Discrete Mathematics 339 (2016) 1861–1870

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Global defensive sets in graphs[☆,☆☆]

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ARTICLE INFO

Article history:
Received 28 November 2014
Accepted 28 January 2016
Available online 3 March 2016

Keywords:
Defensive set
Alliance
Domination
$\mathcal{NP}$-completeness

ABSTRACT

In the paper we study a new problem of finding a minimum global defensive set in a graph which is a generalization of the global alliance problem. For a given graph G and a subset S of a vertex set of G, we define for every subset X of S the predicate $SEC(X) = true$ if and only if $|N[X] \cap S| \geq |N[X] \setminus S|$ holds, where $N[X]$ is a closed neighbourhood of X in graph G. A set S is a *defensive alliance* if and only if for each vertex $v \in S$ we have $SEC(\{v\}) = true$. If S is also a dominating set of G (i.e., $N[S] = V(G)$), we say that S is a *global defensive alliance*.

We introduce the concept of defensive sets in graph G as follows: set S is a *defensive set* in G if and only if for each vertex $v \in S$ we have $SEC(\{v\}) = true$ or there exists a neighbour u of v such that $u \in S$ and $SEC(\{v, u\}) = true$. Similarly, if S is also a dominating set of G, we say that S is a *global defensive set*. We also study the problems of total dominating alliances (*total alliances*) and total dominating defensive sets (*total defensive sets*), i.e., S is a dominating set and the induced graph $G[S]$ has no isolated vertices.

In the paper we proved the $\mathcal{NP}$-completeness for planar bipartite subcubic graphs of the decision versions of the following minimalization problems: a global and total alliance, a global and total defensive set. We proposed polynomial time algorithms solving in trees the problem of finding the minimum total and global defensive set and the total alliance. We obtained the lower bound on the minimum size of a global defensive set in arbitrary graphs and trees.

1. Introduction

1.1. Problem definition

In the following we consider solely simple nonempty graphs and follow the standard notation of the graph theory. For a given simple graph $G = (V, E)$ and a subset S of the vertex set $V(G)$, we define for any non-empty subset X of S the predicate $SEC_S(X) = true$ if and only if $|N[X] \cap S| \geq |N[X] \setminus S|$ holds, where $N[X]$ is a closed neighbourhood of X in graph G, i.e., $N[X] = X \cup N(X)$, where $N(X) = \{v \in V(G) : \exists_{u \in X}\{v, u\} \in E(G)\}$ is the open neighbourhood of X. In the following we will use the notation $SEC(X)$ instead of $SEC_S(X)$ if set S is clearly given. By $G[A]$, where $A \subset V(G)$, we mean a subgraph of G

☆ This project has been partially supported by Narodowe Centrum Nauki under contract DEC-2011/02/A/ST6/00201.

☆☆ The results of this paper were presented at the Seventh Cracow Conference on Graph Theory (September 14–19, 2014, Rytro, Poland).

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http://dx.doi.org/10.1016/j.disc.2016.01.022

induced by set A. By $n(G)$ we denote the number of vertices of G, i.e., $n(G) = |V(G)|$. For the sake of notation simplicity, we will write $N[v]$ and $N[v, u]$ instead of $N[\{v\}]$ and $N[\{v, u\}]$, respectively. Analogously, we will write $SEC(v)$ and $SEC(v, u)$. By a subcubic graph G we mean a graph with the maximum degree of a vertex bounded by 3 (i.e., $\Delta(G) \le 3$).

Definition 1. Set S is a *defensive set* in G if and only if for each vertex $v \in S$ we have $SEC(v) = true$ or there exists a neighbour $u \in S$ of v (i.e., $\{v, u\} \in E(G)$) such that $SEC(v, u) = true$. If S is also a dominating set of G (i.e., $N[S] = V(G)$), we say that S is a *global defensive set*.

By $ds(G)$ we denote the size of the minimum defensive set in G, and by $\gamma_{ds}(G)$ we denote the size of the minimum global defensive set in G.

1.2. Alliances vs. defensive sets

A set S is a *defensive alliance* (or alliance) if and only if for each vertex $v \in S$ we have $SEC(v) = true$. If S is also a dominating set of G, we say that S is a *global defensive alliance* (or global alliance). By $\gamma_a(G)$ we mean the size of the minimum global alliance in G.

The concept of alliances in graphs was introduced in two conference papers: [10] and [8], where the authors defined and studied the problem of alliances and global alliances in graphs, respectively. The problem attracted the attention of researchers due to certain interesting applications in web communities [6,13] or fault-tolerant computing [17,23].

In [9], which was the first paper on global alliances (i.e., global defensive alliances), the authors proved bounds on the minimum global alliance for general graphs (lower bounds: $\frac{\sqrt{4n+1}-1}{2}$ and $\frac{n}{\lceil\frac{\Delta}{2}\rceil+1}$, upper bound: $n - \lceil\frac{\delta}{2}\rceil$), for bipartite graphs (lower bound: $\lceil\frac{2n}{\Delta+3}\rceil$), and trees (lower bound: $\frac{n+2}{4}$, upper bound: $\frac{3n}{5}$), where n is the number of vertices of a graph, and δ and Δ its minimum and maximum degree, respectively. In [19] the authors proved two lower bounds for general graphs: $\lceil\frac{2n}{\Delta+3}\rceil$ and $\lceil\frac{n}{\lambda+2}\rceil$, where λ is the spectral radius of the graph. The lower bounds on the minimum global alliance for planar graphs were given in [5] and [18], where the authors independently proved the lower bounds $\lceil\frac{n+6}{6}\rceil$ and $\lceil\frac{n+12}{8}\rceil$ $(n > 6)$, respectively. In [18] the authors proved the lower bound for triangle-free planar graphs $\lceil\frac{n+8}{6}\rceil$ $(n > 6)$. The lower bound $\lceil\frac{2m}{\Delta_1+\Delta_2+1}\rceil$ for line graphs is given in [22], where m is the number of vertices of $L(G)$ (i.e., the number of edges of G), Δ_1 and Δ_2 are two maximal degrees in graph G. The lower and upper bounds for the Cartesian product of paths and cycles are given in [4]. For more bounds on trees, see [1] and [2]. The exact values of the minimum global alliance were given in [9] for complete graphs, complete bipartite graphs, cycles, paths and wheels, for k-ary trees ($k = 2, 3, 4$) in [4], and independently in [7] (for $k = 2, 3$), and for star graphs in [12]. In [3] the authors proved the $\mathcal{NP}$-completeness of the global alliance problem for general graphs, and in [14] the author proved it even for bipartite or chordal graphs. A quite comprehensive survey of results concerning defensive alliances can be found in [24]. A certain sort of generalization of alliances are the so-called k-alliances, which were studied in [21], whereas global defensive k-alliances were studied in [20].

The concept of defensive sets arises from the concept of alliances, but is a kind of relaxation of the alliance problem. It is a straightforward observation that any (global) alliance is a (global) defensive set. The converse is not true, as shown in Figs. 1 and 2.

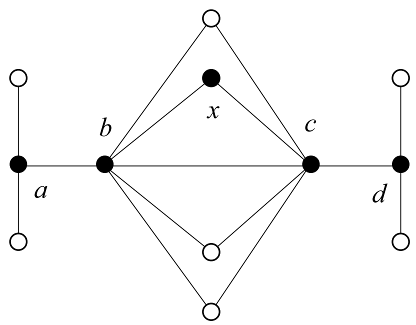

Fig. 1. Dominating set $\{a, b, c, d, x\}$ is a minimum global alliance: $SEC(a) = SEC(b) = SEC(c) = SEC(d) = SEC(x) = true$.

