Generation of random fields to reflect material and geometric imperfections of plates and shells

J. Górski & K. Winkelmann
Gdansk University of Technology, Gdansk, Poland

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ABSTRACT: The paper covers two patterns of random field generation: conditional acceptance – rejection method and Karhunen – Loève expansion. The generation of two-dimensional random fields is essential in plates and shells analysis, allowing for a relevant limit and critical state assessment of geometrically and materially imperfect structures. The features of both generation methods dedicate them to selected problems.

1 INTRODUCTION

The plates and shells analysis makes use of real geometric and material imperfections. Ideal structural models lead to an improper assessment of buckling and limit structural load. On the other hand, virtually assumed imperfections, e.g. natural vibration modes may indicate lowered limit loads. The available measurements and standardized tolerances lead to numerical models reflecting real structures. Therefore random imperfection fields may be generated.

The theoretical background for a random field generation is found in (Adler 1981). The developed methods allow to generate homogeneous and Gaussian fields of numerous applications. However, worldwide literature also proposes methods to generate non-homogeneous and non-Gaussian fields. The paper presents conditional generation method, included in papers (Bielewicz & Gorski 2002, Walukiewicz et al. 1997, Górski 2006), further compared with the widely used Karhunen – Loève expansion (Karhunen 1994, Anders 2000).

2 RANDOM FIELD GENERATION BASED ON CONDITIONAL ACCEPTANCE AND REJECTION METHOD

The conditional acceptance and rejection method of random field generation makes it possible to generate a broad class of 2D and 3D Gaussian random fields. Wide analysis proved this method versatile and applicable for both homogenous and non-homogeneous fields. The tests proved them effective even for fields of advanced Wiener or Braun correlation functions. The algorithm does not specify the field bounds. It is presented in (Górski 2006).

The method employs a discrete random field in the form of multivariate random vectors defined in a number of m grid points. Thus a random vector $\mathbf{X}_{(m\times 1)}$ of a mean $\bar{\mathbf{X}}_{(m\times 1)}$ represents the field. The field covariance function is represented by a covariance matrix $\mathbf{K}_{(m\times m)}$. Due to the conditional method, the random vector $\mathbf{X}_{(m\times 1)}$ is split into two parts: the known $\mathbf{X}_{k(p\times 1)}$ and the unknown $\mathbf{X}_{u(p\times 1)}$, where $n+p=m$. The unknown vector part is determined on the basis of a conditional distribution, given in (Devroye 1986)

$$f(\mathbf{X}_u/\mathbf{X}_k)=f(\mathbf{X})/f(\mathbf{X}_k) \tag{1}$$

where $f(\mathbf{X}_k)$ is a joint probability density function of random variables $\mathbf{X}_k$.

In engineering applications the random variables are strictly bounded. Thus, a bounded Gaussian variable is applied here (Jankowski & Walukiewicz 1997) in the form

$$t=s\exp\left(-s^2/2\right)\Big/\sqrt{2\pi}\,erf(s) \tag{2}$$

where s denotes the bounding parameter of standard deviation.

Finally, the joint probability density functions of a bounded Gaussian variable takes the form

$$f_t(\mathbf{X}_u/\mathbf{X}_k)=(1-t)^{-m/2}\left(\det\mathbf{K}_c\right)^{-1/2}(2\pi)^{-m/2}\times \exp\left(-\frac{1}{2(1-t)}\left(\mathbf{X}_u-\bar{\mathbf{X}}_c\right)^{\mathrm{T}}\mathbf{K}_c^{-1}\left(\mathbf{X}_u-\bar{\mathbf{X}}_c\right)\right) \tag{3}$$

where $\mathbf{K}_c$ is a conditional covariance matrix, and $\bar{\mathbf{X}}_c$ is a conditional expectation vector

$$\mathbf{K}_c = \mathbf{K}_{11} - \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{21} \tag{4}$$

$$\bar{\mathbf{X}}_c = \bar{\mathbf{X}}_u + \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\left(\mathbf{X}_k - \bar{\mathbf{X}}_k\right) \tag{5}$$

The generation algorithm employs a base field of random variables of a defined range. The generation process makes the base field shift in order to cover the entire generation field. The assumed generation method defines the maximum dimension of a covariance matrix related to the base field dimension.

The first algorithm step generates four starting points. The next points are further generated from the base field. The following stage is multiple-repeated. The base field is shifted, next single points of a field are generated. Note that any subsequent field point *i* gives an unknown X_i while all prior generated variables $X_1 \div X_{i-1}$ are known. The one-dimensional reduction of an algorithm essentially improves the computational efficiency.

