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GRAPHS WITH EQUAL DOMINATION AND CERTIFIED DOMINATION NUMBERS

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Abstract. A set D of vertices of a graph $G = (V_G, E_G)$ is a dominating set of G if every vertex in $V_G - D$ is adjacent to at least one vertex in D. The domination number (upper domination number, respectively) of G, denoted by $\gamma(G)$ ($\Gamma(G)$, respectively), is the cardinality of a smallest (largest minimal, respectively) dominating set of G. A subset $D \subseteq V_G$ is called a certified dominating set of G if D is a dominating set of G and every vertex in D has either zero or at least two neighbors in $V_G - D$. The cardinality of a smallest (largest minimal, respectively) certified dominating set of G is called the certified (upper certified, respectively) domination number of G and is denoted by $\gamma_{\rm cer}(G)$ ($\Gamma_{\rm cer}(G)$, respectively). In this paper relations between domination, upper domination, certified domination and upper certified domination numbers of a graph are studied.

Keywords: domination, certified domination.

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1. NOTATION AND DEFINITIONS

We generally follow the notation and terminology of [7]. For a graph G, the set of vertices is denoted by V_G and the edge set by E_G. For a vertex $v \in V_G$, the *open neighborhood* $N_G(v)$ of v is the set of all vertices adjacent to v, and $N_G[v] = N_G(v) \cup \{v\}$ is the *closed neighborhood* of v. The *open neighborhood* of a set $X \subseteq V_G$ is $N_G(X) = \bigcup_{v \in X} N_G(v)$, whereas the *closed neighborhood* of X is the set $N_G[X] = N_G(X) \cup X$. For $X \subseteq V_G$ and $v \in X$, the set $N_G[v] - N_G[X - \{v\}]$ is denoted by $PN_G[v, X]$ and called the *private neighborhood* of v with respect to X. Every vertex belonging to $PN_G[v, X]$ is called a *private neighbor* of v with respect to X. By $PN_G(v, X)$ we denote the set $N_G(v) - N_G[X - \{v\}]$ and call it the *open private neighborhood* (of v with respect to X). The *degree* of a vertex v in G is $d_G(v) = |N_G(v)|$. The number $\min\{d_G(v) \colon v \in V_G\}$ is the minimum degree of G and is denoted by $\delta(G)$. A vertex of degree 0 is called an *isolated vertex*, whereas a vertex of degree one in G is called a *leaf* of G. If v is a leaf, then its only neighbor is called a *support vertex* of v. A support vertex is called *strong* or *weak* depending on whether or not it is adjacent to at least two leaves. We use L_G, S_G, S_G^1 and S_G^2 to denote the set of all leaves, support vertices, weak support vertices and strong support vertices of G, respectively. Finally, the *corona* $H \circ K_1$ of a graph H was defined in [4] as the graph obtained from H by adding exactly one pendant edge to each vertex of H. A graph G is said to be a *corona* if it is the corona $H \circ K_1$ of some graph H. It is obvious that a corona is a graph in which each vertex is a leaf or a weak support vertex.