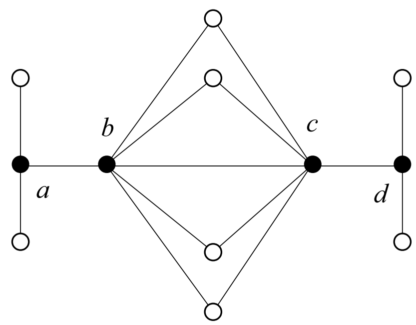

Fig. 2. Dominating set $\{a, b, c, d\}$ is a minimum global defensive set: $SEC(a) = SEC(d) = SEC(b, c) = true$, note that $SEC(b) = SEC(c) = false$.

In the alliance problem at most one vertex can be attacked at the moment. In the defensive set problem, the vertex being under attack (say x) can be defended by itself and its neighbours, otherwise, one of its neighbours (say a) joins 'the war', i.e., the attack can be simultaneously done on two vertices (x and a), and in that case each attack on x and a must be defended. This situation is depicted in Figs. 3 and 4, where white vertices are attacking vertices and black vertices form a defensive set.

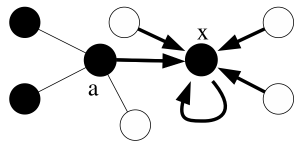

Fig. 3. There is an attack on vertex x that cannot be defended by x and a ($SEC(x) = false$).

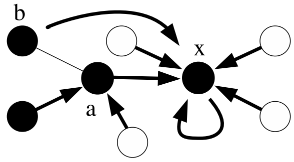

Fig. 4. Each simultaneous attack on x and a can be defended, which is equivalent to $SEC(x, a) = true$.

1.2.1. Total alliances and total defensive sets

Set $X \subset V(G)$ is a total dominating set in a nonempty connected graph G if and only if $N(X) = V(G)$ (every vertex of G is adjacent to some vertex in S), or equivalently, graph $G[X]$ has no isolated vertices and X is a dominating set. The total domination problem is a well-studied graph concept, we refer the reader to [11] for a comprehensive survey of recent results.

Definition 2. An alliance S is a *total alliance* if and only if S is a total dominating set in G. A defensive set S is a *total defensive set* if and only if S is a total dominating set in G.

By $\gamma_{ta}(G)$ and $\gamma_{tds}(G)$ we denote the size of the minimum total alliance and total defensive set, respectively. An example graph with the minimum global alliance (defensive set) smaller than the minimum total alliance (defensive set) is shown in Figs. 5 and 6, where white vertices are attacking vertices and black vertices form an alliance (defensive set). From the definitions we have $\gamma_{ds}(G) \le \gamma_a(G)$, $\gamma_{ds}(G) \le \gamma_{tds}(G)$ and $\gamma_a(G) \le \gamma_{ta}(G)$.

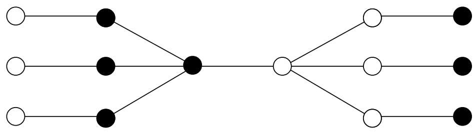

Fig. 5. $\gamma_{ds} = \gamma_a = 7$.

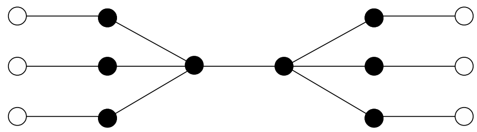

Fig. 6. $\gamma_{tds} = \gamma_{ta} = 8$.

1.3. Edge alliances

In Definition 1, if $SEC(v) = false$ for some $v \in S$, then we require to satisfy $SEC(v, u) = true$ for some $u \in N(v)$. This concept can be restricted in a natural way to the situation when for each edge $e = \{v, u\} \in E(G[S])$ we have $SEC(v, u) = true$. This leads us to a new concept of *edge alliance* in a graph. Let us introduce the definition formally.

Definition 3. Set S is an *edge alliance* if and only if $G[S]$ has no isolated vertices and for each edge $e = \{v, u\} \in E(G[S])$ we have $SEC(v, u) = true$. A set S is a *global edge alliance* if it is also a dominating set of G.

By $\gamma_{ea}(G)$ we denote the size of the minimum edge alliance. Observe that a global edge alliance S is a total dominating set (i.e., $N(S) = V(G)$). What follows from the definitions of total alliance and total defensive set is that each total alliance is a total defensive set, and each global edge alliance is a total defensive set. Figs. 7–9 illustrate some example graphs.

The concept of edge alliances was widely studied in [16], where the authors proved the $\mathcal{NP}$-completeness for bounded degree graphs, showed polynomial time algorithms for some classes (e.g., for trees), and proved some bounds on the minimum global edge alliance number for general graphs and trees.

1.4. Our results

In the paper we study the problems related to the alliance problem. We proved that the following problems are $\mathcal{NP}$-complete for subcubic bipartite planar graphs: a global alliance, a total alliance, a global defensive set and a total defensive set. We constructed polynomial time algorithms solving in trees the problem of finding the minimum total and global defensive set and the total alliance. We obtained a lower bound on the minimum size of a global defensive set in arbitrary graphs and trees.

2. $\mathcal{NP}$-completeness results for subcubic bipartite planar graphs

In this section we will prove the $\mathcal{NP}$-completeness for subcubic bipartite planar graphs of the minimalization problems: a global alliance, a global defensive set, a total alliance and a total defensive set. Each of these problems is a decision problem defined as follows: for a graph G and $k > 0$ we ask if there is $S \subset V(G)$ of at most k vertices satisfying an adequate property. Our reduction is from the total domination problem for subcubic bipartite planar graphs, which is proved to be $\mathcal{NP}$-complete [15].

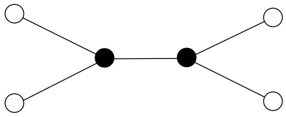

Fig. 7. $\gamma_{tds} = \gamma_{ta} = 2 < \gamma_{ea} = 3$.

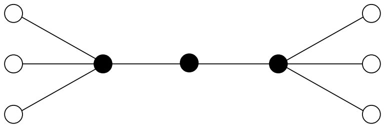

Fig. 8. $\gamma_{tds} = \gamma_{ea} = 3 < \gamma_{ta} = 5$.

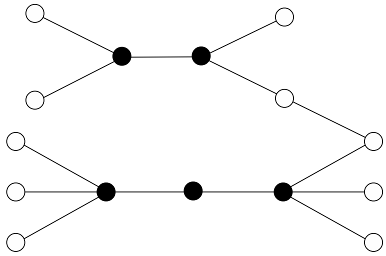

Fig. 9. $\gamma_{tds} = 5 < \gamma_{ea} = 6 < \gamma_{ta} = 7$.

Theorem 1 ([15]). *The total domination problem for subcubic bipartite planar graphs is $\mathcal{NP}$-complete.* □

Proposition 1. *Let G be a subcubic graph and $S \subset V(G)$. Then, S is a total dominating set of G if and only if S is a total alliance in G.*

Proof. Obviously, if S is a total alliance, then S is a total dominating set. Let S be a total dominating set and let $v \in S$. Since $|N[v] \cap S| \geq 2$, by $deg(v) \leq 3$ we have that $|N[v] \cap S| \geq |N(v) \setminus S|$. Thus, $SEC(v) = true$. □

Proposition 2. *Let G be a subcubic graph and $S \subset V(G)$. Then S is an alliance in G if and only if S is a defensive set in G.*

Proof. Obviously, if S is an alliance, then S is a defensive set. Let S be a defensive set and let $v \in S$. If $deg(v) = 1$, then $SEC(v) = true$. If $deg(v) \geq 2$, then $|N[v] \cap S| \geq 2$ and, analogously, $SEC(v) = true$. Thus, S is an alliance. □

By Propositions 1 and 2 as well as by Theorem 1 we have the following

Theorem 2. *The problems of total alliance and total defensive set for subcubic bipartite planar graphs are $\mathcal{NP}$-complete.* □

By Theorem 1 we prove the following

Theorem 3. *The problem of global alliance for subcubic bipartite planar graphs is $\mathcal{NP}$-complete.*

Proof. We construct a polynomial time reduction from the total domination problem to the problem of global alliance. For a given subcubic bipartite planar graph G with p leaves (i.e., pendant vertices) and $n(G) \geq 3$, we construct a graph G' as follows: for every leaf $v \in V(G)$ we attach a gadget H, as shown in Fig. 10. Let $L(G) = \{v \in V(G) : deg(v) = 1\}$ be a set of all pendant vertices of G. In the following all neighbourhoods N are defined in G. Since $n(G) \geq 3$, for any $w \in N(v)$, where $v \in L(G)$, we have that $|N(w)| \geq 2$.

