mathematics

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Generalized Gradient Equivariant Multivalued Maps, Approximation and Degree

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Received: 30 June 2020; Accepted: 28 July 2020; Published: 1 August 2020

Abstract: Consider the Euclidean space $\mathbb{R}^n$ with the orthogonal action of a compact Lie group G. We prove that a locally Lipschitz G-invariant mapping f from $\mathbb{R}^n$ to $\mathbb{R}$ can be uniformly approximated by G-invariant smooth mappings g in such a way that the gradient of g is a graph approximation of Clarke's generalized gradient of f. This result enables a proper development of equivariant gradient degree theory for a class of set-valued gradient mappings.

Keywords: set-valued mapping; G-space; locally Lipschitz mapping; Clarke's generalized gradient; equivariant degree; graph approximation

1. Introduction

It is well known that various versions of degree theory are very useful in nonlinear analysis; see the books [1–3] and their extensive references. A powerful special degree for maps commuting with an action of a Lie group gives many multiplicity results; see the book [4]. In order to extend the degrees to set-valued maps, a graph approximation method in the spirit of [5] appeared to be very successful. In [6], a series of selection and graph approximation results for convex-valued mappings was obtained in the presence of symmetries given by a compact group action as extensions of many classical results. Some of them can be basic tools in the construction of equivariant degree theory. An equivariant version of the Cellina approximation theorem for convex-valued upper semicontinuous mappings was used in [7] to define the equivariant degree, which extended the one from [4,8]. The degree theory was applied in [7] to obtain nontrivial solutions to multivalued boundary value problems. See also [9,10] for further applications of that degree to obtain multiple solutions to some implicit functional differential equations.

On the other hand, in many applications, the considered maps are additionally gradients of smooth functionals, and special invariants for this class are involved like the gradient equivariant degree and Conley index; see [11,12]. There are plenty of papers on various applications, and we mention only a few of them concerning second order ODEs [13], symmetry-breaking [14], bifurcations of the Neumann problem [15] and symmetric Newtonian systems [16]. In the non-smooth case, one can use Clarke's generalized gradient notion [17] for locally Lipschitz functions, which is a convex-valued u.s.c. map in this case (see Definition 3 or [18] for definition of u.s.c. for multifunction). It appears that an approximation result is valid for such functions defined in $\mathbb{R}^n$ (see [19]), which may be treated as an analogue of the Whitney approximation theorem.

The main purpose of this paper is to prove the equivariant version of the approximation theorem of Ćwiszewski and Kryszewski. Then, we use this to provide an equivariant gradient degree theory with all the usual properties including the Hopf classification theorem.

It is worth pointing out that the proof is strictly finite-dimensional, and it is a challenging problem to extend the approximation theorem to Hilbert spaces. Nevertheless, the degree can be defined for compact vector fields, i.e., maps of the form $Id - \partial f$, by standard Leray–Schauder-type techniques.

One can think also of the perturbations of unbounded self-adjoint operators in Hilbert spaces; see [20]. We postpone the details to another paper, as well as applications to set-valued variational problems.

2. Results

2.1. Preliminaries

Let us recall some basic information on group actions. For a more detailed description, see [21,22]. Let G be a group. Recall that a G-set is a pair (X, ξ_X), where X is a set and $\xi_X : G \times X \to X$ is the action of G on X, i.e., a map such that:

(i) $\xi_X(g_1, \xi(g_2, x)) = \xi_X(g_1 g_2, x)$ for $g_1, g_2 \in G$ and $x \in X$;

(ii) $\xi_X(e, x) = x$ for $x \in X$, where $e \in G$ is the group unit.

In the sequel, we write gx instead of $\xi_X(g, x)$, $x \in X$, $g \in G$, unless it leads to ambiguity.

Given G-sets X and Y, a map $f : X \to Y$ is G-equivariant if $f(gx) = gf(x)$ for any $x \in X$ and $g \in G$. If the G-action on Y is trivial, then we say that f is G-invariant.

A subset $A \subset X$ of a G-set X is G-invariant if $gA := \{gx \mid x \in A\} \subset A$ for all $g \in G$. The set $Gx := \{gx \mid g \in G\}$ is called the orbit through $x \in X$, and X/G denotes the set of all orbits. Observe that if $A \subset X$, then the set $GA := \bigcup_{x \in A} Gx = \bigcup_{g \in G} gA$ is G-invariant.

If G is a topological group, X is a topological space and a G-set, then X is a G-space, provided the action ξ_X is (jointly) continuous. We say that a real (resp. complex) Banach space $\mathbb{E}$ is a real (resp. complex) Banach representation of G if $\mathbb{E}$ is a G-space, and for each $g \in G$, the map $\xi_{\mathbb{E}}(g, \cdot) : \mathbb{E} \ni x \mapsto gx$ is linear and bounded. Throughout the whole paper, we assume that G is a compact Lie group.

Set-valued maps are the main object of our studies. Recall that given sets X and Y, a set-valued map φ from X into Y (written $\varphi : X \multimap Y$) is a map that assigns to each $x \in X$ the value $\varphi(x)$, being a non-empty subset of Y. If X and Y are topological spaces and, for any closed (resp. open) set $U \subset Y$, the preimage $\varphi^{-1}(U) := \{x \in X \mid \varphi(x) \cap U \neq \emptyset\}$ is closed (resp. open), then we say that φ is upper (resp. lower) semicontinuous; φ is continuous if it is upper and lower semicontinuous simultaneously.

If Y is a metric space, then $\varphi : X \multimap Y$ is lower semicontinuous if and only if for any $y \in Y$, the function $X \ni x \mapsto d(y, \varphi(x)) := \inf_{z \in \varphi(x)} d(y, z)$ is upper semicontinuous (as a real function) or, equivalently, given $x_0 \in X$ and $y_0 \in \varphi(x_0)$, $\lim_{x \to x_0} d(y_0, \varphi(x)) = 0$.

