is positive if and only if the inequalities (18) and (17) are satisfied for $X=A,B$. Thus, because $\langle\langle R_X^{(q)}\rangle\rangle_\varrho$ is (i) positive for all separable states ϱ (ii) negative for some entangled states ϱ [those that violate inequilities (18) and (17)], we propose to call them *multicopy entanglement witnesses*. In general, it is likely that *multicopy observables* like the above entanglement witnesses can be useful not only in quantum information theory but also in quantum domain, in general. Application of such witnesses is expected when technology allows for precise control of physical interactions among many quantum systems. More formally, we can introduce the following.

Definition 4. An n-copy entanglement witness is an n-copy observable $R^{(n)}$ that satisfies $\langle\langle R^{(n)}\rangle\rangle_\varrho\geqslant 0$ for all separable ϱ, but $\langle\langle R^{(n)}\rangle\rangle_{\varrho_{ent}}<0$ holds for some entangled state ϱ_{ent}.

C. Remarks on state spectrum estimation

It is remarkable that if one wants to determine spectrum $\{p_i\}$ of the state ϱ defined on m-dimensional Hilbert space, then it is sufficient to know $m-1$ values $w_2=\mathrm{Tr}(\varrho^2)$, $w_3=\mathrm{Tr}(\varrho^3)$, . . . ,$w_m=\mathrm{Tr}(\varrho^m)$. This can be seen by realizing that $\mathrm{Tr}(\varrho^k)=(p_1)^k+(p_2)^k+\cdots+(p_m)^k$. Thus, since a finite discrete random variable is determined by its $m-1$ moments, all p_i's can be uniquely determined from the set of given values $\{w_i\}$. But according to Sec. IV C, these happen to be mean values of observables (15), i.e., $w_i=\langle\langle W^{(i)}\rangle\rangle_\varrho\equiv\langle W^{(i)}\rangle_{\varrho^{\otimes i}}$. Thus, we have the following.[2]

Conclusion. In order to determine the spectrum of (completely unknown) state ϱ, it is enough to know mean values w_i of $m-1$ multicopy observables $W^{(k)}$ $(k=2, . . . ,m)$ that correspond to Hermitian parts of shift operations $V^{(k)}$.

Remark. The above conclusion is complementary to what is known in the literature so far. We have had either (i) to perform full tomography: estimation of mean values of m^2 observables, each *via* single copy statistics, or (ii) an asymptotic estimation of Young frames requiring estimation of sequence of n-copy observables with n approaching infinity. The method (ii), has an elegant mathematical background, was shown to be optimal (see Ref. [19]). It involves effectively only m output parameters (because it needs estimation of probabilities of m results of output observable), but requires possibility of collective measurements on arbitrary number of copies to get good accuracy. Quantum tomography (i) requires only single copy per measurement, but one needs m^2-1 observables. In the present approach, we see that measurement of only $m-1$ observables is needed if collective measurements on *finite* numbers of 2,3, . . . ,m copies are possible. In this sense, it represents some kind of compromise between the previous two methods (i) and (ii).

It is quite remarkable that one can design powerful interferometric schemes [13,20] that allow to detect means of those $m-1$ observables in a natural way as a result of interaction with controlled bit ancilla (see also discussion in Sec. V A).

VI. CONCLUSION

We have considered possibility of transformation of getting nth power of the state ϱ provided that operations on n copies of ϱ are allowed. We have shown that it is impossible. We have analyzed the possibility of construction of the structural physical approximation (SPA) of the unphysical map under the condition of preserving (in a well-defined way) the structure coming from the map. We have shown that it is possible to approximate any linear, Hermitian map in such a way. We have also optimized the SPA any nontrivial Hermitian map. On the other hand, it has been pointed out that if joint measurements on n copies of given state ϱ are allowed, then the nonlinear function of the state defined by $\mathrm{Tr}(\varrho^n)$ can be easily measurable. This leads to the notion of multicopy observable with remarkable examples of measurement of Tsallis entropies.

Further, we have introduced the notion of multicopy entanglement witnesses. The examples of the latter have been provided, measuring the degree of violation of separability criterion based on entropic inequalities. Finally, the existence of simple method of spectrum estimation for the unknown state has been pointed out, which requires collective measurements on small number of copies. The number of needed estimated parameters is $2\dim\mathcal{H}-3$, which is less than $(\dim\mathcal{H})^2-1$ required in tomography. One can hope that the multicopy observables idea together with direct physical interpretation of Tsallis entropy in context of multicopy observables can be useful not only for the quantum entanglement theory but also for the quantum information in general.

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[2]The conclusion is an improvement (based on technique from Ref. [13]) of the one made in previous version of the present paper.

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APPENDIX

Here according to a well-known quantum mechanical procedure, known form is Kraus (see for instance, Ref. [27]), we shall recall how any completely positive map Λ satisfying

$$\mathrm{Tr}[\Lambda(\varrho)]\leqslant 1 \quad (A1)$$

can be probabilistically implemented in laboratory. We know that $\Lambda(\cdot)=\Sigma_{i=1}^{k}V_i(\cdot)V_i^{\dagger}$. From Eq. (A1), remembering that $\mathrm{Tr}[\Sigma_{i=1}^{k}V_i\varrho V_i^{\dagger}]=\mathrm{Tr}[\Sigma_{i=1}^{k}V_i^{\dagger}V_i\varrho]$, we get that the positive operator $A_0=\Sigma_{i=1}^{k}V_i^{\dagger}V_i$ has the spectrum form in interval $[0,1]$. Thus we can define $V_0=\sqrt{I-A_0}$ and then, the extended completely positive map $\Lambda'(\cdot)=\Lambda(\cdot)+V_0(\cdot)V^{\dagger}=\Sigma_{i=0}^{k}V_i(\cdot)V_i^{\dagger}$ is trace preserving because $\mathrm{Tr}[\Lambda(\varrho)]=\mathrm{Tr}[\Sigma_{i=0}^{k}V_i\varrho V_i^{\dagger}]=\mathrm{Tr}[\Sigma_{i=0}^{k}V_i^{\dagger}V_i\varrho]=\mathrm{Tr}[(A_0+I-A_0\varrho]=\mathrm{Tr}(I\varrho)=\mathrm{Tr}(\varrho)=1$. But any trace-preserving map can be implemented in lab by the interaction with some additional quantum system (ancilla) and some von-Neumann measurement on this system with outputs $i=0,1,\ldots,k$ (for description, see Ref. [27]). In this case, the ith "event" corresponds to the single map $V_iV_i^{\dagger}$. It can be interpreted as "producing" unnormalized state $V_i\varrho V_i^{\dagger}$. Indeed, though its action results in normalized state $\varrho_i=V_i\varrho V_i^{\dagger}/p_i$ this occurs only with probability $p_i=\mathrm{Tr}(V_i\varrho V_i^{\dagger})$. In this sense, the original map Λ can be implemented: we apply some special von-Neumann measurement on the ancilla and keep the system if only the ith event with $i\neq 0$ occurs. If the singled-out event corresponding to $i=0$ occurs, we "discard" our system. This gives the new state $\varrho'=\Lambda(\varrho)/p$ with probability $p=\mathrm{Tr}[\Lambda(\varrho)]$.

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