GENERAL DYNAMIC PROJECTING OF MAXWELL EQUATIONS

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Abstract: A complete – system of Maxwell equations is splitting into independent subsystems by means of a special dynamic projecting technique. The technique relies upon a direct link between field components that determine correspondent subspaces. The explicit form of links and corresponding subspace evolution equations are obtained in conditions of certain symmetry, it is illustrated by examples of spherical and quasi-one-dimensional waves.

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1. Introduction

The main obstacle in the modeling of electromagnetic phenomena is the multicomponent nature of the basic Maxwell system. A solution of the problem of evolution of a system with a multicomponent state includes a classification of basic states as eigenstates of the evolution operator or modes [1]. There are works that investigate the mode content on the basis of the following algorithm, applicable in the case of problems that may be formulated as a system of differential equations with constant coefficients (see *e.g.* [1–3]):

1. Each component of the state vector ψ is subjected to the Fourier transformation

$$\psi \to F\psi \tag{1}$$

2. The evolution operator L is then transformed as

$$L \to L = FLF^{-1} \tag{2}$$

resulting in a matrix, dependent on $\vec{k}$.

3. The eigenvectors of the matrix $L(\vec{k})$ form the matrix $\Psi(\vec{k})$. The fact that the eigenproblem is homogeneous, allows choosing one of components as 1.
4. This matrix defines the projecting operators [4]

$$(\tilde{P}^s)_{ij} = \Psi_{is}\Psi^{-1}_{sj} \tag{3}$$

5. Its Fourier transforms, named dynamic projecting operators allow splitting the evolution problem and the space of the initial condition that defines evolution in each mode subspace [4].

The algorithm is transparent and effective in a case of low dimensions for a wide class of Cauchy problems at infinite space. Even in a two- or three-dimensional cases it leads to the appearance of complicated integral operators as projectors of matrix elements. A large number of vector components also include a cumbersome technique or approximations when the eigenvalues are evaluated approximately. Special efforts are necessary if the boundary conditions are taken into account as, for example in waveguides [1].

An alternative idea to carry out such splitting refers to direct manipulation with the evolution operator without application of Fourier transformations [5–7].

Such studies are continued in this paper, considering the important example of a full system of Maxwell equations. A complete system of Maxwell equations is splitting into independent subsystems by means of the dynamic projecting technique. The above-mentioned formalism, based on Fourier transformations, is used at the initial stage, however, the operator relations are used at the next step. Finally, the technique relies upon direct connections between field components that determine correspondent subspaces. The links are effectively used in conditions of some symmetry, it is illustrated by examples of spherical and quasi-one-dimensional waves.

The first section of the paper contains a complete system of Maxwell equations and the matrix form of equations, which will be further considered. We also define the linear operator $\boldsymbol{L}$, which contains spatial derivatives. In Section 2, we determine the projecting operators for the operator $\boldsymbol{L}$ and also the operator's eigenvalues and eigenvectors. Afterwards, we derive equations resulting from applying the projector operators on the matrix form of the considered portion of Maxwell equations. In Section 3 we use the projection technique for the Maxwell equations with the assumption of linear dependence of electric induction on the electric field and the magnetic induction on the magnetic field. Section 4 contains examples of equations and their solutions obtained by means of application of projecting operators.

2. Basic equations and starting points

Our starting point are Maxwell's equations for a non-magnetic medium in the Lorentz-Heaviside unit system

$$\frac{1}{c}\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} = \vec{0} \tag{4}$$

$$\frac{1}{c}\frac{\partial\vec{D}}{\partial t} - \vec{\nabla}\times\vec{H} = -\frac{4\pi}{c}\vec{J}_f \tag{5}$$

$$\vec{\nabla}\vec{B} = 0 \tag{6}$$

$$\vec{\nabla}\vec{D} = 4\pi\rho_f \tag{7}$$

where

$$\vec{B} = \vec{H} + 4\pi\vec{M} \tag{8}$$

$$\vec{D} = \vec{E} + 4\pi\vec{P} \tag{9}$$

Using following form of operator $\boldsymbol{L}$:

$$\boldsymbol{L} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} = \\ 0 & 0 & 0 & \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ 0 & 0 & 0 & -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & -\frac{\partial}{\partial y} & 0 & 0 & 0 \\ -\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 \\ \frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

the system of equations (1,2) can be written in the matrix notation:

$$\frac{1}{c}\frac{\partial\phi}{\partial t} + \boldsymbol{L}\psi = -\frac{4\pi}{c}\vec{J}_{ex} \tag{11}$$

where

$$\phi = \begin{pmatrix} B_x \\ B_y \\ B_z \\ D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \vec{B} \\ \vec{D} \end{pmatrix} \qquad \psi = \begin{pmatrix} H_x \\ H_y \\ H_z \\ E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} \vec{H} \\ \vec{E} \end{pmatrix} \qquad \vec{J}_{ex} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ J_{f,x} \\ J_{f,y} \\ J_{f,z} \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{J}_f \end{pmatrix} \tag{12}$$

For given operator $\boldsymbol{L}$ we can determine a system of operators that has the following properties:

$$\begin{gathered} \boldsymbol{P}_i\boldsymbol{P}_j = \delta_{ij}\boldsymbol{P}_i \\ \sum_i \boldsymbol{P}_i = \boldsymbol{I} \end{gathered} \tag{13}$$

where $i = 1,...,n$, n is the dimension of matrix $\boldsymbol{L}$ and the dimensions of operators $\boldsymbol{P}_i$ and $\boldsymbol{L}$ are equal. The operators that fulfill these properties define the projector operators $\boldsymbol{P}_i$.

3. Determination of operator eigenvalues, eigenvectors and projecting operators for a full system of Maxwell's equations

Let us determine projection operators for $\boldsymbol{L}$. For this purpose we use the spatial Fourier transform (and the inverse Fourier transform). We start our studies with representing all perturbations as a sum of planar waves:

$$f(\vec{r},t) = \int_{R^3} \tilde{f}(\vec{k},t)\exp(-i\vec{k}\cdot\vec{r})d\vec{k} \tag{14}$$

$\tilde{f}(\vec{k},t)$ denotes the Fourier-transforms of $f(\vec{r},t)$, $\tilde{f}(\vec{k},t) = \frac{1}{(2\pi)^3}\int_{R^3} f(\vec{r},t)e^{i\vec{k}\cdot\vec{r}}d\vec{r}$ and $\vec{r} = (x,y,z)$, $\vec{k} = (k_x,k_y,k_z)$. Operator eigenvalues of operator $\boldsymbol{L}$ take the following form

$$\begin{aligned}\lambda_{1,2} &= 0\\ \lambda_{3,4} &= \sqrt{\Delta}\\ \lambda_{5,6} &= -\sqrt{\Delta}\end{aligned} \tag{15}$$

where the designation $\sqrt{\Delta}$ (the square root of the Laplacian) is the integral operator, which corresponds to $-i\sqrt{k_x^2+k_y^2+k_z^2}$ in the space of Fourier-transforms. In the quasi-one-dimensional geometry, for example, when the electromagnetic wave propagates along axis OX, if components of $\vec{k}$ satisfy the condition $k_x^2 \gg k_y^2+k_z^2$, we can interpret the operator $\sqrt{\Delta}$ in the following way:

$$\sqrt{\Delta} \approx \partial/\partial x + 0.5\epsilon\Delta_\perp \int dx \tag{16}$$

where $\Delta_\perp = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and $\epsilon = \frac{k_y^2+k_z^2}{k_x^2} << 1$ is the diffraction parameter. Or, when we know that the considered function f has a spherical symmetry with respect to point $r_0 = (x_0,y_0,z_0)$ and depends only on the radial coordinate, then $\sqrt{\Delta}f = \frac{1}{r}\frac{\partial}{\partial r}(rf)$, $r = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$.