A similar characterization of upper semicontinuity is not true, i.e., the lower semicontinuity of $d : X \ni x \mapsto d(y, \varphi(x)) \in \mathbb{R}$ does not imply in general that φ is upper semicontinuous. However, if φ has closed values, is locally compact, i.e., each point $x \in X$ has a neighbourhood U such that $\overline{\varphi(U)}$ is compact, and d is lower semicontinuous, then φ is upper semicontinuous (with compact values). The graph $\mathrm{Gr}\,(\varphi) := \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$ of an upper semicontinuous map φ with closed values is closed; $\varphi : X \multimap Y$ is upper semicontinuous with compact values if and only if the projection $\mathrm{Gr}\,(\varphi) \to X$ is perfect (recall that a continuous map $f : X \to Y$ is perfect if it is closed and $f^{-1}(y)$ is compact for any $y \in Y$). We say that a map φ is compact if it is upper semicontinuous and the closure of the image $\varphi(X) := \bigcup_{x \in X} \varphi(x)$ is compact. For other details on set-valued maps, see [23] or [18].

Let X, Y be metric spaces and $\varepsilon > 0$. Recall the notion of graph approximations.

Definition 1. *A continuous map $f : X \to Y$ is an ε-approximation of $\varphi : X \multimap Y$ if for every x, there exists x' such that $d(x, x') < \varphi$ and $f(x) \in B_\varepsilon(\varphi(x'))$.*

One can formulate the above condition as follows: $f(x) \in B_\varepsilon(\varphi(B_\varepsilon(x)))$ for all $x \in X$. It is also equivalent to the condition that the graph of f is contained in the ε-neighbourhood of the graph of φ. It is well known that such approximations are a good tool for extending topological invariants, and they have been proven to exist for example for convex-valued u.s.c. maps; see [5]. We are interested in equivariant versions of the results.

Definition 2. *Let X and Y be G-sets. A set-valued map $\varphi : X \multimap Y$ is G-equivariant (resp. G-invariant) if $\varphi(gx) = g\varphi(x)$ (resp. $\varphi(gx) = \varphi(x)$) for all $g \in G$ and $x \in X$.*

Note that φ is G-equivariant if and only if $\varphi(gx) \subset g\varphi(x)$ for all $g \in G$ and $x \in X$ (or $g\varphi(x) \subset \varphi(gx)$ for all $g \in G$ and $x \in X$). Moreover, it is easy to see that φ is G-equivariant if and only if its graph $\mathrm{Gr}\,(F)$ is a G-invariant subset of $X \times Y$ with a natural action $g(x, y) := (gx, gy)$, $x \in X$, $y \in Y$. Observe that if $\varphi : X \multimap Y$, where Y is a topological space, is G-equivariant, then so is its closure, i.e., the map $\overline{\varphi} : X \multimap Y$ given by $\overline{\varphi}(x) := \overline{\varphi(x)}$, $x \in X$.

Let us collect simple examples of G-equivariant set-valued maps.

Example 1. *(1) Let $\varphi : [0, T] \times U \multimap \mathbb{E}$, where $T > 0$ and $U \subset \mathbb{E}$ is an open G-invariant subset of a Banach G-representation $\mathbb{E}$, be G-equivariant, i.e., $\varphi(t, gx) = g\varphi(t, x)$ for $0 \le t \le T$ and $x \in U$. Consider a differential inclusion (under suitable assumptions assuring the existence of solutions):*

$$\begin{cases} x'(t) \in \varphi(t, x(t)), \\ x(0) = x_0. \end{cases}$$

Consider the space $C([0, T], \mathbb{E})$ of continuous maps from $[0, T]$ to $\mathbb{E}$ with the G-action defined by $(g, x) \mapsto gx$, where $(gx)(t) := g(x(t))$ for $x \in C([0, T], \mathbb{E})$, $g \in G$ and $t \in [0, T]$. If $x : [0, T] \to \mathbb{E}$ is a solution to the above problem, i.e., there is an integrable function $y : [0, T] \to \mathbb{E}$ such that $x(t) = x_0 + \int_0^t y(s)\,ds$, $t \in [0, T]$, then gx is also a solution to this problem with the initial condition gx_0. Therefore, the solution map $P : U \multimap C([0, T], \mathbb{E})$ that assigns to each initial value $x_0 \in U$ the set of all solutions is G-equivariant, whenever well defined.

(2) Let $f : \mathbb{E} \to \mathbb{R} \cup \{\infty\}$, where $\mathbb{E}$ is a real Banach representation of G, be a convex function. For each $x_0 \in dom(f) := \{x \in \mathbb{E} \mid f(x) < \infty\}$, the subdifferential:

$$\partial f(x_0) := \{p \in \mathbb{E}^* \mid f(x) \ge f(x_0) + \langle p, x - x_0\rangle \text{ for all } x \in \mathbb{E}\}$$

is defined. Let f be G-invariant. Then, the map $\partial f : dom(f) \multimap \mathbb{E}^*$ is G-equivariant. Clearly, $dom(f)$ is invariant, and if $p \in \partial f(x_0)$, then for all $x \in \mathbb{E}$, $\langle p, x - x_0\rangle \le f(x) - f(x_0)$. Hence:

$$\langle gp, x - gx_0\rangle = \langle p, g^{-1}x - x_0\rangle \le f(g^{-1}x) - f(x_0) = f(x) - f(gx_0),$$

which gives the assertion. In a similar manner, one shows that the Clarke generalized gradient $\partial f : U \to \mathbb{E}^*$, where U is a G-invariant open in $\mathbb{E}$ and $f : U \to \mathbb{R}$ is a G-invariant locally Lipschitz function, is G-equivariant.

2.2. Gradient approximations

In $\mathbb{R}^n$, we shall use various analytic characterizations of graph-approximations. For $A \in \mathbb{R}^n$, we recall the notion of a support function $\sigma_A : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ given by:

$$\sigma_A(v) := \sup_{a \in A} \langle a, v\rangle, \quad v \in \mathbb{R}^n.$$

One easily checks that σ_A is finite and continuous if A is bounded. Moreover, $\overline{\mathrm{conv}} A = \{x \in \mathbb{R}^n \mid \forall v \in \mathbb{R}^n \langle x, v\rangle \le \sigma_A(v)\}$.