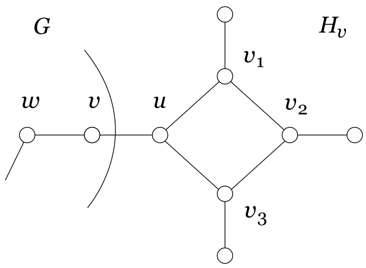

Fig. 10. H_v is attached to a pendant vertex v.

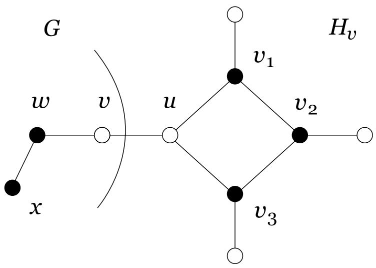

Fig. 11. v is not in an alliance, w, x are in an alliance.

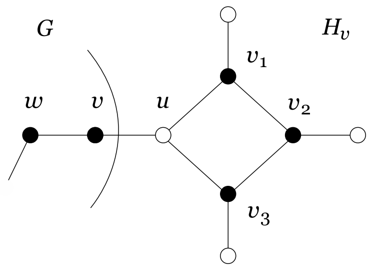

Fig. 12. Both vertices v and w are in an alliance.

We prove that there is a total dominating set U of G of at most k vertices if and only if there is a global alliance A in G' such that $|A| \leq k + 3p$, where $p = |L(G)|$.

($\Rightarrow$) Let $U \subset V(G)$ be a total dominating set of G such that $|U| \leq k$. By Proposition 1 we have that U is a global alliance in G. Let $A = U \cup \bigcup_{v \in L(G)} \{v_1, v_2, v_3\}$ (see Fig. 10). Hence, for any $v \in L(G)$ we get two possibilities, as shown in Figs. 11 and 12, where black vertices in G are from U. Obviously, we have $SEC(v_i) = true$, for $i = 1, 2, 3$, and since $\{v_1, v_2, v_3\}$ dominates H_v, we get that A is a global alliance in G' and $|A| \leq k + 3p$.

($\Leftarrow$) Let A be a global alliance in G' such that $|A| \leq k + 3p$. Since A is dominating set of G', we have $|A \cap V(H_v)| \geq 3$, for every $v \in L(G)$. Moreover, if for some $v \in L(G)$, vertex u is a neighbour of v in G' such that $u \in A \cap V(H_v)$, then $|A \cap V(H_v)| \geq 4$.

For any $v \in L(G)$ and any of its neighbours u such that $u \in A \cap V(H_v)$, let us define a local replacement $A' = A \cap (V(G') \setminus V(H_v)) \cup \{w\} \cup \{v_1, v_2, v_3\}$. If $v \notin A$, then $x \in A$, for some $x \in N(w)$, where $w \in N(v)$. Fig. 11 illustrates set A' after the local replacement operation. If $v \in A$, then set A' after the local replacement is depicted in Fig. 12. Obviously, A' is a global alliance and since $|A \cap V(H_v)| \geq 4$, $|A'| \leq |A| \leq k + 3p$.

Now, let us assume that for all $v \in L(G)$ we have $u \notin A'$, possibly after a certain number of successive local replacement operations, as previously described. Let us define a global transformation $A'' = A' \cap V(G) \cup \bigcup_{v \in L(G)} \{v_1, v_2, v_3\}$. Obviously, A'' is a global alliance and $|A''| \leq |A'| \leq k + 3p$.

Let $U = A'' \cap V(G)$. Since A'' is a dominating set of G', U is a dominating set of G. For any pendant vertex $v \in V(G)$, if $v \in U$, then $w \in U$, where $w \in N(v)$, as shown in Fig. 12. Let $v \in U$ and $deg(v) \geq 2$. Since $SEC(v) = true$, $|N[v] \cap U| \geq |N[v] \setminus U|$. Hence, there is a neighbour of v belonging to U. Thus, U is a total dominating set of G and $|U| \leq k$. □

By Proposition 2 we have the following

Theorem 4. *The problem of global defensive set for subcubic bipartite planar graphs is $\mathcal{NP}$-complete.* □

3. Polynomial time algorithms for trees

In [4] the authors constructed $O(n \log \Delta)$-time algorithm for finding the minimum global alliance in trees. In this section we present polynomial time algorithms for finding a minimum total alliance, a minimum global defensive set and a minimum total defensive set using the bottom-up technique.

3.1. General sketch of the algorithms

We construct the optimal solution for a given tree T using the bottom-up technique in accordance with the defined orientation of T. First, we orient all edges of T in an *in-tree* manner with a leaf as root, i.e., we choose any leaf r as root and orient all edges of tree T towards the root r. As a result, for each vertex $v \in V(T) \setminus \{r\}$, there is exactly one oriented edge outcoming from a vertex v towards r, let us denote this edge by $e_v = \{v, r_v\}$. By T_v we denote a subtree of T rooted at v and consisting of all (oriented) edges that lead to vertex v. By T_v^* we mean a tree T_v with an attached edge e_v, i.e., $T_v^* = T_v \cup e_v$. Let $p = deg(v) - 1$ and let $N_v^b = \{v_1, \dots, v_p\}$ be a set of vertices adjacent to v and different from r_v.

The key idea of the approach is to use the recursive schema, in which we build a data structure A_v, related to the vertex v, from data structures $A_{v_1}, \dots, A_{v_p}$ related to the children of vertex v. We will use some auxiliary data structures (B_v) to clarify the process of building A_v from $A_{v_1}, \dots, A_{v_p}$. It is important to ensure that one can apply the data structures associated with all children of vertex v to build A_v. The algorithm makes use of the *bottom-up* technique, as follows:

1. Starting from leaves, first build A_v, and go towards root r.
2. Traversing tree T for each vertex $v \neq r$:
   (i) construct an *auxiliary* data structure B_v using $A_{v_1}, \dots, A_{v_p}$,
   (ii) construct A_v from B_v.
3. Use A_s, where s is the only neighbour of root r, to find an optimal solution for the problem in tree T.

The total time complexity of the algorithm depends on the time complexity of the construction of structures A_v and B_v, and we derive it for a particular problem. In fact, by this schema we calculate in most cases the size of the optimal solution. The construction of an optimal solution may be possible by using additional data structures for remembering the appropriate information while building structures A_v and B_v, which, however, does not change the time complexity of the algorithm.

3.2. $O(n \log \Delta)$-time algorithm for finding a minimum total alliance

In this section we construct data structures A_v, B_v^0 and B_v^1 for the algorithm solving the problem of finding a minimum total alliance in a tree, which is a slight generalization of the algorithm for global alliances in trees. In the following we use the symbol ∞ to denote some illegal cases, and we assume $\infty \geq a$, $\infty \pm a = \infty$ and $\min\{\infty, a\} = a$, where a is a number or ∞. Let $v \in V(T) \setminus \{r\}$, and $p = \deg(v) - 1$.