Given a graph G, we say that a subset $D \subseteq V_G$ is a *dominating set* of G if every vertex belonging to $V_G - D$ is adjacent to at least one vertex in D. The *domination number* (*upper domination number*, respectively) of G, denoted by $\gamma(G)$ ($\Gamma(G)$, respectively), is the cardinality of a smallest (largest minimal, respectively) dominating set of G. A dominating (minimal dominating, respectively) set of G of minimum (maximum, respectively) cardinality is called a γ-set (Γ-set, respectively) of G. A subset $D \subseteq V_G$ is called a *certified dominating set* of G if D is a dominating set of G and every vertex belonging to D has either zero or at least two neighbors in $V_G - D$. The cardinality of a smallest (largest minimal, respectively) certified dominating set of G is called the *certified* (*upper certified*, respectively) *domination number* of G and is denoted by $\gamma_{\rm cer}(G)$ ($\Gamma_{\rm cer}(G)$, respectively). A certified dominating (minimal certified dominating, respectively) set of G of minimum (maximum, respectively) cardinality is called a $\gamma_{\rm cer}$-set ($\Gamma_{\rm cer}$-set, respectively) of G. For example, it is easy to observe that for the most common graph families, we have $\gamma(K_n) = \gamma_{\rm cer}(K_n) = \Gamma(K_n) = \Gamma_{\rm cer}(K_n) = 1$ if $n \neq 2$, $\gamma(P_n) = \gamma_{\rm cer}(P_n) = \lceil n/3 \rceil$ and $\Gamma_{\rm cer}(P_n) = \lfloor (n-1)/2 \rfloor = \Gamma(P_n) - 1$ if $n \geq 5$, $\gamma(C_n) = \gamma_{\rm cer}(C_n) = \lceil n/3 \rceil$ and $\Gamma(C_n) = \Gamma_{\rm cer}(C_n) = \lfloor n/2 \rfloor$ if $n \geq 3$, $\Gamma(K_{1,n}) = n$ (if $n \geq 1$) and $\gamma(K_{1,n}) = \gamma_{\rm cer}(K_{1,n}) = \Gamma_{\rm cer}(K_{1,n}) = 1$ if $n \geq 2$, $\gamma(K_{m,n}) = \gamma_{\rm cer}(K_{m,n}) = 2$ and $\Gamma(K_{m,n}) = \Gamma_{\rm cer}(K_{m,n}) = n$ if $2 \leq m \leq n$.

It is obvious that for any graph G we have the inequalities $\gamma(G) \leq \gamma_{\rm cer}(G) \leq \Gamma_{\rm cer}(G)$, while the parameters $\gamma_{\rm cer}(G)$ and $\Gamma(G)$, and also the parameters $\Gamma_{\rm cer}(G)$ and $\Gamma(G)$ are not comparable. For example, the inequalities $\gamma(G) \leq \Gamma(G) \leq \gamma_{\rm cer}(G) \leq \Gamma_{\rm cer}(G)$ hold for the graph G in Figure 1. In this case it is easy to check that $\gamma(G) = 5$, $\Gamma(G) = 6$, $\gamma_{\rm cer}(G) = 7$, $\Gamma_{\rm cer}(G) = 8$, and the sets $\{v_1, v_2, v_3, v_5, v_6\}$, $\{v_5, v_8, v_{10}, v_{11}, v_{12}, v_{13}\}$, $\{v_1, v_2, v_3, v_5, v_6, v_{12}, v_{13}\}$, $\{v_1, v_2, v_3, v_5, v_8, v_{10}, v_{12}, v_{13}\}$ are examples of γ-, Γ-, $\gamma_{\rm cer}$-, and $\Gamma_{\rm cer}$-sets of G, respectively. The graph F in Figure 1 illustrates the inequalities $\gamma(F) \leq \gamma_{\rm cer}(F) \leq \Gamma(F) \leq \Gamma_{\rm cer}(F)$. One can check that $\gamma(F) = 3$, $\gamma_{\rm cer}(F) = 4$, $\Gamma(F) = 5$, $\Gamma_{\rm cer}(F) = 6$, and the sets $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_5, v_9\}$, $\{v_4, v_6, v_7, v_8, v_9\}$ and $\{v_1, v_2, v_4, v_6, v_7, v_9\}$ are examples of γ-, $\gamma_{\rm cer}$-, Γ-, and $\Gamma_{\rm cer}$-sets of F, respectively. Finally, the inequalities $\gamma(H) \leq \gamma_{\rm cer}(H) \leq \Gamma_{\rm cer}(H) \leq \Gamma(H)$ hold for the graph H in Figure 1, where $\gamma(H) = 5$, $\gamma_{\rm cer}(H) = 6$, $\Gamma_{\rm cer}(H) = 7$, $\Gamma(H) = 8$, and $\{v_4, v_5, v_7, v_{12}, v_{14}\}$, $\{v_4, v_5, v_6, v_7, v_{12}, v_{14}\}$, $\{v_4, v_5, v_8, v_9, v_{10}, v_{12}, v_{14}\}$ and $\{v_1, v_2, v_3, v_6, v_8, v_9, v_{10}, v_{13}\}$ are examples of γ-, $\gamma_{\rm cer}$-, $\Gamma_{\rm cer}$- and Γ-sets of H, respectively.