We also define:

$$|||A||| := \sup_{u \in \overline{B_1}(0)} \inf_{a \in A} \langle a, u\rangle,$$

or equivalently:

$$|||A||| = \inf_{a \in \mathrm{conv} A} \sup_{||u|| \le 1} \langle a, u\rangle = \inf_{a \in \mathrm{conv} A} ||a||.$$

Lemma 1. *([19], Lemma 2.2) Let $U \subset \mathbb{R}^n$, $\varphi : U \multimap \mathbb{R}^n$ with closed convex values, $g : U \to \mathbb{R}^n$, and $\varepsilon : U \to (0, +\infty)$. The following conditions are equivalent:*

(i) *g is an ε-approximation of φ;*
(ii) *for every $x \in U$, there is $\overline{x} \in U \cap B_{\varepsilon(x)}(x)$ such that:*

$$\|g(x) - \varphi(\overline{x})\| < \varepsilon(x);$$

(iii) *for every $x \in U$, there is $\overline{x} \in U \cap B_{\varepsilon(x)}(x)$ such that:*

$$\forall ||u|| \leq 1 \quad \langle g(x), u\rangle - \sigma_{\varphi(\overline{x})}(u) < \varepsilon(x).$$

Let $U \subset \mathbb{R}^n$ be an open subset, and consider a locally Lipschitz function $f : U \to \mathbb{R}$. A generalized directional Clarke derivative:

$$f^o(x; v) := \limsup_{y \to x, H \to 0^+} \frac{f(y + hv) - f(y)}{h}$$

is defined for all $x \in U, v \in \mathbb{R}^n$. Clarke's generalized gradient at x is the set:

$$\partial f(x) := \{p \in \mathbb{R}^n \mid \text{for all } v \in \mathbb{R}^n \quad \langle p, v\rangle \leq f^o(x; v)\}.$$

It is well known that the map $\partial f : U \multimap \mathbb{R}^n$ is an u.s.c. mapping with compact convex values (see [17] for this and more of the properties of the generalized gradient). One observes also that $\sigma_{\partial f(x)}(v) = f^o(x; v)$ for $v \in \mathbb{R}^n$. Recall only the local u.s.c. property:

Definition 3. *The map ∂f is upper semicontinuous at $x \in X$ if:*

$$\forall_{\varepsilon > 0} \exists_{\delta(x) > 0} \forall_{x' \in X} \ \|x' - x\| < \delta(x) \Rightarrow \partial f(x') \subset B_\varepsilon(\partial f(x)). \tag{1}$$

The following approximation result was proved in [19]:

Theorem 1. *Let $f : U \to \mathbb{R}$ be locally Lipschitz and $\varepsilon : U \to (0, \infty)$ a continuous function. Then, there exists a C^∞-map $g : U \to \mathbb{R}$ such that:*

(i) *$|f(x) - g(x)| < \varepsilon(x)$ for all $x \in U$,*
(ii) *$\nabla g(x) \in B_{\varepsilon(x)}(\partial f(B_{\varepsilon(x)}(x)))$ for all $x \in U$.*

In other words, g is a uniform approximation of f, and ∇g is an ε-approximation of ∂f. We shall call such pairs $(g, \nabla g)$ Whitney approximations.

Definition 4. *Suppose $f : X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$ compact, is a locally Lipschitz function. If $g \in C^\infty(X, \mathbb{R})$ satisfies:*

1. *$|f(x) - g(x)| \leq \varepsilon(x)$ for all $x \in \mathbb{R}^n$ (uniform ε-approximation),*
2. *$\nabla g(x) \in B_{\varepsilon(x)}(\partial f(B_{\varepsilon(x)}(x)))$ for all $x \in \mathbb{R}^n$ (ε-approximation (on the graph)),*

for a continuous function $\varepsilon : X \to (0, \infty)$, then we say that $(g, \nabla g)$ is the ε-WT-approximation (ε-Whitney-type-approximation) of $(f, \partial f)$.

Let us assume that V is an n-dimensional orthogonal representation of a compact Lie group G and $U \subset V$ is open and G-invariant. Assume that a locally Lipschitz function $f : U \to \mathbb{R}$ is G-invariant, i.e., $f(hx) = f(x)$ for all $h \in G$. It is easy to verify that ∂f is then equivariant, i.e., $\partial(gx) = g(\partial(x))$ as sets.

We now modify the proof from [19] in order to obtain an equivariant version of Theorem 1.

Let us start from the remark that since G acts orthogonally, there exists an invariant C^∞-mollifier, i.e., a nonnegative invariant function $\omega : \mathbb{R}^n \to \mathbb{R}$ with the support $\operatorname{supp}\omega \subset \overline{B_1}(0)$ and such that:

$$\int_{\mathbb{R}^n} \omega(x)dx = 1.$$

We can take, e.g., a function given by:

$$\omega_1(x) = \begin{cases} e^{\frac{1}{\|x\|^2-1}} & \text{if} \quad \|x\| < 1, \\ 0 & \text{if} \quad \|x\| \geq 1 \end{cases},$$

and then, we normalize it:

$$\omega(x) := \frac{\omega_1(x)}{\int_{\mathbb{R}^n} \omega_1(x)dx}.$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be an invariant L-Lipschitz map with a compact support, and let $\lambda > 0$. We define the λ-regularization of f by the formula:

$$g_\lambda(x) := \int_{\mathbb{R}^n} f(x - \lambda z)\omega(z)dz.$$

The following is a straightforward consequence of the definition.

Proposition 1. *The regularization g_λ satisfies the following properties:*

(i) g_λ *is* C^∞*-smooth;*
(ii) g_λ *is G-invariant;*
(iii) g_λ *is a uniform* λL*-approximation of* f*;*
(iv) $\operatorname{supp} g_\lambda \subset \overline{B_\lambda}(\operatorname{supp} f)$.

The following technical lemma was proven in detail in [19], Lemma 3.4.

Lemma 2. *Let $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz functions of common rank L and with compact supports. For any $\varepsilon > 0$, there exists sufficiently small $\mu > 0$ with the property that for every $x \in \mathbb{R}^n$, there exists $\overline{x} \in B_\varepsilon(x)$ such that:*

$$\forall u \in \overline{B_1(0)} \qquad \sup_{z \in B_\mu(x), t \in (0,\mu)} \frac{f_i(z + tu) - f_i(z)}{t} < f_i^o(\overline{x}; u) + \frac{\varepsilon}{2}$$

for each $i = 1, \ldots, k$.

A local approximation result is a consequence of the above.