Let us formally define matrices A_v, B_v^0 and B_v^1 for this problem as follows:

(A_v) Let A_v be a matrix of the size 2×2 and let $A_v[j, h] = \min\{|S \setminus \{r_v\}| : S \text{ is an alliance in } T_v^* \wedge C_{jh}\}$, for $j, h \in \{0, 1\}$, where C_{jh} is an additional condition defined as follows:
 (00) C_{00} if and only if S is a total alliance in T_v and $v, r_v \notin S$,
 (01) C_{01} if and only if for every $u \in N_v^b$ set $S \cap V(T_u)$ is a total alliance in T_u, and $v \notin S$, $r_v \in S$,
 (10) C_{10} if and only if S is a total alliance in T_v, and $v \in S$, $r_v \notin S$,
 (11) C_{11} if and only if S is a total alliance in T_v^*, and $v, r_v \in S$.
 Let us observe that $j = 1$ if and only if $v \in S$, and $h = 1$ if and only if $r_v \in S$. If $A_v[j, h]$ cannot be legally defined, we preset the value as ∞. If v is a leaf, then by definition we initially put $A_v[0, 0] = \infty$, $A_v[0, 1] = 0$, $A_v[1, 0] = \infty$, $A_v[1, 1] = 1$.

(B_v^j) Let B_v^j (for $j \in \{0, 1\}$) be a matrix of the size $p \times 3$, where for $i = 1, \dots, p$ and $j = 1, 2$ we set $B_v^j[i, 0] = i$ and $B_v^j[i, 1] = 1$, if $A_{v_i}[1, j] > A_{v_i}[0, j]$, otherwise, $B_v^j[i, 1] = 0$, and finally

- if $B_v^j[i,1]=1$, then $B_v^j[i,2]=A_{v_i}[1,j]-A_{v_i}[0,j]$,
- if $B_v^j[i,1]=0$, then $B_v^j[i,2]=A_{v_i}[0,j]-A_{v_i}[1,j]$.

Let us observe that if $A_{v_i}[0,j]=\infty$ or $A_{v_i}[1,j]=\infty$, then $B_v^j[i,2]=\infty$. Let us define $a_v^j=\sum_{i=1}^{p}(1-B_v^j[i,1])$ and $b_v^j=\sum_{i=1}^{p}\min\{A_{v_i}[0,j],A_{v_i}[1,j]\}$. Let us observe that if for some i we have $A_{v_i}[0,j]=A_{v_i}[1,j]=\infty$, then $b_v^j=\infty$. Let $\hat{B}_v^j$ be a matrix obtained from B_v^j by sorting rows $B_v^j[i]$ (for $i=1,\dots,p$) in a non-decreasing order with respect to the third value of the row, i.e., $B_v^j[i,2]$. Thus we get $\hat{B}_v^j[1,2]\le\hat{B}_v^j[2,2]\le\cdots\le\hat{B}_v^j[p,2]$. The construction of matrix $\hat{B}_v^j$ can be done in $O(\deg(v)\log\deg(v))$ time.

Now, we construct matrix A_v from $\hat{B}_v^j$ and b_v^j satisfying the conditions C_{jh}.

(00) We have to ensure that v is dominated by at least one v_i, where $i=1,\dots,p$. If $a_v^0>0$ (v is dominated), then $A_v[0,0]=b_v^0$, otherwise $A_v[0,0]=b_v^0+\hat{B}_v^0[1,2]$ (the minimum cost of dominated v).

(01) Since v is dominated by r_v, just take the best solution: $A_v[0,1]=b_v^0$.

(10) We have to ensure that $SEC(v)=true$ and v is dominated by at least one v_i, where $i=1,\dots,p$. If $a_v^1+1\ge p+1-a_v^1$ (i.e., $2a_v^1\ge p$) ($SEC(v)=true$), then $A_v[1,0]=b_v^1+1$. Otherwise, if $2a_v^1<p$, then let t be the smallest integer such that $2t\ge p-2a_v^1$. Thus, $a_v^1+t+1\ge p+1-a_v^1-t$. Let k be the smallest integer such that $t=\sum_{i=1}^{k}\hat{B}_v^1[i,1]$, and let $c_v=\sum_{i=1}^{k}\hat{B}_v^1[i,2]\cdot\hat{B}_v^1[i,1]$ (the minimum cost of making $SEC(v)=true$). Hence, we put $A_v[1,0]=b_v^1+c_v+1$.

(11) We have to ensure that $SEC(v)=true$. Analogously, if $a_v^1+2\ge p-a_v^1$ (i.e., $2a_v^1+2\ge p$) ($SEC(v)=true$), then $A_v[1,1]=b_v^1+1$. Otherwise, if $2a_v^1+2<p$, then let t be the smallest integer such that $2t\ge p-2a_v^1-2$. Thus, $a_v^1+t+2\ge p-a_v^1-t$. Let k be the smallest integer such that $t=\sum_{i=1}^{k}\hat{B}_v^1[i,1]$, and let $c_v=\sum_{i=1}^{k}\hat{B}_v^1[i,2]\cdot\hat{B}_v^1[i,1]$ (the minimum cost of making $SEC(v)=true$). Hence, we put $A_v[1,1]=b_v^1+c_v+1$.

The construction of matrix A_v can be done in $O(\deg(v)\log\deg(v))$.

Finally, $\gamma_{ta}(T)=\min\{A_s[1,1]+1,A_s[1,0]\}$, where $\{s\}=N(r)$. The time complexity of the algorithm is obviously $O(n\log\Delta)$.

3.3. $O(n\Delta^2\log\Delta)$-*time algorithm for finding a minimum global defensive set*

In this section we construct data structures A_v and B_v for the algorithm solving the problem of finding a minimum global defensive set in a tree. In the following we use the symbol ∞ to denote some illegal cases, and assume $\infty\ge a$, $\infty\pm a=\infty$ and $\min\{\infty,a\}=a$, where a is a number or ∞.

Let $v\in V(T)\setminus\{r\}$, $p=\deg(v)-1$ and $q=\deg(r_v)-1$. Let us define a tree T_v^q obtained from T_v^* by attaching q pendant vertices $L_q=\{u_1,\dots,u_q\}$ to vertex r_v.

In the following, for the sake of notation simplicity, we will use *gds* instead of *global defensive set*. Let us formally define data structure A_v and $B_v=(B_v^0,B_v^1)$ for this problem as follows.

(A_v) Let $A_v=(a_v^{00},a_v^{01},a_v^{10},a_v^{11},A_v^{11})$, where a_v^{jh} is an integer or ∞ (for $j,h\in\{0,1\}$), and A_v^{11} is a matrix of the size $(q+1)\times 1$, defined as follows:

(00) $a_v^{00}=\min\{|S\setminus\{r_v\}|: S \text{ is a } gds \text{ in } T_v\wedge v\notin S\wedge r_v\notin S\}$,

(01) $a_v^{01}=\min\{|S\setminus\{r_v\}|: S \text{ is a } gds \text{ in } T_v^*\wedge v\notin S\wedge r_v\in S\}$,

(10) $a_v^{10}=\min\{|S\setminus\{r_v\}|: S \text{ is a } gds \text{ in } T_v^*\wedge v\in S\wedge r_v\notin S\}$,

(11) $a_v^{11}=\min\{|S\setminus\{r_v\}|: S \text{ is a } gds \text{ in } T_v^*\wedge v\in S\wedge r_v\in S\}$

$A_v^{11}[k]=\min\{|S\setminus(L_q\cup\{r_v\})|: S \text{ is a } gds \text{ in } T_v^q\wedge v\in S\wedge r_v\in S\wedge|L_q\cap S|=k\}$, for any $k=0,\dots,q$.

Let us observe that for a_v^{jh}, we have $j=1$ if and only if $v\in S$, and $h=1$ if and only if $r_v\in S$.