To simplify the notation of projection operators let us introduce the notation:

$$\boldsymbol{P}_d = \frac{1}{\Delta}\begin{pmatrix}\frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x\partial y} & \frac{\partial^2}{\partial x\partial z}\\ \frac{\partial^2}{\partial y\partial x} & \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial y\partial z}\\ \frac{\partial^2}{\partial z\partial x} & \frac{\partial^2}{\partial z\partial y} & \frac{\partial^2}{\partial z^2}\end{pmatrix} \tag{17}$$

$$\boldsymbol{P}_r = \frac{1}{\sqrt{\Delta}}\begin{pmatrix}0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x}\\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0\end{pmatrix} \tag{18}$$

$$\mathbf{0} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} \tag{19}$$

With such markings, the projection operators $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ take the form:

$$\boldsymbol{P}_1 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P}_d \end{pmatrix} \tag{20}$$

$$\boldsymbol{P}_2 = \begin{pmatrix} \boldsymbol{P}_d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{21}$$

Every two operators $\boldsymbol{P}_3$, $\boldsymbol{P}_4$ and $\boldsymbol{P}_5$, $\boldsymbol{P}_6$ generate a two-dimensional subspace, in this connection, their appearance depends on the choice of eigenvectors within their subspace, but their sum ($\boldsymbol{P}_3+\boldsymbol{P}_4$ and $\boldsymbol{P}_5+\boldsymbol{P}_6$) will always take the same form:

$$\boldsymbol{P}_+ = \boldsymbol{P}_3+\boldsymbol{P}_4 = \frac{1}{2}\begin{pmatrix} -\boldsymbol{P}_r^2 & \boldsymbol{P}_r \\ -\boldsymbol{P}_r & -\boldsymbol{P}_r^2 \end{pmatrix} \tag{22}$$

$$\boldsymbol{P}_- = \boldsymbol{P}_5+\boldsymbol{P}_6 = \frac{1}{2}\begin{pmatrix} -\boldsymbol{P}_r^2 & -\boldsymbol{P}_r \\ \boldsymbol{P}_r & -\boldsymbol{P}_r^2 \end{pmatrix} \tag{23}$$

The form of eigenvectors is obtained from the equality:

$$\boldsymbol{P}_i\psi = \psi_i = \begin{pmatrix} \vec{H}_i \\ \vec{E}_i \end{pmatrix} \tag{24}$$

Let us use the notation:

$$\boldsymbol{P}_i\phi = \phi_i = \begin{pmatrix} \vec{B}_i \\ \vec{D}_i \end{pmatrix} \tag{25}$$

Indices $i=1,2,3,4,5,6$ at individual vectors will mean vectors, which have been obtained after applying projection operators $\boldsymbol{P}_i$.

3.1. Projection with operator $\boldsymbol{P}_1$

Applying operator $\boldsymbol{P}_1$ on the system (11) and using the equation (7) after simplification we get the equation of continuity:

$$\frac{\partial\rho_f}{\partial t} = -\vec{\nabla}\vec{J}_f \tag{26}$$

The eigenvector will have the form:

$$\psi_1 = \begin{pmatrix} H_{x,1} \\ H_{y,1} \\ H_{z,1} \\ E_{x,1} \\ E_{y,1} \\ E_{z,1} \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}\vec{\nabla}\vec{E} \tag{27}$$

And the following equality will be fulfilled for vector ϕ_1:

$$\phi_1 = \begin{pmatrix} B_{x,1} \\ B_{y,1} \\ B_{z,1} \\ D_{x,1} \\ D_{y,1} \\ D_{z,1} \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \vec{\nabla}\vec{D} \tag{28}$$