Proposition 2. *Let $f_i (i = 1, \ldots, k)$ be G-invariant functions as in Lemma 2. Then, for any $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that:*

(i) *for any $\lambda \leq \lambda_0$, the regularization g_λ^i of f_i is a uniform ε-approximation of f_i, for each $i = 1, \ldots, k$;*
(ii) *for every $x \in \mathbb{R}^n$, there exists $\overline{x} \in B_\varepsilon(x)$ such that for all $\lambda \leq \lambda_0$ and $i = 1, \ldots, k$:*

$$\|\nabla g_\lambda^i(x) - \partial f_i(\overline{x})\| < \varepsilon.$$

Proof. We put $\lambda_0 := \min\{\mu, \frac{\varepsilon}{L}\} > 0$, and let $\lambda \leq \lambda_0$. Then, Property (i) follows from Proposition 1. We prove (ii) by means of Property (iii) from Lemma 1. For this, we fix a point x and $u \in \overline{B_1(0)}$. There exists a $\delta = \delta(\lambda, x, u) < \mu$ such that, for all i, we have:

$$\forall h \in (0, \delta) \qquad \langle \nabla g^i_\lambda(x), u \rangle < \frac{g^i_\lambda(x + hu) - g^i_\lambda(x)}{h} + \frac{\varepsilon}{2}.$$

On the other hand, we have:

$$\frac{g^i_\lambda(x + hu) - g^i_\lambda(x)}{h} = \frac{1}{h} \int_{\overline{B_1(0)}} [f_i(x - \lambda z + hu) - f_i(x - \lambda z)]\omega(z)dz,$$

and by the use of Lemma 2, we find $\overline{x}$ such that:

$$\frac{g^i_\lambda(x + hu) - g^i_\lambda(x)}{h} < f^o_i(\overline{x}; u) + \frac{\varepsilon}{2}.$$

Therefore, we have the inequality $\langle \nabla g^i_\lambda(x), u \rangle < f^o_i(\overline{x}; u) + \varepsilon$. This ends the proof because of the equality $\sigma_{\partial f(x)}(u) = f^o(x; u)$. $\square$

Theorem 2. *Let $f : U \to \mathbb{R}$ be a locally G-invariant map as in Theorem 1. Then, for every $\varepsilon > 0$, there exists a C^∞-smooth G-invariant map $g : U \to \mathbb{R}$ such that $(g, \nabla g)$ is an ε-WT-approximation of $(f, \partial f)$.*

Proof. Since U is G-invariant, there exists an increasing family of G-invariant compact sets $\{K_k\}_{k=0}^\infty$ such that:

$$K_0 = \emptyset; \quad \forall k \geq 1 \quad \emptyset \neq K_k \subset K_{k+1} \quad \text{and} \quad U = \bigcup_{k=1}^\infty K_k.$$

For each k = 1, the set $P_k := \operatorname{int} K_{k+1} \setminus K_{k-1}$ is also G-invariant. Therefore, the family $\{P_k\}$ is an open G-invariant covering of U. Let $\{\varphi_k\}_{k=1}^\infty$ be a C^∞-smooth and G-invariant partition of unity subordinated to $\{P_k\}$.

Thus, the functions $\varphi_k \cdot f$ $(k \geq 1)$ are globally Lipschitz, G-invariant, and they can be considered as defined on $\mathbb{R}^n$.

We define a sequence of positive numbers:

$$\varepsilon_1 := \min\left\{\frac{1}{2}\min_{x \in K_2} \varepsilon(x), \mathrm{dist}(\mathrm{supp}\,'_3, \mathrm{K}_2), \mathrm{dist}(\mathrm{supp}\,'_2, \mathrm{K}_1)\right\};$$

$$\varepsilon_k := \min\left\{\frac{1}{2}\min_{x \in K_{k+1}} \varepsilon(x), \mathrm{dist}(\mathrm{supp}\,'_{\mathrm{k}-1}, \mathbb{R}^{\mathrm{n}} \setminus \mathrm{K}_\mathrm{k}), \mathrm{dist}(\mathrm{supp}\,'_{\mathrm{k}+2}, \mathrm{K}_{\mathrm{k}+1})\right\} \quad (k \geq 2).$$

Now, we can apply Proposition 2 for each $k \geq 1$. We find λ_k such that for any $0 < \lambda \leq \lambda_k$ invariant λ-regularizations: g^k_λ of $\varphi_k \cdot f$ and g^{k+1}_λ of $\varphi_{k+1} \cdot f$ are uniform $\varepsilon_k/2$-approximations. Moreover, for every $x \in P_k \cup P_{k+1}$, there is $\overline{x} \in B_{\varepsilon_k}(x) \cap (P_k \cup P_{k+1})$ such that:

$$\|\nabla g^k_\lambda(x) - \partial(\varphi_k \cdot f)(\overline{x})\| < \frac{\varepsilon_k}{2},$$

$$\|\nabla g^{k+1}_\lambda(x) - \partial(\varphi_{k+1} \cdot f)(\overline{x})\| < \frac{\varepsilon_k}{2}$$

for all $0 < \lambda \leq \lambda_n$. We can assume that $\operatorname{supp} g^k_\lambda \subset P_k$, and the sequence $\{\lambda_k\}_{k=1}^\infty$ is nonincreasing.

The desired map $g : U \to \mathbb{R}$ is defined by the formula:

$$g(x) := \sum_{k=1}^\infty g^k_{\lambda_k}(x).$$

It is obviously well defined, since at most two terms are different from zero at given x. By definition, it is C^∞-smooth and G-invariant. We have to check the approximation conditions. For a fixed $x \in U$, there is a unique $k \geq 0$ such that $x \in K_{k+1} \setminus K_k$. If $k = 0$, then $g(y) = g^1_{\lambda_1}$ in some neighbourhood of x, and the claim is clear from the choice of ε_1.