If any $\min(\cdot)$ cannot be legally defined, we preset the value as ∞. If v is a leaf, then by definition we initially put $a_v^{00}=\infty$, $a_v^{01}=0$, $a_v^{10}=1$, $a_v^{11}=1$ and $A^{11}[k]=1$ for $2k+2\ge q$, and $A^{11}[k]=\infty$ for $2k+2<q$.

(B_v^0) Let B_v^0 be a matrix of the size $p\times 3$, where for $i=1,\dots,p$ we set $B_v^0[i,0]=i$ and $B_v^0[i,1]=1$, if $a_{v_i}^{10}>a_{v_i}^{00}$, otherwise, $B_v^0[i,1]=0$, and finally

- if $B_v^0[i,1]=1$, then $B_v^0[i,2]=a_{v_i}^{10}-a_{v_i}^{00}$,
- if $B_v^0[i,1]=0$, then $B_v^0[i,2]=a_{v_i}^{00}-a_{v_i}^{10}$.

Let us observe that if $a_{v_i}^{10}=\infty$ or $a_{v_i}^{00}=\infty$, then $B_v^0[i,2]=\infty$. Let us define $a_v^0=\sum_{i=1}^{p}(1-B_v^0[i,1])$ and $b_v^0=\sum_{i=1}^{p}\min\{a_{v_i}^{10},a_{v_i}^{00}\}$. Let us observe that if for some i we have $a_{v_i}^{10}=a_{v_i}^{00}=\infty$, then $b_v^0=\infty$. Let c_v^0 be the minimum value of $B_v^0[i,2]$ such that $B_v^0[i,1]=1$.

(B_v^1) Let B_v^1 be a matrix of the size $(p+1)\times p\times 6$, where for $k=0,\dots,p$ and $i=1,\dots,p$ we set $B_v^1[k,i,0]=i$, and

(a) if $A_{v_i}^{11}[k]\le a_{v_i}^{01}$ and $A_{v_i}^{11}[k]\le a_{v_i}^{11}$, then $B_v^1[k,i,1]=0$ and $B_v^1[k,i,2]=A_{v_i}^{11}[k]$, moreover we set $B_v^1[k,i,3]=B_v^1[k,i,4]=B_v^1[k,i,5]=0$,

(b) if $a^{11}_{v_i} \le a^{01}_{v_i}$ and $a^{11}_{v_i} < A^{11}_{v_i}[k]$, then $B^1_v[k,i,1] = 0$ and $B^1_v[k,i,2] = a^{11}_{v_i}$, and $B^1_v[k,i,3] = 1$, $B^1_v[k,i,4] = A^{11}_{v_i}[k] - a^{11}_{v_i}$, $B^1_v[k,i,5] = 0$,
(c) if $a^{01}_{v_i} < A^{11}_{v_i}[k]$ and $A^{11}_{v_i}[k] \le a^{11}_{v_i}$, then $B^1_v[k,i,1] = 1$ and $B^1_v[k,i,2] = a^{01}_{v_i}$, and $B^1_v[k,i,3] = 0$, $B^1_v[k,i,4] = A^{11}_{v_i}[k] - a^{01}_{v_i}$, $B^1_v[k,i,5] = 0$,
(d) if $a^{01}_{v_i} < a^{11}_{v_i} < A^{11}_{v_i}[k]$, then $B^1_v[k,i,1] = 1$ and $B^1_v[k,i,2] = a^{01}_{v_i}$, and $B^1_v[k,i,3] = 1$, $B^1_v[k,i,4] = a^{11}_{v_i} - a^{01}_{v_i}$, $B^1_v[k,i,5] = A^{11}_{v_i}[k] - a^{11}_{v_i}$.

Now, we construct A_v from B_v.

(00) We have to ensure that v is dominated by at least one v_i, where $i = 1, \ldots, p$. If $a^0_v > 0$, then $a^{00}_v = b^0_v$, otherwise, $a^{00}_v = b^0_v + c^0_v$.
(01) Since v is dominated by r_v, just take the best solution: $a^{01}_v = b^0_v$.
(10) We have to ensure that vertex v satisfies the defensive set property. For every $k = 1, \ldots, p$, let us define $s_k = \min\{|S \setminus \{r_v\}| : S \text{ is a } gds \text{ in } T^*_v \wedge v \in S \wedge r_v \notin S \wedge |N^b_v \cap S| = k\}$, or $s_k = \infty$, if there is no such S. Obviously, $a^{10} = \min\{s_1, \ldots, s_p\}$.

For $k = 1, \ldots, p$ we calculate s_k or prove that there is $l > k$, such that $s_l \le s_k$. Let $a = \sum_{i=1}^{p}(1 - B^1_v[k-1,i,1])$, and $b = \sum_{i=1}^{p} B^1_v[k-1,i,2]$. We have to ensure that exactly k vertices from $\{v_1, \ldots, v_p\}$ satisfy the defensive set property, and the rest of them is outside the defensive set.

If $a > k$, then it is easy to observe that for some $l \ge a$ we have $s_l \le s_k$. Thus, without loss of generality we can assume that $a \le k$.

If $a = k$, then we have two cases: (1) there is i such that $B^1_v[k-1,i,1] = B^1_v[k-1,i,3] = 0$ (case (a) from the definition of B^1_v), and thus $s_k = b + 1$, (2) for all $i = 1, \ldots, p$ we have $B^1_v[k-1,i,1] = 1$ or $B^1_v[k-1,i,3] = 1$ (case (b), (c) or (d) from the definition of B^1_v). Let $U = \{1, \ldots, p\}$, $c = \min\{B^1_v[k-1,i,4] + B^1_v[k-1,i,5] : i \in U\}$ and $U_c = \{i \in U : B^1_v[k-1,i,4] + B^1_v[k-1,i,5] = c\}$. If for every $i \in U_c$ we have $B^1_v[k-1,i,1] = 1$ (case (c) or (d) in the definition of B^1_v), then $s_{k+1} \le s_k$. Hence, without loss of generality we can assume that for some $i \in U_c$ we have $B^1_v[k-1,i,1] = 0$ and $B^1_v[k-1,i,3] = 1$ (case (b)). Thus, we can put $s_k = b + c + 1$.