Other properties:

$$\begin{aligned} \vec{\nabla}\vec{E}_1 &= \vec{\nabla}\vec{E} \\ \vec{\nabla}\vec{D} = \vec{\nabla}\vec{D}_1 &= \vec{\nabla}\vec{E}_1 + 4\pi\vec{\nabla}\vec{P}_1 \\ \vec{\nabla}\vec{J}_f &= \vec{\nabla}\vec{J}_{f,1} \\ \vec{B}_1 = \vec{H}_1 &= \vec{0} \\ \vec{\nabla}\times\vec{E}_1 &= \vec{0} \\ \vec{\nabla}\times\vec{D}_1 &= \vec{0} \end{aligned} \tag{29}$$

where $\vec{D}_1 = \boldsymbol{P}_d\vec{D}$ and $\vec{E}_1 = \boldsymbol{P}_d\vec{E}$.

3.2. Projection with operator $\boldsymbol{P}_2$

Applying operator $\boldsymbol{P}_2$ on system (11) we obtain the equality:

$$\frac{\partial\vec{B}_2}{\partial t} = \vec{0} \tag{30}$$

therefore the magnetic induction vector $\vec{B}_2$ does not change over time, where $\vec{B}_2 = \boldsymbol{P}_d\vec{B}$ and $\vec{H}_2 = \boldsymbol{P}_d\vec{H}$. Taking into account equation (6) we get the equation of identity. Eigenvector ψ_2 will have the form:

$$\psi_2 = \begin{pmatrix} H_{x,2} \\ H_{y,2} \\ H_{z,2} \\ E_{x,2} \\ E_{y,2} \\ E_{z,2} \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ 0 \\ 0 \\ 0 \end{pmatrix} \vec{\nabla}\vec{H} \tag{31}$$

And for vector ϕ_2 we will have that

$$\phi_2 = \begin{pmatrix} \vec{B}_2 \\ \vec{D}_2 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} \vec{\nabla} \\ \vec{0} \end{pmatrix} \vec{\nabla}\vec{B} \tag{32}$$

Other properties:

$$
\begin{gathered}
\vec{\nabla}\vec{H}_2=\vec{\nabla}\vec{H}\\
\vec{\nabla}\vec{B}=\vec{\nabla}\vec{B}_2=\vec{\nabla}\vec{H}_2+4\pi\vec{\nabla}\vec{M}_2\\
\vec{D}_2=\vec{E}_2=\vec{J}_{f,2}=\vec{0}\\
\vec{\nabla}\times\vec{H}_2=\vec{0}\\
\vec{\nabla}\times\vec{B}_2=\vec{0}
\end{gathered}
\tag{33}
$$

3.3. Results for other projector operators

We cannot give the form of other projector operators until we specify how eigenvectors unbutton adequate subspaces. Nonetheless, not knowing the form of eigenvectors, we can specify the general properties of the subspace generated by operators $\boldsymbol{P}_+$ and $\boldsymbol{P}_-$. We can choose such eigenvectors and adequate operators $\boldsymbol{P}_3$, $\boldsymbol{P}_4$, $\boldsymbol{P}_5$ and $\boldsymbol{P}_6$, which will generate subspaces with the same properties as operators $\boldsymbol{P}_+$ and $\boldsymbol{P}_-$.

For any choice of eigenvectors $i=3,4,5,6$ will be fulfilled by the following dependences:

$$\vec{\nabla}\times\vec{H}_i=-\lambda_i\vec{E}_i \tag{34}$$

$$\vec{\nabla}\times\vec{E}_i=\lambda_i\vec{H}_i \tag{35}$$

It follows from these equalities that:

$$\Delta\vec{H}_i=-\vec{\nabla}\times(\vec{\nabla}\times\vec{H}_i)(36)$$

$$\Delta\vec{E}_i=-\vec{\nabla}\times(\vec{\nabla}\times\vec{E}_i)(37)$$

therefore

$$\vec{\nabla}(\vec{\nabla}\vec{H}_i)=\vec{0} \text{ we can choose such } \vec{H}_i \text{ for which } \vec{\nabla}\vec{H}_i=0 \tag{38}$$

and

$$\vec{\nabla}(\vec{\nabla}\vec{E}_i)=\vec{0} \text{ we can choose such } \vec{E}_i \text{ for which } \vec{\nabla}\vec{E}_i=0 \tag{39}$$