If $k \geq 1$, then in some neighbourhood of x contained in $B_{\varepsilon_k}(x)$, we have $g(y) = g^k_{\lambda_k}(y) + g^{k+1}_{\lambda_{k+1}}(y)$ and $\varphi_k(y) + \varphi_{k+1}(y) = 1$. Therefore:

$$|g(x) - f(x)| \leq |g^k_{\lambda_k}(x) - \varphi_k(x)f(x)| + |g^{k+1}_{\lambda_{k+1}}(x) - \varphi_{k+1}(x)f(x)| < \varepsilon_n \leq \varepsilon(x).$$

By the properties of the generalized gradient, we obtain:

$$\|\nabla g(x) - \partial f(\overline{x})\| = \|(\nabla g^k_{\lambda_k}(x) + \nabla g^{k+1}_{\lambda_{k+1}}(x)) - (\partial(\varphi_k f)(\overline{x}) + \partial(\varphi_{k+1} f)(\overline{x}))\| \leq$$

$$\leq \|\nabla g^k_{\lambda_k}(x) - \partial(\varphi_k f)(\overline{x}\| + \|\nabla g^{k+1}_{\lambda_{k+1}}(x) - \partial(\varphi_{k+1} f)(\overline{x})\| < \varepsilon_n \leq \varepsilon(x).$$

Because of Lemma 1, the last condition means that ∇g is a graph ε-approximation of ∂f. This ends the proof. □

Theorem 3. *Let X be a compact G-invariant subset of V. Suppose that $f : X \to \mathbb{R}$ is locally Lipschitz and G-invariant. Then, for all $\varepsilon > 0$, there exists a $\delta > 0$ with $\delta < \varepsilon$ and such that, for every two $WT_{\frac{\delta}{2}}$-approximations $(g_1, \nabla g_1), (g_2, \nabla g_2)$ of $(f, \partial f)$, the pair $(g_t, \nabla g_t)$ is an WT_ε-approximation of $(f, \partial f)$, where:*

$$g : \mathbb{R}^n \times I \to \mathbb{R}, \quad g(x,t) = g_t(x) = t g_1(x) + (1-t) g_2(x),$$

is a linear homotopy.

Proof. For all $x \in X$, $\partial f(\cdot)$ is upper semicontinuous (1) at x, so we can apply Definition 3 with a given ε and obtain an appropriate, small $\delta(x)$.

We can assume that $\delta(x) < |x|$ for $x \neq 0$, and $\delta(0) = \delta_0 > 0$. Then, we have $x \to 0 \Rightarrow \delta(x) \to 0$. For all $x \in X$, we observe that $x' \in B(x, \delta(x))$ implies $\partial f(x') \subset B_\varepsilon(\partial f(x))$. The family $\{B(x, \delta(x))\}_{x \in X}$ is an open covering of X. We can choose a finite subcovering:

$$\eta = \{O_{x_1}, O_{x_2}, ..., O_{x_n}\}, \text{ where } O_{x_k} = B(x_k, \delta(x_k)).$$

Let $l > 0$ be a Lebesgue number for the covering η, i.e.,:

$$\|x' - x''\| < l \Rightarrow \exists_{i \in \{1,2,\ldots,n\}}\ x', x'' \in O_{x_i} = B(x_i, l),$$

and let $\delta = 2\min\{l, \frac{1}{2}\varepsilon\}$.

Fix $x \in X$, and suppose that $(g_1, \nabla g_1), (g_2, \nabla g_2) \in C^\infty(\mathbb{R}^n)$ are $WT_{\frac{\delta}{2}}$-approximations of $(f, \partial f)$. Then, we have:

$$\begin{cases} |f(x) - g_1(x)| \leq \frac{\delta}{2} \\ \exists_{\tilde{x} \in X}\ \|\tilde{x} - x\| < \frac{\delta}{2} \text{ and } \nabla g_1(x) \in B_{\frac{\delta}{2}}(\partial f(\tilde{x})), \end{cases} \tag{2}$$

$$\begin{cases} |f(x) - g_2(x)| \leq \frac{\delta}{2} \\ \exists_{\bar{x} \in X}\ \|\bar{x} - x\| < \frac{\delta}{2} \text{ and } \nabla g_2(x) \in B_{\frac{\delta}{2}}(\partial f(\bar{x})), \end{cases} \tag{3}$$

and:

$$\nabla g_t(x) = t \nabla g_1(x) + (1-t) \nabla g_2(x).$$

Then, from (2) and (3), we get:

$$\begin{array}{rcll} |f(x) - g_t(x)| & = & |tf(x) + (1-t)f(x) - tg_1(x) - (1-t)g_2(x)| & \leq \\ & \leq & t|f(x) - g_1(x)| + (1-t)|f(x) - g_2(x)| & < \frac{\delta}{2} < \varepsilon. \end{array}$$

From (2) and (3), we have:

$$\|\tilde{x} - \bar{x}\| \leq \|\tilde{x} - x\| + \|x - \bar{x}\| \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

so:

$$\exists_{i \in \{1,2,\ldots,n\}}\ \tilde{x}, \bar{x} \in O_{x_i} \subset B\left(x_i, \frac{\delta}{2}\right). \tag{4}$$

We can use the upper semicontinuity property of ∂f in x_i:

$$\forall_{\varepsilon > 0} \exists_{\delta > 0}\ \forall_{x' \in X} \|x' - x\| < \delta \Rightarrow \partial f(x') \subset B_{\frac{\varepsilon}{2}}(\partial f(x))$$

for points $\tilde{x}, \bar{x}$, which are close enough to x_i:

$$\partial f(\tilde{x}) \subset B_{\frac{\varepsilon}{2}}\left(\partial f(x_i)\right)$$

$$\partial f(\bar{x}) \subset B_{\frac{\varepsilon}{2}}\left(\partial f(x_i)\right)$$

Now, we have:

$$\nabla g_j(x) \in B_{\frac{\delta}{2}}\left(\partial f(\tilde{x})\right) \subset B_{\frac{\delta}{2}}\left(B_{\frac{\varepsilon}{2}}\left(\partial f(x_i)\right)\right) \subset B_{\varepsilon}\left(\partial f(x_i)\right),\ j = 1, 2.$$

However, $B_{\varepsilon}\left(\partial f(x_i)\right)$ is a convex set, and therefore:

$$\nabla g_t(x) = t\nabla g_1(x) + (1-t)\nabla g_2(x) \in B_{\varepsilon}\left(\partial f(x_i)\right).$$

Finally, from (2) and (4), we have:

$$\|x - x_i\| \leq \|x - \tilde{x}\| + \|\tilde{x} - x_i\| < \frac{\delta}{2} + \frac{\delta}{2} < \delta < \varepsilon.$$

This ends the proof. □

2.3. Equivariant Gradient Degree

In this section, we outline an axiomatic approach to the equivariant gradient degree. To this end, some preliminaries are in order.