If $a < k$, then we have two cases: ($1'$) there is i such that $B^1_v[k-1,i,1] = B^1_v[k-1,i,3] = 0$ (case (a)), or ($2'$) for all $i = 1, \ldots, p$ we have $B^1_v[k-1,i,1] = 1$ or $B^1_v[k-1,i,3] = 1$ (case (b), (c) or (d)). In the first case ($1'$) let $\hat{B}_v$ be a matrix of the size $p \times 6$ obtained from $B^1_v[k-1]$ by sorting rows $B^1_v[k-1,i]$ (for $i = 1, \ldots, p$) in a non-decreasing order with respect to the value $B^1_v[k-1,i,4]$. Thus, we get $\hat{B}[1,4] \le \hat{B}[2,4] \le \cdots \le \hat{B}[p,4]$. The construction can be done in $O(p \log p)$ time. Let k_0 be the smallest integer such that $k - a = \sum_{i=1}^{k_0} \hat{B}_v[i,1]$, and let $c = \sum_{i=1}^{k_0} \hat{B}_v[i,4] \cdot \hat{B}_v[i,1]$. Hence, we put $s_k = b + c + 1$. In the second case ($2'$) if $2k \ge p$ (i.e., $SEC(v) = true$), then we put analogously as in case ($1'$) $s_k = b + c + 1$. Let $k < 2p$. Then, we have to ensure that $SEC(v, v_{i_0}) = true$ for some $v_{i_0} \in N^b_v$. We consider two subcases: ($2'_1$) $B^1_v[k-1,i_0,1] = 0$ and $B^1_v[k-1,i_0,3] = 1$ (case (b)), the minimal additional cost of ensuring $SEC(v, v_{i_0}) = true$ is $c_1 = B^1_v[k-1,i_0,4] + c'_1$, ($2'_2$) $B^1_v[k-1,i_0,1] = 1$ (case (c) or (d)), the minimal additional cost of ensuring $SEC(v, v_{i_0}) = true$ is $c_2 = B^1_v[k-1,i_0,4] + B^1_v[k-1,i_0,5] + c'_2$. In both cases, by c'_l (for $l = 1, 2$) we mean an additional cost of ensuring that exactly k vertices from N^b_v are in a defensive set. Thus, we can put $s_k = \min\{b + c_1 + 1, b + c_2 + 1\}$. In the subcase ($2'_1$) we take any i_0 such that $B^1_v[k-1,i_0,4] = \min\{B^1_v[k-1,i,4] : i \in \{1, \ldots, p\} \wedge B^1_v[k-1,i,1] = 0\}$. Let $c'_1 = \sum_{i=1}^{k_0} \hat{B}_v[i,4] \cdot \hat{B}_v[i,1]$ be calculated analogously as in case ($1'$). Let $U_0 = \{i \in \{1, \ldots, p\} : B^1_v[k-1,i,1] = 1\}$. In the subcase ($2'_2$) we have to find a subset $U \subset U_0$, $|U| = k - a$, and a vertex $i_0 \in U$ such that the sum $\sum_{i \in U} B^1_v[k-1,i,4] + B^1_v[k-1,i_0,5]$ is minimized. For every $t \in U_0$ we calculate $c^t_2 = B^1_v[k-1,t,4] + B^1_v[k-1,t,5] + \sum_{i=1,i\ne t}^{k_0} \hat{B}_v[i,4] \cdot \hat{B}_v[i,1]$, where k_0 is the smallest integer such that $k - a - 1 = \sum_{i=1,i\ne j}^{k_0} \hat{B}_v[i,1]$, and $\hat{B}_v$ is constructed analogously as in case ($1'$). Finally, $c_2 = \min\{c^j_2 : j = 1, \ldots, p\}$. Thus, we constructed a^{10}_v and the construction can be done in $O(p^2 \log p)$ time.
(11) The construction of a^{11}_v is analogous as in case (10).

Now, for any $l = 0, \ldots, q$ we construct $A^{11}_v[l]$ in time $O(p^2 \log p)$.

We have to ensure that vertex r_v satisfies the defensive set property (i.e., $SEC(r_v) = true$ or $SEC(v, r_v) = true$) and vertex v satisfies one of the following: $SEC(v) = true$, $SEC(v, r_v) = true$ or $SEC(v, v_i) = true$ for some $i \in \{1, \ldots, p\}$. The proof goes analogously as in case (10): for every $k = 1, \ldots, p$, let us define $s_k = \min\{|S \setminus \{r_v\}| : S \text{ is a gds in } T^q_v \wedge v \in S \wedge r_v \in S \wedge |N^b_v \cap S| = k - 1 \wedge |L_q \cap S| = l\}$, or $s_k = \infty$, if there is no such S. Analogously as in case (10), we have to ensure that exactly $k - 1$ vertices from $\{v_1, \ldots, v_p\}$ satisfy the defensive set property, and the rest of them is outside the defensive set. Let us observe that $SEC(r_v) = true$ if and only if $2l + 2 \ge q$ and $SEC(v, r_v) = true$ if and only if $2k + 2l \ge q + p$. Hence, we have two positive cases: (1) $SEC(v, r_v) = true$, and (2) $SEC(r_v) = true$ and $SEC(v, r_v) = false$. In case (1) vertex v satisfies the defensive set property and it suffices to ensure the defensive set property for $k - 1$ vertices from the set $\{v_1, \ldots, v_p\}$, analogously as in case (10). In case (2) we have to ensure that

$SEC(v) = true$ or $SEC(v, v_i) = true$ for some $i \in \{1, \dots, p\}$, and we must ensure the defensive set property for $k-1$ vertices from the set $\{v_1, \dots, v_p\}$, which can be done in the same manner as in case (10). The construction of matrix A_v^{11} can be done in $O(qp^2 \log p)$.

Finally, $\gamma_{ds}(T) = \min\{a_s^{10}, a_s^{01} + 1, a_s^{11} + 1\}$, where $\{s\} = N(r)$. Since the construction of data structure A_v can be done in $O(qp^2 \log p)$, the time complexity of the algorithm is obviously $O(n\Delta^2 \log \Delta)$.

3.4. $O(n\Delta^2 \log \Delta)$-time algorithm for finding a minimum total defensive set

The construction of the exact algorithm for finding a minimum total defensive set goes very similar to the algorithm from Section 3.3, and we can use some ideas from Section 3.2. Thus, there is an $O(n\Delta^2 \log \Delta)$-time algorithm for the total defensive set problem.

4. Lower bounds on the minimum global defensive set

For a given set $S \subset V$ and any $U \subset S$ let us define $N^+[U] = N[U] \cap S$, $N^-[U] = N[U] \cap (V \setminus S)$. For a defensive set S of graph $G = (V, E)$ we define a partition of S into two sets, $S^0 = \{v \in S : SEC(v) = false\}$ and $S^1 = \{v \in S : SEC(v) = true\}$. Observe that $S^0 = \emptyset$ if and only if S is an alliance.

Proposition 3. *Let S be a defensive set of a graph G and $X_1, X_2 \subset S$. Then $|N^-[X_1]| + |N^-[X_2]| \geq |N^-[X_1 \cup X_2]|$.*

Proof. $|N^-[X_1]| + |N^-[X_2]| \geq |N^-[X_1] \cup N^-[X_2]| \geq |(N[X_1] \cup N[X_2]) \cap (V \setminus S)| = |N^-[X_1 \cup X_2]|$. □

Proposition 4. *Let S be a defensive set of a graph G. Then for every $w \in S$ there is $|N^-[w]| \leq |S|$.*

Proof. If $w \in S^1$, then $|N^-[w]| \leq |N^+[w]| \leq |S|$. If $w \in S^0$, then there is a neighbour $u \in N(w) \cap S$ such that $|N^-[w, u]| \leq |N^+[w, u]| \leq |S|$. Thus, $|N^-[w]| \leq |N^-[w, u]| \leq |S|$. □

4.1. Lower bound in general graphs

In this section we will prove a lower bound on the minimum size of a global defensive set in an arbitrary graph.

By definition we have $\gamma_{ds}(G) \leq \gamma_a(G)$ for any graph G. Observe that for a tree T depicted in Fig. 14, we have $\gamma_{ds}(T) = 5$, and each of three lower bounds on $\gamma_a(T)$ (from [9]) is greater than 5: $\frac{n}{\lceil \frac{\Delta}{2} \rceil + 1} > 6$, $\lceil \frac{2n}{\Delta + 3} \rceil = 6$, and $\frac{n+2}{4} > 6$. In [9] the authors proved the following

Theorem 5 ([9]). *For any graph G with n vertices $\gamma_a(G) \geq \frac{\sqrt{4n+1}-1}{2}$.* □

For a defensive set S of graph $G = (V, E)$ let us define $E^1(S) = \{\{v, u\} \in E(G) : v \in S \wedge u \in S \wedge SEC(v, u) = true\}$, and let $r(S)$ be the maximum cardinality of a subset of $E^1(S)$ of independent edges.

Lemma 5. *For any global defensive set S of graph $G = (V, E)$*

$$|S| \geq \frac{\sqrt{4n(G) + (r(S) - 1)^2} + r(S) - 1}{2}.$$

Proof. Let S be any defensive set of G, and let $n = n(G)$, $s = |S|$ and $r = r(S)$.