In addition:

$$
\begin{gathered}
\vec{\nabla}\vec{B}_+=\vec{\nabla}(\vec{B}_3+\vec{B}_4)=0\\
\vec{\nabla}\vec{B}_-=\vec{\nabla}(\vec{B}_5+\vec{B}_6)=0\\
\vec{\nabla}\vec{H}_+=\vec{\nabla}(\vec{H}_3+\vec{H}_4)=0\\
\vec{\nabla}\vec{H}_-=\vec{\nabla}(\vec{H}_5+\vec{H}_6)=0\\
\vec{\nabla}\vec{M}_+=\vec{\nabla}(\vec{M}_3+\vec{M}_4)=0\\
\vec{\nabla}\vec{M}_-=\vec{\nabla}(\vec{M}_5+\vec{M}_6)=0
\end{gathered}
\tag{40}
$$

$$
\begin{gathered}
\vec{\nabla}\vec{D}_+=\vec{\nabla}(\vec{D}_3+\vec{D}_4)=0\\
\vec{\nabla}\vec{D}_-=\vec{\nabla}(\vec{D}_5+\vec{D}_6)=0\\
\vec{\nabla}\vec{E}_+=\vec{\nabla}(\vec{E}_3+\vec{E}_4)=0\\
\vec{\nabla}\vec{E}_-=\vec{\nabla}(\vec{E}_5+\vec{E}_6)=0\\
\vec{\nabla}\vec{P}_+=\vec{\nabla}(\vec{P}_3+\vec{P}_4)=0\\
\vec{\nabla}\vec{P}_-=\vec{\nabla}(\vec{P}_5+\vec{P}_6)=0
\end{gathered}
\tag{41}
$$

Let us introduce the notations:

$$\begin{aligned}
\lambda_+ &= \lambda_3 = \lambda_4 \\
\lambda_- &= \lambda_5 = \lambda_6 \\
\psi_+ &= \psi_3 + \psi_4 \\
\psi_- &= \psi_5 + \psi_6 \\
\phi_+ &= \phi_3 + \phi_4 \\
\phi_- &= \phi_5 + \phi_6
\end{aligned} \tag{42}$$

Then, applying operator $\boldsymbol{P}_+$ on the system (11) we will obtain:

$$\frac{\partial \phi_+}{\partial t} + c\lambda_+\psi_+ = -4\pi \boldsymbol{P}_+ \left(\frac{\partial}{\partial t} \begin{pmatrix} \vec{M} \\ \vec{P} \end{pmatrix} + \vec{J}_{ex} \right) = -4\pi \left(\frac{\partial}{\partial t} \begin{pmatrix} \vec{M}_+ \\ \vec{P}_+ \end{pmatrix} + \boldsymbol{P}_+ \vec{J}_{ex} \right) \tag{43}$$

$$\boldsymbol{P}_+ \vec{J}_{ex} = \frac{1}{2} \begin{pmatrix} \boldsymbol{P}_r \\ -\boldsymbol{P}_r^2 \end{pmatrix} \vec{J}_f \tag{44}$$

Similarly, applying operator $\boldsymbol{P}_-$ on the system (11) we obtain:

$$\frac{\partial \phi_-}{\partial t} + c\lambda_-\psi_- = -4\pi \boldsymbol{P}_- \left(\frac{\partial}{\partial t} \begin{pmatrix} \vec{M} \\ \vec{P} \end{pmatrix} + \vec{J}_{ex} \right) = -4\pi \left(\frac{\partial}{\partial t} \begin{pmatrix} \vec{M}_- \\ \vec{P}_- \end{pmatrix} + \boldsymbol{P}_- \vec{J}_{ex} \right) \tag{45}$$

$$\boldsymbol{P}_- \vec{J}_{ex} = \frac{1}{2} \begin{pmatrix} -\boldsymbol{P}_r \\ -\boldsymbol{P}_r^2 \end{pmatrix} \vec{J}_f \tag{46}$$