Recall that the Euler ring $U(G)$ of a compact Lie group G as an abelian group is equal to $\mathbb{Z}(\Phi(G))$, where $\Phi(G)$ is the set of all conjugacy classes of normal subgroups of G. The multiplicative structure of $U(G)$ is quite involved (see [22] or [4] for details on the ring structure).

Let V be an orthogonal G-representation. Denote by $C^2_G(V, \mathbb{R})$ the space of G-invariant real C^2-functions on V. Clearly, if $\varphi \in C^2_G(V, \mathbb{R})$, then the gradient map $\nabla\varphi$ is G-equivariant. Let $\Omega \subset V$ be an open, bounded and G-invariant subset of V and $f : V \to V$. A pair (f, Ω) is said to be a G-gradient Ω-admissible pair if $f(x) \neq 0$ for all $x \in \operatorname{bd}\Omega$, and there exists $\varphi \in C^2_G(V, \mathbb{R})$ such that $\nabla\varphi = f$. Denote by $M^G_{\nabla}(V, V)$ the set of all G-gradient Ω-admissible pairs, and put $M^G_{\nabla} := \cup_V M^G_{\nabla}(V, V)$. In an obvious way, one can define a G-gradient Ω-admissible homotopy between two G-gradient Ω-admissible maps.

The notion of equivariant gradient degree we consider here was defined originally by K. Gęba [12]. We use an axiomatic approach here adapted from [8].

Theorem 4. *There exists a unique map* $\deg_G^{\nabla} : M_{\nabla}^{G} \to U(G)$*, which assigns to every* $(\nabla\varphi, \Omega) \in M_{\nabla}^{G}$ *an element* $\deg_G^{\nabla}(\nabla\varphi, \Omega) \in U(G)$*:*

$$\nabla_G \deg(f, \Omega) = \sum_{(H_i) \in \Phi(G)} n_{H_i}(H_i),$$

satisfying the following properties ((H_i) *are generators of the Euler ring, and* n_{H_i} *are integer coefficients that may be nonzero only if the orbit type* (H_i) *is present in the representation* V*):*

1. *(Existence) If* $\deg_G^{\nabla}(f, \Omega) \neq 0$*, i.e., there is a nonzero coefficient* n_{H_i}*, then* $\exists x \in \Omega$ *such that* $\nabla\varphi(x) = 0$ *and* $(G_x) \geq (H_i)$*.*
2. *(Additivity) Let* Ω_1, Ω_2 *be two disjoint, open, G-invariant subsets of* Ω *and* $\nabla\varphi^{-1}(0) \cap (\overline{\Omega}) \subset \Omega_1 \cup \Omega_2$*. Then:*

$$\deg_G^{\nabla}(\nabla\varphi, \Omega) = \deg_G^{\nabla}(\nabla\varphi, \Omega_1) + \deg_G^{\nabla}(\nabla\varphi, \Omega_2).$$

3. *(Homotopy) If* $\nabla_v \Psi : [0, 1] \times \overline{\Omega} \to V$ *is an admissible gradient G-equivariant homotopy, then:*

$$\deg_G^{\nabla}(\nabla_v \Psi_t, \Omega) = constant.$$

4. *(Normalization) Let* $\varphi \in C_G^2(V, \mathbb{R})$ *be a special* Ω*-Morse function such that* $(\nabla\varphi)^{-1}(0) \cap \Omega = G(v_0)$ *and* $G_{v_0} = H$*. Then,*

$$\deg_G^{\nabla}(\nabla\varphi, \Omega) = (-1)^{m^-(\nabla^2\varphi(v_0))} \cdot (H),$$

where "$m^-(\cdot)$*" stands for the total dimension of eigenspaces for negative eigenvalues of a (symmetric Hessian) matrix.*

5. *(Multiplicativity) For all* $(\nabla\varphi_1, \Omega_1), (\nabla\varphi_2, \Omega_2) \in M_{\nabla}^{G}$*,*

$$\deg_G^{\nabla}(\nabla\varphi_1 \times \nabla\varphi_2, \Omega_1 \times \Omega_2) = \deg_G^{\nabla}(\nabla\varphi_1, \Omega_1) \star \deg_G^{\nabla}(\nabla\varphi_2, \Omega_2),$$

where the multiplication "$\star$*" is taken in the Euler ring* $U(G)$*.*

6. *(Suspension) If* W *is an orthogonal G-representation and* $\mathcal{B}$ *an open bounded invariant neighbourhood of* $0 \in W$*, then:*

$$\deg_G^{\nabla}(\nabla\varphi \times Id_W, \Omega \times \mathcal{B}) = \deg_G^{\nabla}(f, \Omega).$$

7. *(Hopf property) Assume that* $B(V)$ *is the unit ball of an orthogonal G-representation* V*, and for* $(\nabla\varphi_1, \mathcal{B}(V)), (\nabla\varphi_2, \mathcal{B}(V)) \in M_{\nabla}^{G}$*, one has:*

$$\deg_G^{\nabla}(\nabla\varphi_1, B(V)) = \deg_G^{\nabla}(\nabla\varphi_2, B(V)).$$

Then, $\nabla\varphi_1$ *and* $\nabla\varphi_2$ *are G-gradient* $B(V)$*-admissible homotopic.*

Let us remark that in the case of the trivial action of G, the above degree reduces to the classical Brouwer degree, as well as the usual non-gradient G-equivariant degree (comp. [8]). This follows from the celebrated A. Parusiński theorem (see [24]), which says that every two gradient maps that are $B(V)$-admissible homotopic are also gradient $B(V)$-admissible homotopic. However, this is no longer true for equivariant maps.

Example 2. *Take* $V = \mathbb{C}$ *with the natural* S^1*-action. Clearly,* $\Phi = Id$ *and* $\Psi = -Id$ *are* S^1*-equivariant. Moreover,* $h(t, z) = e^{i\pi t} z$ *is a* $B(V)$*-admissible* S^1*-equivariant homotopy between* Φ *and* Ψ*. Clearly, this homotopy is not a gradient. Moreover, it follows from [11] that such a gradient* S^1*-equivariant homotopy does not exist.*

Since the equivariant gradient degree satisfies the Hopf property with respect to gradient G-homotopies, as well as the ordinary equivariant degree with respect to equivariant homotopies, the gradient degree is more delicate homotopy invariant.

The ordinary equivariant degree for multivalued convex-valued u.s.c. maps was defined in [7] by means of the equivariant version of the Cellina approximation theorem (see [6]). The approximation results from the previous section enable us to develop the equivariant degree theory in quite an analogous way.