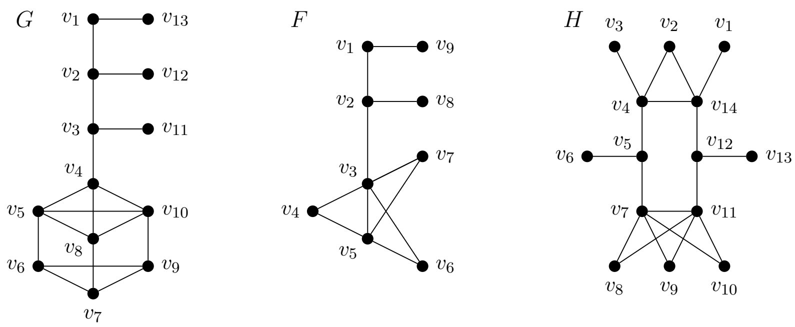

Fig. 1. Graphs G, F, and H

Domination in graphs is one of the most fundamental and well-studied concepts in graph theory. The reader is referred to [6, 7] and [8] for more details on these important topics. The previously mentioned certified domination was introduced by Dettlaff *et al.* [2] in order to describe some possible relations in social networks. In this paper we continue the study of certified dominating sets and certified domination numbers of graphs. For different classes of graphs G we establish conditions for the equality of the domination number $\gamma(G)$ and the certified domination number $\gamma_{\rm cer}(G)$ of a graph G. Furthermore, we characterize all graphs G for which $\gamma(H) = \gamma_{\rm cer}(H)$ for each induced and connected subgraph $H \neq K_2$ of G. The last part of the paper is concerned with main properties of the upper certified domination number $\Gamma_{\rm cer}(G)$ of G and its relations to $\gamma_{\rm cer}(G)$ and $\Gamma(G)$. We conclude with some open problems.

2. GRAPHS G FOR WHICH $\gamma_{\rm cer}(G) = \gamma(G)$

In this section we study basic properties which guarantee equalities of domination and certified domination numbers. We begin with the following necessary and sufficient condition for the equality of domination and certified domination numbers of a graph.

Theorem 2.1. *Let G be a connected graph of order at least three. Then $\gamma(G) = \gamma_{\rm cer}(G)$ if and only if G has a γ-set D such that every vertex in D has at least two neighbors in $V_G - D$.*

Proof. Assume that $\gamma(G) = \gamma_{\rm cer}(G)$. Let D be a $\gamma_{\rm cer}$-set of G. Then the equality $\gamma(G) = \gamma_{\rm cer}(G)$ guarantees that D is also a γ-set of G. Now, let v be a vertex in D. Since v is not an isolated vertex, the set $N_G(v) \cap (V_G - D)$ is nonempty (as otherwise $D - \{v\}$ would be a smaller dominating set of G). Consequently, since D is a certified dominating set, we have $|N_G(v) \cap (V_G - D)| \geq 2$.

On the other hand, if D is a γ-set of G and $|N_G(v) \cap (V_G - D)| \geq 2$ for every $v \in D$, then D is also a certified dominating set of G. Hence $\gamma_{\rm cer}(G) \leq |D| = \gamma(G) \leq \gamma_{\rm cer}(G)$ and therefore $\gamma(G) = \gamma_{\rm cer}(G)$. $\square$

From Theorem 2.1 we immediately obtain the following corollary.

Corollary 2.2. *Let G be a connected graph of order at least three. If G has an independent γ-set that contains no leaf of G, then $\gamma(G) = \gamma_{\rm cer}(G)$.*

It was already proved in [2] that $\gamma(G) = \gamma_{\rm cer}(G)$ for all graphs G without leaves. Here we present another proof of that result.