In the vacuum the above equation will take the form:

$$\frac{\partial \psi_+}{\partial t} + c\lambda_+\psi_+ = \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix} \tag{47}$$

$$\frac{\partial \psi_-}{\partial t} + c\lambda_-\psi_- = \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix} \tag{48}$$

4. The case of linear dependence of electromagnetic induction of the electric field and magnetic induction of the magnetic field

Consider the case where the relations between $\vec{B}$ and $\vec{H}$ and $\vec{D}$ and $\vec{E}$ are linear:

$$\vec{B} = \mu \vec{H} \tag{49}$$

$$\vec{D} = \epsilon \vec{E} \tag{50}$$

then, we can write the right side of equations (4), (5) using the vector:

$$\tilde{\psi} = \begin{pmatrix} B_x \\ B_y \\ B_z \\ E_x \\ E_y \\ E_z \end{pmatrix} \tag{51}$$

For operator $\widetilde{\boldsymbol{L}}$:

$$\widetilde{\boldsymbol{L}} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & 0 & 0 & \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ 0 & 0 & 0 & -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{1}{\epsilon\mu}\frac{\partial}{\partial z} & -\frac{1}{\epsilon\mu}\frac{\partial}{\partial y} & 0 & 0 & 0 \\ -\frac{1}{\epsilon\mu}\frac{\partial}{\partial z} & 0 & \frac{1}{\epsilon\mu}\frac{\partial}{\partial x} & 0 & 0 & 0 \\ \frac{1}{\epsilon\mu}\frac{\partial}{\partial y} & -\frac{1}{\epsilon\mu}\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \end{pmatrix} \tag{52}$$

the system of equations (4), (5) can be written in the matrix form:

$$\frac{1}{c}\frac{\partial\tilde{\psi}}{\partial t} + \widetilde{\boldsymbol{L}}\tilde{\psi} = -\frac{4\pi}{c\epsilon}\vec{J}_{ex} \tag{53}$$

4.1. Projection operators

In the considered case the operator eigenvalues of operator $\widetilde{\boldsymbol{L}}$ take the form:

$$\begin{aligned} \tilde{\lambda}_{1,2} &= 0 \\ \tilde{\lambda}_{3,4} &= \frac{1}{\sqrt{\epsilon\mu}}\sqrt{\Delta} \\ \tilde{\lambda}_{5,6} &= -\frac{1}{\sqrt{\epsilon\mu}}\sqrt{\Delta} \end{aligned} \tag{54}$$

With such designations the projector operators take the form:

$$\widetilde{\boldsymbol{P}}_1 = \boldsymbol{P}_1 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P}_d \end{pmatrix} \tag{55}$$

$$\widetilde{\boldsymbol{P}}_2 = \boldsymbol{P}_2 = \begin{pmatrix} \boldsymbol{P}_d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{56}$$

$$\widetilde{\boldsymbol{P}}_+ = \widetilde{\boldsymbol{P}}_3 + \widetilde{\boldsymbol{P}}_4 = \frac{1}{2}\begin{pmatrix} -\boldsymbol{P}_r^2 & \sqrt{\epsilon\mu}\boldsymbol{P}_r \\ -\frac{1}{\sqrt{\epsilon\mu}}\boldsymbol{P}_r & -\boldsymbol{P}_r^2 \end{pmatrix} \tag{57}$$

$$\widetilde{\boldsymbol{P}}_- = \widetilde{\boldsymbol{P}}_5 + \widetilde{\boldsymbol{P}}_6 = \frac{1}{2}\begin{pmatrix} -\boldsymbol{P}_r^2 & -\sqrt{\epsilon\mu}\boldsymbol{P}_r \\ \frac{1}{\sqrt{\epsilon\mu}}\boldsymbol{P}_r & -\boldsymbol{P}_r^2 \end{pmatrix} \tag{58}$$

All the results obtained for the operators $\widetilde{\boldsymbol{P}}_1$ and $\widetilde{\boldsymbol{P}}_2$ will be the same as those previously obtained for $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$.