Let V be a finite dimensional orthogonal representation of a compact Lie group G, and let Ω be an open, bounded G-invariant subset of V. Let $f : V \to \mathbb{R}$ be a locally Lipschitz G-invariant map such that its Clarke's generalized gradient $\partial f : V \to V$ is Ω-admissible, i.e., $0 \notin \partial f(x)$ for $x \in \operatorname{bd}\Omega$. Since the map ∂f is u.s.c., the image of the boundary $\partial f(\operatorname{bd}\Omega)$ is compact, and its distance from zero is positive. The same is true for a sufficiently fine neighbourhood of $\operatorname{bd}\Omega$. Thus, there exists an $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, each ε-WT-approximation of $(f, \partial f)$ is Ω-admissible. Moreover, Theorem 3 assures that there exists a smaller $\delta > 0$ such that all δ-approximations are Ω-admissible homotopic by a linear homotopy. Take such a small δ.

Definition 5. *We define the equivariant degree of Ω-admissible pair $(f, \partial f)$ as follows:*

$$\deg_G^{\nabla}((f, \partial f), \Omega) := \deg_G^{\nabla}(\nabla g, \Omega),$$

where $(g, \nabla g)$ is a δ-WT-approximation (with g being G-invariant) of $(f, \partial f)$.

The properties of the degree are consequences of Theorem 4. The first three of them are the most important for applications. Properties 1 and 2 are quite straightforward. The homotopy property needs some comment. Recall that the generalized gradient of a locally Lipschitz map is a multivalued u.s.c. map with convex compact values. Considering homotopies as continuous families of maps, we need here a stronger regularity than the u.s.c. property, which may admit quite a rapid change of values.

Definition 6. *A gradient G-homotopy is a pair $(f_t, \partial f_t)$, where $f_t : V \to \mathbb{R}$ are locally Lipschitz G-invariant maps and the family of maps $\partial f_t : V \to V$ is continuous with respect to $t \in [0,1]$, as well as the family f_t.*

Observe that in the most popular case of linear homotopy $f_t(x) = tf_1(x) + (1-t)f_2(x)$, the Clarke gradient with respect to the x variable satisfies the condition:

$$\partial f_t(x) \subset t\partial f_1(x) + (1-t)\partial f_2(x).$$

In some cases, an equality holds, e.g., if at least one of the f_i is of class C^1, or if both of them are convex functions.

Theorem 5. *Let the linear homotopy of locally Lipschitz maps $f_t(x) = tf_1(x) + (1-t)f_2(x)$ be Ω-admissible, and let the condition $\partial f_t(x) = t\partial f_1(x) + (1-t)\partial f_2(x)$ be satisfied for all x and $t \in [0,1]$. Then:*

$$\deg_G^{\nabla}(\nabla_v \Psi_t, \Omega) = constant.$$

Proof. The proof is similar to the proof of Theorem 3. For all $x \in X$, $\partial f_i(\cdot)$ $(i = 1,2)$ are upper semicontinuous (1) at x, so we can apply Definition 3 with a given ε and obtain an appropriate $\delta(x)$.

Let $\delta(x) < |x|$ and $\delta(0) = \delta_0 > 0$. Then, $x \to 0 \Rightarrow \delta(x) \to 0$.

For all $x \in X$, we have that $x' \in B(x, \delta(x))$ implies $\partial f_i(x') \subset B_\varepsilon(\partial f_i(x))$ and $\{B(x, \delta(x))\}_{x \in X}$ is an open covering of X. We can choose a finite subcovering:

$$\eta = \{O_{x_1}, O_{x_2}, ..., O_{x_n}\}, \text{ where } O_{x_k} = B(x_k, \delta(x_k)).$$

Let $l > 0$ be a Lebesgue number for the covering η, i.e.,:

$$\|x' - x''\| < l \Rightarrow \exists_{i\in\{1,2,\ldots,n\}}\ x', x'' \in O_{x_i} = B(x_i, l)$$

and let $\delta = 2\min\{l, \frac{1}{2}\varepsilon\}$.

Fix $x \in X = \overline{\Omega}$, and suppose that $(g_1, \nabla g_1), (g_2, \nabla g_2) \in C^\infty(\mathbb{R}^n)$ are $WT_{\frac{\delta}{2}}$-approximations of $(f_1, \partial f_1), (f_2, \partial f_2)$, respectively. Then, we have:

$$\begin{cases} |f_1(x) - g_1(x)| \le \frac{\delta}{2} \\ \exists_{\tilde{x}\in X}\ \|\tilde{x} - x\| < \frac{\delta}{2} \text{ and } \nabla g_1(x) \in B_{\frac{\delta}{2}}\left(\partial f_1(\tilde{x})\right), \end{cases} \tag{5}$$

$$\begin{cases} |f_2(x) - g_2(x)| \le \frac{\delta}{2} \\ \exists_{\bar{x}\in X}\ \|\bar{x} - x\| < \frac{\delta}{2} \text{ and } \nabla g_2(x) \in B_{\frac{\delta}{2}}\left(\partial f_2(\bar{x})\right), \end{cases} \tag{6}$$

and:

$$\nabla g_t(x) = t\nabla g_1(x) + (1-t)\nabla g_2(x).$$

Then, from (5) and (6), we get:

$$\begin{array}{rcll} |f_t(x) - g_t(x)| & = & |tf_1(x) + (1-t)f_2(x) - tg_1(x) - (1-t)g_2(x)| & \le \\ & \le & t|f_1(x) - g_1(x)| + (1-t)|f_2(x) - g_2(x)| & < \frac{\delta}{2} < \varepsilon. \end{array}$$

From (5) and (6), we have:

$$\|\tilde{x} - \bar{x}\| \le \|\tilde{x} - x\| + \|x - \bar{x}\| \le \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

so:

$$\exists_{i\in\{1,2,\ldots,n\}}\ \tilde{x}, \bar{x} \in O_{x_i} \subset B\left(x_i, \frac{\delta}{2}\right). \tag{7}$$