If $r = 0$, then S is an alliance, and by Theorem 5 we have $|S| \geq \frac{\sqrt{4n+1}-1}{2}$.

Now, let $r > 0$. The thesis is equivalent to $s^2 - (r-1)s - n \geq 0$. Let us assume that $s^2 - (r-1)s < n$. Hence, $|N^-[S]| = |V(G) \setminus S| = n - s > s^2 - r \cdot s$.

Let $M^1 = \{\{v_1, u_1\}, \dots, \{v_r, u_r\}\}$ be any maximum cardinality subset of $E^1(S)$ of independent edges. For any edge $\{v_i, u_i\} \in M^1$ ($i \in \{1, \dots, r\}$) we have $SEC(v_i, u_i) = true$. Thus, $|N^-[v, u]| \leq |N^+[v, u]| \leq s$.

By Proposition 3 we have $|N^-[S \setminus U]| + \sum_{i=1}^{r} |N^-[v_i, u_i]| \geq |N^-[S]|$, where $U = \bigcup_{i=1}^{r} \{v_i, u_i\}$. Thus we get $|N^-[S \setminus U]| > s(s - 2r)$.

By Proposition 4 for every vertex $w \in S \setminus U$ we have $|N^-[w]| \leq s$. Thus, $|N^-[S \setminus U]| \leq \sum_{w \in S \setminus U} |N^-[w]| \leq s(s - 2r)$, a contradiction. □

Corollary 6. *For any graph G with n vertices*

$$\gamma_{ds}(G) \geq max\{L(S) : S \text{ is a global defensive set and } \gamma_{ds}(G) = |S|\},$$

where

$$L(S) = \frac{\sqrt{4n(G) + (r(S) - 1)^2} + r(S) - 1}{2},$$

and this bound is tight for any $r \geq 1$.

Proof. The bound is tight for complete bipartite graphs $K_{r,l}$, where $l \geq r$, with attached $l(l+r)$ pendant vertices ($K^*_{r,l}$) or, more precisely, $l+r$ pendant vertices to each vertex of the second partition (with l vertices). Thus, $n(K^*_{r,l}) = l+r+l(l+r)$ and $\gamma_{ds}(K^*_{r,l}) = l+r$, where $S = V(K_{r,l})$ is a defensive set with $l+r$ vertices, and there is no smaller one. Obviously $r(S) = r$. Let us notice (without the proof) that $\gamma_a(K^*_{r,l}) = l+r+l\lfloor \frac{l}{2} \rfloor > \gamma_{ds}(K^*_{r,l})$, for $r = 1$ and $l \geq 2$ or $2 \leq r \leq l$. □

Theorem 6. *For any graph G with $n(G) = k^2 + i$, where $k \geq 1$ and $1 \leq i \leq 2k+1$, the following tight bounds hold:*

(1) *if $\gamma_{ds}(G) < \gamma_a(G)$, then $\gamma_{ds}(G) \geq \sqrt{n}$,*
(2) *if $\gamma_{ds}(G) = \gamma_a(G)$ and $i \in \{k+1, \ldots, 2k+1\}$, then $\gamma_{ds}(G) \geq \sqrt{n}$,*
(3) *if $\gamma_{ds}(G) = \gamma_a(G)$ and $i \in \{1, \ldots, k\}$, then $\gamma_{ds}(G) \geq \sqrt{n-i} = \lfloor \sqrt{n} \rfloor$.*

Proof. (1) If $\gamma_{ds}(G) < \gamma_a(G)$, then $r(S) \geq 1$ for any defensive set S such that $\gamma_{ds}(G) = |S|$. Hence, by Corollary 6 we have $\gamma_{ds}(G) \geq \sqrt{n}$.

The bound is tight for trees T_k ($k \geq 2$) such that $T_k = K^*_{1,k}$, for which $\gamma_{ds}(T_k) = \sqrt{n(T_k)} = k+1$. Graph T_4 is shown in Fig. 14, for which have $n(T_4) = 25$, $\gamma_{ds}(T_4) = 5$ and $\gamma_a(T_4) = 13$. Obviously, by [9] we have $\gamma_a(T_k) \geq \frac{n(T_k)+2}{4} = \frac{(k+1)^2+2}{4}$. Thus, $\gamma_a(T_k) > \gamma_{ds}(T_k)$ for any $k \geq 2$.

(2) Since $i \in \{k+1, \ldots, 2k+1\}$ we have $\lceil \frac{\sqrt{4k^2+4i+1}-1}{2} \rceil = k+1 = \lceil \sqrt{n} \rceil$. Thus, by Theorem 5 we get $\gamma_{ds}(G) \geq \sqrt{n}$.

The bound is tight for cliques of size $k+1$ with attached $k(k+1)$ pendant vertices (G^*_k) or, more precisely, k pendant vertices to each vertex of the clique. Hence, $n = (k+1)^2$ and $\gamma_a(G_k) = \gamma_{ds}(G_k) = k+1 = \sqrt{n}$.

(3) Since $i \in \{1, \ldots, k\}$ we have $\lceil \frac{\sqrt{4k^2+4i+1}-1}{2} \rceil = k = \sqrt{n-i} = \lfloor \sqrt{n} \rfloor$. Thus, by Theorem 5 we get $\gamma_{ds}(G) \geq \sqrt{n-i}$.

The bound is tight for cliques of size k with attached k^2 pendant vertices (G_k) or, more precisely, k pendant vertices to each vertex of the clique. Hence, $n = k^2 + k$ and $\gamma_a(G_k) = \gamma_{ds}(G_k) = k$. The example of such a graph for $k = 4$ is depicted in Fig. 13. □

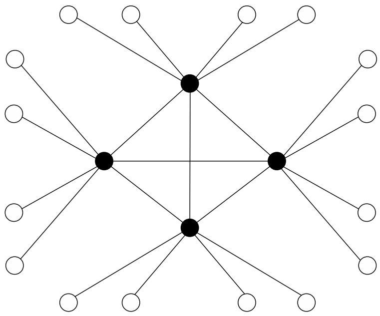

Fig. 13. $\gamma_{ds} = \gamma_a = 4, n = 20$.

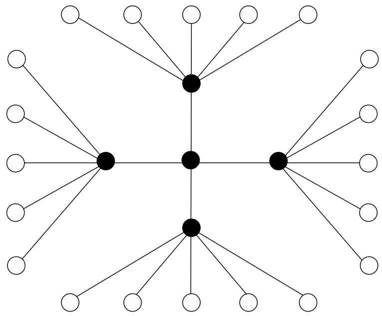

Fig. 14. $\gamma_{ds}(T_4) = 5, n = 25, \Delta = 6$.

4.2. Lower bound in trees

By Theorem 6 we have that the minimum global defensive set is at least $\lceil \sqrt{n} \rceil$ or $\lfloor \sqrt{n} \rfloor$, depending on n and the graph properties. For global alliances the lower bound $\frac{n+2}{4}$ for trees is proved in [9], but this bound does not hold for global defensive sets due to the trees depicted in Fig. 14.

Theorem 7 ([9]). *For any tree T with n vertices $\gamma_a(T) \geq \frac{n+2}{4}$.* □

Since for any $n > 10$ there is $\lceil \frac{n+2}{4} \rceil \geq \lceil \sqrt{n} \rceil$, by Theorem 6 we have the following corollary

Corollary 7. *Given a tree T with $n(T) > 10$, $\gamma_{ds}(T) \geq \sqrt{n(T)}$ and this bound is tight.* □

For any non-empty tree $T = (V, E)$ by $L(T) \subset V(T)$ we denote a set of all leaves (pendant vertices) of T. Let us denote $C(T) = V(T) \setminus L(T)$.