For elements of eigenvectors generated by $\widetilde{\boldsymbol{P}}_i$ from vector $\tilde{\psi}$, where $i = 3,4,5,6$, the following relationships are fulfilled:

$$\begin{aligned} \vec{\nabla} \times \vec{\tilde{E}}_i &= \tilde{\lambda}_i \vec{\tilde{B}}_i \\ \vec{\nabla} \times \vec{\tilde{B}}_i &= -\epsilon\mu\tilde{\lambda}_i \vec{\tilde{E}}_i \end{aligned} \tag{59}$$

From these equalities it follows that:

$$\begin{aligned} \Delta\vec{\tilde{B}}_i &= -\vec{\nabla} \times (\vec{\nabla} \times \vec{\tilde{B}}_i) \\ \Delta\vec{\tilde{E}}_i &= -\vec{\nabla} \times (\vec{\nabla} \times \vec{\tilde{E}}_i) \end{aligned} \tag{60}$$

And for operators $\widetilde{\boldsymbol{P}}_+$ and $\widetilde{\boldsymbol{P}}_-$ we receive the following equations:

$$\frac{\partial\tilde{\psi}_+}{\partial t} + \frac{c}{\sqrt{\epsilon\mu}}\lambda_+\tilde{\psi}_+ = -2\pi\begin{pmatrix} \sqrt{\epsilon\mu}\boldsymbol{P}_r \\ -\boldsymbol{P}_r^2 \end{pmatrix}\frac{\vec{J}_f}{\epsilon} \tag{61}$$

$$\frac{\partial\tilde{\psi}_-}{\partial t} + \frac{c}{\sqrt{\epsilon\mu}}\lambda_-\tilde{\psi}_- = 2\pi\begin{pmatrix} \sqrt{\epsilon\mu}\boldsymbol{P}_r \\ \boldsymbol{P}_r^2 \end{pmatrix}\frac{\vec{J}_f}{\epsilon} \tag{62}$$

where $\tilde{\psi}_+ = \widetilde{\boldsymbol{P}}_+\tilde{\psi}$ and $\tilde{\psi}_- = \widetilde{\boldsymbol{P}}_-\tilde{\psi}$. Besides, for the subspace generated by $\widetilde{\boldsymbol{P}}_+$ and $\widetilde{\boldsymbol{P}}_-$, the following equalities will be satisfied:

$$\begin{aligned} \vec{\nabla}\vec{\tilde{B}}_+ &= \vec{\nabla}(\vec{\tilde{B}}_3 + \vec{\tilde{B}}_4) = 0 \\ \vec{\nabla}\vec{\tilde{E}}_+ &= \vec{\nabla}(\vec{\tilde{E}}_3 + \vec{\tilde{E}}_4) = 0 \\ \vec{\nabla}\vec{\tilde{B}}_- &= \vec{\nabla}(\vec{\tilde{B}}_5 + \vec{\tilde{B}}_6) = 0 \\ \vec{\nabla}\vec{\tilde{E}}_- &= \vec{\nabla}(\vec{\tilde{E}}_5 + \vec{\tilde{E}}_6) = 0 \end{aligned} \tag{63}$$

5. Examples

As the first example let us consider equations in a region with no currents ($\vec{J}_g = \vec{0}$). For this case we can see from equation (26) that charges for the first mode are invariant with respect to time.