The key issue in the generation process is the interval (a_i, b_i), defined separately for a point, bounding the generated variates. The interval is related to the *i*-th point standard deviation

$$\left(\int_{a_i}^{b_i} \left(x_i - \bar{x}_i\right)^2 f\left(x_i\right) dx_i \right)^{1/2} = \sigma_i \tag{6}$$

It is worth highlighting that the conditional random field generation method fits any correlation function type, homogeneous or non-homogeneous, of strong or weak correlation. The field envelope introduction opens the method to highly developed boundary conditions (constraints), due to considered structural problems. The field envelope makes it possible to generate random fields defined by correlation functions adjusted to experimental data, e.g. real structural geometric imperfections. While the method is working for Gaussian symmetric fields, the introduction of non-symmetric field envelopes allows the non-symmetric field generation as well.

Random field generation based on the bounded base field makes it possible not to limit the field dimension. The generation block, the base field, is adjustable to cylindrical surfaces, e.g. tanks, of a closed-type random field. This is the most essential and unique feature of the method, as compared to other numerical tools.

3 RANDOM FIELD GENERATION BY KARHUNEN – LOÈVE EXPANSION

The random field generation software commonly applies orthogonal Karhunen – Loève approach (Karhunen 1947, Anders 2000)). The expansion applies the three-dimensional decomposition of a second-order random field covariance function. The field generation takes the form

$$w = \bar{w}(1 + \alpha) \tag{7}$$

where *w* denotes a random variable, $\bar{w}$ denotes its mean value, and α – its variation.

The Karhunen theorem (1947) states (Adler 1981, Anders 2000) that for a second-order random field of a zero mean value and a correlation function $C_\alpha(\mathbf{x}_1, \mathbf{x}_2)$ the following relation is proper

$$\alpha(\mathbf{x}, \omega) = \sum_{m=1}^{\infty} \sqrt{\lambda_m}\, \xi_m(\omega)\, \varphi_m(\mathbf{x}) \tag{8}$$

$$C_\alpha(\mathbf{x}_1, \mathbf{x}_2) = \sum_{m=1}^{\infty} \lambda_m\, \varphi_m(\mathbf{x}_1)\, \varphi_m(\mathbf{x}_2) \tag{9}$$

$$\lambda_m \varphi_m(\mathbf{x}_1) = \int_V C_\alpha(\mathbf{x}_1, \mathbf{x}_2)\, \varphi_m(\mathbf{x}_2)\, d\mathbf{x}_2, \quad m = 1, 2, 3, \ldots \tag{10}$$

where $\varphi_1, \varphi_2, \ldots$ are the subsequent eigenfunctions of $C_\alpha(\mathbf{x}_1, \mathbf{x}_2)$, they are forming an orthogonal base $(\varphi_m, \varphi_n) = \delta_{mn}$, $m, n = 1, 2, 3, \ldots$, and $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of the functions $(\lambda_1 \geq \lambda_2 \geq \ldots > 0)$.

The functions $\xi_1, \xi_2, \ldots$, may be presented in the form

$$\xi_m(\omega) = \frac{1}{\sqrt{\lambda_m}} \int_V \alpha(\mathbf{x}, \omega)\, \varphi_m(\mathbf{x})\, d\mathbf{x} \tag{11}$$

The Karhunen – Loève expansion is basically intended for the Gaussian field $\alpha(\mathbf{x}, \omega)$ generation. The random variables $\xi_1, \xi_2, \ldots$ are assumed to be Gaussian as well.

In order to compare both methods a simplified procedure was proposed by (Ghanem & Spanos 1991). It assumes a homogeneous correlation function for the field (Anders 2000)

$$C_\alpha(x_2 - x_1,\, y_2 - y_1) = \sigma_\alpha^2 \left(-\frac{|x_2 - x_1|}{b_x} - \frac{|y_2 - y_1|}{b_y} \right) \tag{12}$$

where σ_α is the standard deviation, whereas the parameters b_x and b_y are the correlation ranges.

The defined function is not isotropic. Moreover, in the case $b_x = b_y = b$ while the point position is marked by a unit vector (n_x, n_y), the formula should be replaced by a new one (Anders 2000)

$$C_\alpha(s) = \sigma_\alpha^2 \left(-\frac{|s|}{b}\left(|n_x| + |n_y|\right) \right) \tag{13}$$

The Wiener – Khintchine relations (Vanmarcke 1983) specify a spectral density of a homogeneous correlation function C_α by means of

$$S(v_x, v_y, v_z) = \sigma_\alpha^2 \frac{b_x b_y b_z}{\pi^3 (b_x^2 v_x^2 + 1)(b_y^2 v_y^2 + 1)(b_z^2 v_z^2 + 1)} \tag{14}$$