We can use the upper semicontinuity of ∂f_j in x_i:

$$\forall_{\varepsilon>0}\exists_{\delta>0}\ \forall_{x'\in X}\|x' - x\| < \delta \Rightarrow \partial f_j(x') \subset B_{\frac{\varepsilon}{2}}(\partial f_j(x))$$

for points $\tilde{x}, \bar{x}$:

$$\partial f_1(\tilde{x}) \subset B_{\frac{\varepsilon}{2}}\left(\partial f_1(x_i)\right)$$
$$\partial f_2(\bar{x}) \subset B_{\frac{\varepsilon}{2}}\left(\partial f_2(x_i)\right)$$

Now, we have:

$$\nabla g_j(x) \in B_{\frac{\delta}{2}}\left(\partial f_j(\tilde{x})\right) \subset B_{\frac{\delta}{2}}\left(B_{\frac{\varepsilon}{2}}\left(\partial f_j(x_i)\right)\right) \subset B_\varepsilon\left(\partial f_j(x_i)\right), \ j = 1,2.$$

This means that there exist $y_1 \in \partial f_1(x_i), y_2 \in \partial f_2(x_i)$ such that:

$$\|\nabla g_1(x) - y_1\| < \varepsilon, \quad \|\nabla g_2(x) - y_2\| < \varepsilon.$$

Therefore:

$$\|\nabla g_t(x) - (ty_1 + (1-t)y_2)\| < \varepsilon$$

and $ty_1 + (1-t)y_2 \in \partial f_t$ by our assumption. Thus:

$$\nabla g_t(x) = t\nabla g_1(x) + (1-t)\nabla g_2(x) \in B_\varepsilon\left(\partial f_t(x_i)\right).$$

Finally, from (5) and (7), we have:

$$\|x - x_i\| \leq \|x - \tilde{x}\| + \|\tilde{x} - x_i\| < \frac{\delta}{2} + \frac{\delta}{2} < \delta < \varepsilon.$$

This ends the proof. □

The above theorem is satisfactory in many applications. In particular, the Hopf property follows from this weak version of the homotopy property.

Theorem 6. *Assume that $\Omega = B(V)$ is the unit ball of an orthogonal representation V and $(f_1, \partial f_1), (f_2, \partial f_2))$ are Ω-admissible pairs such that:*

$$\deg_G^{\nabla}((f_1, \partial f_1), \Omega) = \deg_G^{\nabla}((f_2, \partial f_2), \Omega).$$

Then, the pairs are Ω-admissible homotopic.

Proof. It is enough to observe that if we choose a δ-WT-approximation $(g, \nabla g)$ of $(f, \partial f)$ defining the equivariant degree, then the linear formula:

$$f_t(x) = tg(x) + (1-t)f(x)$$

defines an Ω-admissible homotopy satisfying the assumptions of Theorem 5. Therefore, we apply this to both pairs. Now, we can apply the Hopf property for smooth maps ∇g_i, $i = 1, 2$. The transitivity property of the homotopy relation for single-valued maps ends the proof. □

We are ready to prove the full homotopy property now.

Theorem 7. *Let $(f_t, \partial f_t), t \in [0,1]$ be an Ω-admissible gradient G-homotopy. Then:*

$$\deg_G^{\nabla}((f_0, \partial f_0), \Omega) = \deg_G^{\nabla}((f_1, \partial f_1), \Omega).$$

Proof. First, we observe that because of the continuity of the homotopy with respect to $t \in [0,1]$, for each $\varepsilon > 0$ and every fixed t, there exist a $\delta(t) > 0$ and a Whitney approximation $(g_t, \nabla g_t)$ of $(f_t, \partial f_t)$, which is also an ε-Whitney approximation of $(f_\tau, \partial f_\tau)$ for $\tau \in [t - \delta(t), t + \delta(t)]$. Therefore, $(g_t, \nabla g_t)$ is linearly homotopic to $(f_{t-\delta}, \partial f_{t-\delta})$, as well as to $(f_{t+\delta}, \partial f_{t+\delta})$. Thus, by Theorem 5, the gradient degree is constant in the interval $[t - \delta(t), t + \delta(t)]$. Now, we consider the open covering of $[0,1]$ by the intervals $(t - \delta(t), t + \delta(t))$ (for $t = 0, 1$, we put half-open intervals). We find a finite subcovering and apply the transitivity of the homotopy property to finish the proof. □

3. Discussion and Conclusions

The main technical result of this paper is contained in Theorem 2. It says that for a symmetric (G-invariant, where G acts orthogonally on $\mathbb{R}^n$) and locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, there exists an arbitrarily fine smooth and G-invariant uniform approximation g of f such that the gradient ∇g is a graph approximation of Clarke's generalized gradient ∂f. This is the equivariant version of the result from [19]. It is known that a version of the Whitney smooth approximation theorem is valid not only in finite-dimensional spaces, but also in Hilbert spaces. However, the presented proof does not work in infinite dimensions because it uses the compactness of closed balls. Moreover, the generalized Clarke gradient is only a weakly upper semicontinuous map with bounded closed values in general. Thus, it as an open question if a similar approximation theorem is true in general (it is also an open problem in the non-equivariant case).

As an application, we developed a gradient equivariant degree theory for the class of set-valued equivariant maps, which are generated by locally Lipschitz potentials together with the Hopf-type classification theorem. It should be mentioned that thanks to our approximation theorem, the classes of homotopy equivalence of such maps can be represented by smooth equivariant gradient maps, and these are different from the usual equivariant homotopy classes.

Further applications into nonlinear analysis are expected, for example some multiplicity results for Hamiltonian systems as those considered for C^2-maps in [8,16] and some references therein. By means of an infinite-dimensional extension of our degree theory by Leray–Schauder-type finite-dimensional approximations, we can expect similar results under a bit weaker smoothness assumptions. In search of periodic solutions for differential equations (inclusions), a natural action of the group S^1 is considered besides some additional spatial symmetries as one can see in the book [4] and the references therein. Compact multivalued perturbations of the identity operator in a Hilbert space will be considered, as well as perturbations of some unbounded self-adjoint operators (comp. [20]). We treat our paper as the preliminary step towards this direction.

The authors are grateful to the anonymous reviewers for careful reading and the improvements of the manuscript.

Author Contributions: This is a joint research project of both authors. Conceptualization, Z.D.; methodology, Z.D.; formal analysis, Z.D. and T.G.; investigation, Z.D. and T.G.; writing–original draft preparation, Z.D.; writing–review & editing, T.G.; supervision, Z.D.; project administration, T.G. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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