5.1. Spherical geometry

Now let us consider a problem in which we have no currents, no charges and the electric fields changes with a spherical symmetry, then equations (43) and (45) take the form:

$$\frac{\partial E_+}{\partial t} + c\sqrt{\Delta}E_+ = 0 \tag{64}$$

$$\frac{\partial E_-}{\partial t} - c\sqrt{\Delta}E_- = 0 \tag{65}$$

The integral operator for such a problem takes the following form $\sqrt{\Delta}f = \frac{1}{r}\frac{\partial}{\partial r}(rf)$, where $r = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$ and $r_0 = (x_0, y_0, z_0)$ is the point, where the source changing electric fields is located. An analytical solution of these equations will take the form:

$$\begin{aligned} E_+(r,t) &= \frac{F(t-r/c)}{r} \\ E_-(r,t) &= \frac{F(t+r/c)}{r} \end{aligned} \tag{66}$$

5.2. Quasi-one-dimensional geometry

Spherical geometry cannot be used, when we consider propagation of X-rays. In such a case we need to consider Gaussian beam propagation along a specified direction (for example the OX axis), and during the propagation the beam will slowly expand in a direction perpendicular to the direction of propagation of X-rays. To describe such a problem we need to use an approximate form of the integral operator:

$$\sqrt{\Delta} \approx \partial/\partial x + 0.5\Delta_\perp \int dx \tag{67}$$

and then find solutions of the equations:

$$\frac{\partial E_+}{\partial t} + c\frac{\partial E_+}{\partial x} + 0.5c\Delta_\perp \int E_+ dx = 0 \tag{68}$$

$$\frac{\partial E_-}{\partial t} - c\frac{\partial E_-}{\partial x} - 0.5c\Delta_\perp \int E_- dx = 0 \tag{69}$$

Let us consider equation (68). We can rearrange it to the form:

$$\frac{\partial^2 E_+}{\partial t \partial x} + c\frac{\partial^2 E_+}{\partial x^2} + 0.5c\Delta_\perp E_+ = 0 \tag{70}$$

Using the ansatz

$$E_+(\vec{r},t) = A(\vec{r},t)\exp\left(i\left(k_0 x - \omega_0 t\right)\right) \qquad k_0 = \frac{\omega_0}{c} \tag{71}$$

we get an equation for function $A(\vec{r},t)$, which only varies slowly with the variables $\vec{r}$ and t. Leaving only expressions that have the greatest impact on changes of function $A(\vec{r},t)$ and assuming that function $A(\vec{r},t)$ does not change in time leads us to the equation:

$$\frac{\partial A}{\partial x} = \frac{ic}{2\omega_0}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)A \tag{72}$$

This equation has important applications in the science of optics, where it provides solutions that describe the propagation of electromagnetic waves in the form of either paraboloidal waves or Gaussian beams. Solutions for this equation can be obtained with the help of the so-called Kirchhoff propagator when the boundary condition $A(0,y,z)$ is known

$$A(x,y,z) = \iint\limits_S A(0,\xi,\eta)\,G(x,y,z,\zeta,\eta)\,d\xi\,d\eta \tag{73}$$

where $G(x,y,z,\zeta,\eta)$ is a Kirchhoff propagator, which has the form:

$$G(x,y,z,\zeta,\eta) = -\frac{i}{2\pi\sigma^2}\exp\left(i\,\frac{(y-\zeta)^2+(z-\eta)^2}{2\sigma^2}\right) \qquad \sigma^2 = \frac{xc}{\omega_0} \tag{74}$$

For the boundary condition $A(0,y,z) = \exp\left(-\alpha(y^2+z^2)\right)$ the solution obtained via equation (73) is presented below:

$$A(x,y,z) = \frac{1}{1+i\cdot 2\alpha\sigma^2}\exp\left(-\alpha\,\frac{y^2+z^2}{1+i\cdot 2\alpha\sigma^2}\right) \tag{75}$$

Numerical examples of a solution of equation (72) with other boundary conditions can be found in publications [8, 9].

6. Concluding remarks

The developments of the theory are first of all in more advanced material relations compared to (49)–(50) of this paper. An obvious option lies in a different geometry that changes the action of the operators used in the calculations of Sections 5.1 and 5.2.

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