Assuming the solution in a rectangular domain

$$A = \left[-\frac{l_x}{2}, \frac{l_x}{2} \right] \times \left[-\frac{l_y}{2}, \frac{l_y}{2} \right] \tag{15}$$

the final two-dimensional problem can be solved as one-dimensional. The solution is presented in (Ghanem & Spanos 1981) and may be presented as

$$\varphi_i^s = \left[\frac{l_s}{2} \left(1 + \frac{\sin(w_i^s l_s)}{w_i^s l_s} \right) \right]^{-\frac{1}{2}} \cos(w_i^s t), \quad t \in [-\frac{l_s}{2}, \frac{l_s}{2}], \; i = 1,3,5,... \tag{16}$$

$$\varphi_i^s = \left[\frac{l_s}{2} \left(1 + \frac{\sin(w_i^s l_s)}{w_i^s l_s} \right) \right]^{-\frac{1}{2}} \cos(w_i^s t), \quad t \in [-l_s/2, -l_s/2], \; i = 1,3,5,... \tag{17}$$

$$\lambda_i^s = \frac{2b_s^2}{1 + b_s^2 (w_i^s)^2}, \quad i = 1,2,3,... \tag{18}$$

where the parameters $w_i^s, i = 1,2,3,...$ come from the solution of

$$\left(\tan w_i^s \frac{l_s}{2} + w_i^s b_s \right) \left(\tan w_i^s \frac{l_s}{2} - \frac{1}{w_i^s b_s} \right) = 0 \tag{19}$$

When $l_x = l_y$ and $b_x = b_y$

$$\varphi(x,y,z) = \frac{1}{\sqrt{2}} \left[\varphi_i^x(x) \varphi_j^x(y) + \varphi_j^x(x) \varphi_i^x(y) \right] \tag{20}$$

4 AN EXAMPLE OF RANDOM FIELD SIMULATION BY MEANS OF THE PRESENTED GENERATION TECHNIQUES

The following example compares the results of both generation methods. The 10th field was taken for the computations, taking $l_x = l_y = 1.0$ m, $b_x = b_y = 1.0$ m, $\sigma_\alpha = 1.0$ m and assuming the prescribed disretization into 20 and 40 elements

Thus, the dimensions of a random field grid are $\Delta_x = \Delta_y = 1.0/20 = 0.05$ [m] and $1.0/40 = 0.025$ [m]. The dimension of a covariance matrix for both cases covers 441 × 441 and 1681 × 1681 elements, respectively. The assumed number of series expansion terms is $m = 4$. Each element grid was generated using a number of 500 realizations.

The corresponding computations employing identical numerical data of a generated field were conducted with the use of the conditional rejection and acceptance method. Two generation variants were further compared.

Figure 1 compares graphically diagonal elements of a theoretical covariance matrix of 441 × 441 elements with the generation results of both methods. Figure 2 compares in detail the first row elements of the matrix.

Figure 3 compares graphically diagonal elements of a theoretical covariance matrix of 1681 × 1681

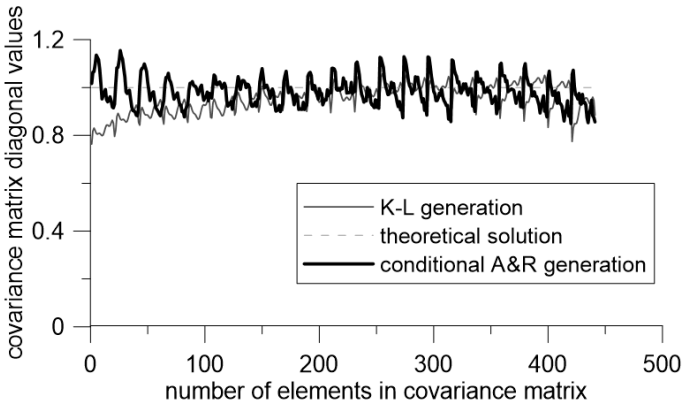

Figure 1. Comparison of diagonal elements of a covariance matrix (441 × 441 elements).

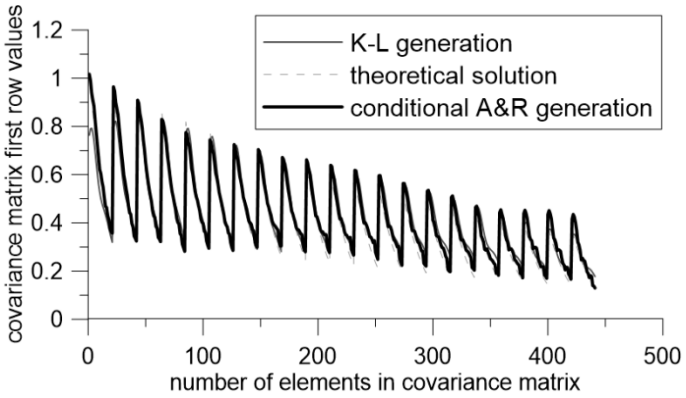

Figure 2. Comparison of the first row elements of a covariance matrix (441 × 441 elements).