Corollary 2.3. *If G is a graph in which $\delta(G) \geq 2$, then:*

(1) *G has a γ-set D such that every vertex in D has at least two neighbors in $V_G - D$;*
(2) *$\gamma(G) = \gamma_{\rm cer}(G)$.*

Proof. For a γ-set X of G, let $q(X)$ denote the set $\{x \in X : |N_G(x) \cap (V_G - X)| \leq 1\}$. Now, let D be a γ-set such that $|q(D)|$ is as small as possible. We claim that $q(D) = \emptyset$ and so D is the required set. Indeed, suppose to the contrary that $|q(D)| > 0$, and let v be any vertex in $q(D)$. Since $d_G(v) \geq \delta(G) \geq 2$, the set $N_G(v)$ cannot be a subset of D, as otherwise $D - \{v\}$ would be a dominating set of G. Consequently $|N_G(v) \cap (V_G - D)| = 1$, say $N_G(v) \cap (V_G - D) = \{v'\}$. Again, since $d_G(v) \geq 2$ and $|N_G(v) \cap (V_G - D)| = 1$, $N_G(v) \cap D \neq \emptyset$ and v' is the only private neighbor of v with respect to D. Thus $N_G(v') - \{v\}$ is a non-empty subset of $V_G - D$ and $D' = (D - \{v\}) \cup \{v'\}$ is a minimum dominating set of G and the set

$$q(D') = \{x \in D' : |N_G(x) \cap (V_G - D')| \leq 1\}$$

is of size smaller than $|q(D)|$ (as $q(D') \subseteq q(D) - \{v\}$, since v' is a private neighbor of v with respect to D and $d_G(v') \geq 2$), a contradiction, which completes the proof of (1). The property (2) follows from (1) and Theorem 2.1. □

It has been proved in [5] that if D is a unique γ-set of a graph G, then every vertex in D that is not an isolated vertex has at least two private neighbors other than itself, that is, in the set $V_G - D$. From this and from Theorem 2.1 we have the following corollary.

Corollary 2.4 ([2]). *If a graph G has a unique γ-set, then $\gamma(G) = \gamma_{\rm cer}(G)$.*

The main properties of graphs having unique γ-sets have been studied in [5] and partialy in [3]. It was also observed in [5] that if D is a γ-set of a graph G and $\gamma(G - x) > \gamma(G)$ for every $x \in D$, then D is the unique γ-set of G. Thus, by Corollary 2.4, we have the next corollary.

Corollary 2.5. *If a graph G has a γ-set D such that $\gamma(G - x) > \gamma(G)$ for every $x \in D$, then $\gamma(G) = \gamma_{\rm cer}(G)$.*

The next theorem provides another sufficient condition for the equality of domination and certified domination numbers of a graph.

Theorem 2.6. *Let G be a connected graph of order at least three. If $\gamma(G - v) \geq \gamma(G)$ for every vertex v belonging to at least one γ-set of G, then $\gamma(G) = \gamma_{\rm cer}(G)$.*

Proof. Similarly as in the proof of Corollary 2.3, if X is a γ-set of G, then by $q(X)$ we denote the set $\{x \in X\colon |N_G(x) \cap (V_G - X)| \leq 1\}$. Assume that $\gamma(G - x) \geq \gamma(G)$ for every vertex x belonging to at least one γ-set of G. To prove that $\gamma(G) = \gamma_{\rm cer}(G)$, by Theorem 2.1, it remains to show that $q(X) = \emptyset$ for some γ-set X of G.

Let D be a γ-set such that $|q(D)|$ is as small as possible. We claim that $q(D) = \emptyset$ and so D is the required set. Indeed, suppose to the contrary that $|q(D)| > 0$, and let v be any vertex belonging to $q(D)$. Since v is not an isolated vertex, the minimality of D implies that $N_G(v) \cap (V_G - D) \neq \emptyset$ (as otherwise $D - \{v\}$ would be a smaller dominating set of G). Thus $|N_G(v) \cap (V_G - D)| = 1$, and let v' be the only vertex in $N_G(v) \cap (V_G - D)$. Then the set $D' = (D - \{v\}) \cup \{v'\}$ is a γ-set of G.