Proposition 8. *Let S be a global defensive set S in a tree T. Then $T[S]$ is a tree if and only if $C(T) \subset S$, which is equivalent to $V(T) \setminus S \subset L(T)$.*

Proof. Let $x \in C(T) \setminus S$ and $T_1 = T \setminus \{x\}$. Hence, T_1 is disconnected. Since S is a dominating set and $T[S]$ is a subtree of T_1, $T[S]$ is disconnected.

Let $C(T) \subset S$. Hence, $V(T) \setminus S \subset V(T) \setminus C(T) = L(T)$ and so, $T[S]$ is connected. □

We will characterize all trees with $\gamma_{ds}(T) = \lfloor \sqrt{n(T)} \rfloor < \lceil \sqrt{n(T)} \rceil$ with $n(T) \leq 10$ vertices.

Lemma 9. *The only trees such that* $\gamma_{ds}(T) = \lfloor\sqrt{n}\rfloor < \lceil\sqrt{n}\rceil$ *are shown in* Figs. 15–18.

Proof. By Theorem 6(3) we have that every such a tree must have 2, 5, 6 or 10 vertices. If $\gamma_{ds}(T) \geq 4$ and $n(T) \leq 10$, then obviously $\gamma_{ds}(T) \geq \lceil\sqrt{n}\rceil$. Let us notice that if $\gamma_{ds}(T) = 1$, then $n(T) \leq 2$, and if $\gamma_{ds}(T) \leq 2$, then $n(T) \leq 6$. It is easy to verify that trees depicted in Figs. 15–17 are the only trees satisfying $\gamma_{ds}(T) \leq 2$ and $\gamma_{ds}(T) = \lfloor\sqrt{n}\rfloor < \lceil\sqrt{n}\rceil$.

Let $\gamma_{ds}(T) = 3$. Thus, from $\gamma_{ds}(T) = \lfloor\sqrt{n}\rfloor < \lceil\sqrt{n}\rceil$ we have that $n = 10$. If S is a global defensive set in a tree T with 10 vertices, and $|S| = 3$, then $T[S]$ is connected, otherwise $n(T) \leq 8$. Hence, by Proposition 8 we have $V(T) \setminus S \subset L(T)$. Thus, at least one vertex from S must have three neighbouring leaves from $V(T) \setminus S$ (otherwise $n(T) < 10$). This, however, leads us easily to the graph depicted in Fig. 18. □

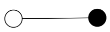

Fig. 15. $\gamma_{ds} = \lfloor\sqrt{2}\rfloor = 1$.

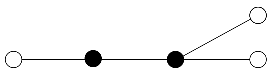

Fig. 16. $\gamma_{ds} = \lfloor\sqrt{5}\rfloor = 2$.

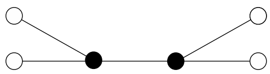

Fig. 17. $\gamma_{ds} = \lfloor\sqrt{6}\rfloor = 2$.

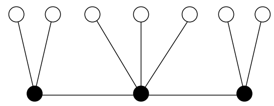

Fig. 18. $\gamma_{ds} = \lfloor\sqrt{10}\rfloor = 3$.

By Corollary 7 and Lemma 9 we have

Theorem 8. *For any tree T with n vertices that is non-isomorphic to one of the trees shown in* Figs. 15–18 *there is* $\gamma_{ds}(T) \geq \sqrt{n}$ *and this bound is tight. For any tree T from* Figs. 15–18 *there is* $\gamma_{ds}(T) = \lfloor\sqrt{n}\rfloor < \lceil\sqrt{n}\rceil$. □

References

[1] M. Bouzefrane, M. Chellali, A note on global alliances in trees, Opuscula Math. 31 (2) (2011) 153–158.
[2] M. Bouzefrane, M. Chellali, T.W. Haynes, Global defensive alliances in trees, Util. Math. 82 (2010) 241–252.
[3] A. Cami, H. Balakrishnan, N. Deo, R. Dutton, On the complexity of finding optimal global alliances, J. Combin. Math. Combin. Comput. 58 (2006) 23–31.
[4] C.-W. Chang, M.-L. Chia, C.-J. Hsu, D. Kuo, L.-L. Lai, F.-H. Wang, Global defensive alliances of trees and cartesian product of paths and cycles, Discrete Appl. Math. 160 (2012) 479–487.
[5] R.I. Enciso, R.D. Dutton, Lower bounds for global alliances on planar graphs, Congr. Numer. 187 (2007) 187–192.
[6] G.W. Flake, S. Lawrence, C.L. Giles, Efficient identification of web communities, in: Proc. of the 6th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, Boston, MA, 2000, pp. 150–160.
[7] A. Harutyunyan, Some bounds on global alliances in trees, Discrete Appl. Math. 161 (2013) 1739–1746.
[8] T.W. Haynes, S.T. Hedetniemi, M.A. Henning, Global defensive alliances, in: Proc. of. the 17th Int. Symposium on Computer Information Science, 2002, pp. 303–307.
[9] T.W. Haynes, S.T. Hedetniemi, M.A. Henning, Global defensive alliances in graphs, Electron. J. Combin. 10 (2003) 139–146. Research Paper 47.
[10] S.M. Hedetniemi, S.T. Hedetniemi, P. Kristiansen, Introduction to alliances in graphs, in: Proc. of. the 17th Int. Symposium on Computer Information Science, 2002, pp. 308–312.
[11] M.A. Henning, A. Yeo, Total Domination in Graphs, in: Springer Monographs in Mathematics, 2013.
[12] C.-J. Hsu, F.-H. Wang, Y.-L. Wang, Global defensive alliances in star graphs, Discrete Appl. Math. 157 (2009) 1924–1931.
[13] H. Ino, M. Kudo, A. Nakamura, Partitioning of web graphs by community topology, in: Proc. of the 14th International Conference on World Wide Web, 2005, pp. 661–669.
[14] L.H. Jamieson, Algorithms and complexity for alliances and weighted alliances of various types (Ph.D. thesis), Clemson University, Clemson, SC, USA, 2007.
[15] A. Kosowski, M. Małafiejski, P. Żyliński, Cooperative mobile guards in grids, Comput. Geom. 37 (2) (2007) 59–71.
[16] R. Lewoń, A. Małafiejska, M. Małafiejski, K. Wereszko, Global edge alliances in graphs, in preparation.
[17] D. Peleg, Local majorities coalitions and monopolies in graphs: A review, Theoret. Comput. Sci. 282 (2) (2002) 213–257.
[18] J.A. Rodríguez-Velázquez, J.M. Sigarreta, Global alliances in planar graphs, AKCE Int. J. Graphs Comb. 4 (1) (2007) 83–98.
[19] J.A. Rodríguez-Velázquez, J.M. Sigarreta, Spectral study of alliances in graphs, Discuss. Math. Graph Theory 27 (1) (2007) 143–157.
[20] J.A. Rodríguez-Velázquez, J.M. Sigarreta, Global defensive k-alliances in graphs, Discrete Appl. Math. 157 (2009) 211–218.
[21] J.A. Rodríguez-Velázquez, I.G. Yero, J.M. Sigarreta, Defensive k-alliances in graphs, Appl. Math. Lett. 22 (2009) 96–100.
[22] J.M. Sigarreta, J.A. Rodríguez, On defensive alliances and line graphs, Appl. Math. Lett. 19 (2006) 1345–1350.
[23] P.K. Srimani, Z. Xu, Distributed protocols for defensive and offensive alliances in network graphs using self-stabilization, in: Proc. of the International Conference on Computing: Theory and Applications, Kolkata, India, 2007, pp. 27–31.
[24] I.G. Yero, J.A. Rodríguez-Velázquez, Defensive alliances in graphs: a survey, 2013, Aug-9, pp. 1–25. arXiv:1308.2096v1.