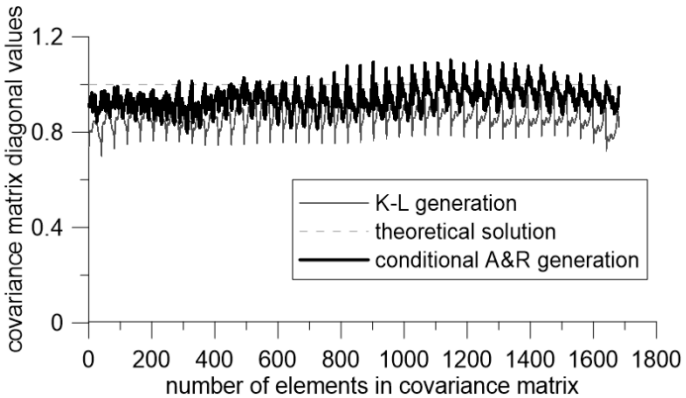

Figure 3. Comparison of diagonal elements of a covariance matrix (1681 × 1681 elements).

elements with the generation results of both methods. Figure 4 compares in detail the first row elements of the matrix.

Additionally the global error of the covariance matrix was calculated (Górski 2006)

$$G_{bl} = \frac{\left(\|\mathbf{K}\| - \|\hat{\mathbf{K}}\|\right)}{\left(\|\mathbf{K}\|\right)} \times 100\% \tag{21}$$

where $\hat{\mathbf{K}}$ is the covariance matrix estimator

$$\hat{\mathbf{K}} = \frac{1}{NR-1}\sum_{i=1}^{NR}(\mathbf{w}_i - \hat{\mathbf{w}})(\mathbf{w}_i - \hat{\mathbf{w}})^{\mathrm{T}} \tag{22}$$

and $\hat{\mathbf{w}}$ is the expected value vector estimator

$$\hat{\mathbf{w}} = \frac{1}{NR}\sum_{i=1}^{NR}\mathbf{w}_i \tag{23}$$

In the case of 20 × 20 field elements the global covariance matrix relative error is 1.76% for conditional generation and 0.08% for Karhunen – Loève generation, respectively.

The corresponding variance estimation errors are 1.59% and 11.37%, respectively.

In the case of 40 × 40 field elements the respective errors are 9.39% and 0.06% for global covariance matrices, 5.94% and 0.06% for variance errors.

5 CONCLUSIONS

The errors of field generation are comparable for both methods. The generation time is also similar, because full covariance matrices are not covered here, no time-consuming computations, e.g. eigenvalue problem or full matrix inversion, occur in the procedure.

The conditional generation method is more general, capturing the engineering cases. It has been applied for the liquid fuel storage tanks (Górski & Mikulski 2008), silos (Górski et al. 2015), granular material behaviour (Tejchman & Górski 2009), composite structures (Winkelmann & Górski 2014, Winkelmann & Sabik 2014). The three-dimensional random field generation was conducted in (Przewłócki & Górski 2001).

The ongoing research is aimed at expanding the method to cover any random field, including the non-symmetric one. The envelopes are of special interest here, allowing to properly reflect the structural boundary conditions.

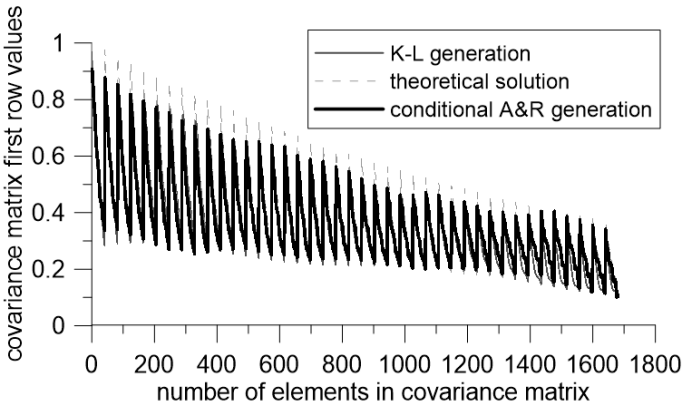

Figure 4. Comparison of the first row elements of a covariance matrix (1681 × 1681 elements).

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