We now claim that v' is a private neighbor of v with respect to D. Suppose, contrary to our claim, that $v' \notin PN_G[v, D]$. Then $v' \in N_G(D - \{v\})$ and either $N_G(v) \cap D \neq \emptyset$ or $N_G(v) \cap D = \emptyset$. The first case is impossible (as otherwise $D - \{v\}$ would be a smaller dominating set of G). Thus $N_G(v) \cap D = \emptyset$, but now $D - \{v\}$ is a dominating set of $G - v$ and therefore, $\gamma(G - v) \leq |D - \{v\}| < |D| = \gamma(G)$, contrary to our assumption. This proves that v' is a private neighbor of v with respect to D.

Next, since the private neighbor v' of v with respect to D is the only neighbor of v in $V_G - D$ and since G is a connected graph of order at least three, the sets $N_G(v) - \{v'\}$ and $N_G(v') - \{v\}$ are subsets of D and $V_G - D$, respectively, and at least one of them is non-empty. If $N_G(v) - \{v'\} \neq \emptyset$, then $PN_G[v', D'] = \{v'\}$ and $D' - \{v'\}$ is a dominating set of $G - v'$, and then $\gamma(G - v') \leq |D' - \{v'\}| < |D'| = \gamma(G)$, contrary to our assumption. Thus assume that $N_G(v) - \{v'\} = \emptyset$. Then $N_G(v') - \{v\}$ is a non-empty subset of $V_G - D'$, and therefore

$$
\begin{aligned}
|N_G(v') \cap (V_G - D')| &= |\{v\}| + |(N_G(v') - \{v\}) \cap (V_G - D')| \\
&= 1 + |N_G(v') - \{v\}| \geq 2.
\end{aligned}
$$

This implies that $v' \notin q(D')$. Now, since $v \notin q(D')$ (as $v \notin D'$), neither v' nor v belongs to $q(D')$, and therefore $q(D') \subseteq q(D) - \{v\}$. Consequently, we have $|q(D')| < |q(D)|$, a contradiction to the choice of D, and this completes the proof. □

The fundamental relations between the classes of graphs considered in Corollaries 2.3–2.5 and Theorems 2.1 and 2.6 are illustrated by examples in Figure 2.

A graph G is said to be P_4-*free* if P_4 is not an induced subgraph of G. We also say that a graph G is $\gamma\gamma_{\rm cer}$-perfect if $\gamma(H) = \gamma_{\rm cer}(H)$ for each induced and connected subgraph $H \neq K_2$ of G. It follows from this definition that a graph G is $\gamma\gamma_{\rm cer}$-perfect if and only if each component of G, different from K_2, is $\gamma\gamma_{\rm cer}$-perfect. The path P_4 is the smallest non-$\gamma\gamma_{\rm cer}$-perfect graph. The union $K_2 \cup C_4$ is a $\gamma\gamma_{\rm cer}$-perfect graph, while the union $K_2 \cup C_5$ is not a $\gamma\gamma_{\rm cer}$-perfect graph. We now study the equality $\gamma(G) = \gamma_{\rm cer}(G)$ for P_4-free graphs.

Theorem 2.7. *If G is a connected P_4-free graph and $G \neq K_2$, then $\gamma(G) = \gamma_{cer}(G)$.*

Proof. The result is obvious if $G = K_1$. Thus assume that G is a connected P_4-free graph of order at least three. We shall prove that $\gamma(G) = \gamma_{\rm cer}(G)$. Since $\gamma(G) \leq \gamma_{\rm cer}(G)$, it suffices to show that some γ-set of G is a certified dominating set of G.

Similarly as in the proof of Corollary 2.3 and Theorem 2.6, let D be a γ-set of G such that D contains no leaf of G and $q(D) = \{x \in D\colon |N_G(x) \cap (V_G - D)| \leq 1\}$ is as small as possible (since G is connected and $G \neq K_2$, such D exists). We claim that $q(D) = \emptyset$. Indeed, suppose to the contrary that $|q(D)| > 0$, and let v be any vertex belonging to $q(D)$. By the same reason as in the proof of Theorem 2.6, there exists v' which is the only neighbor of v in $V_G - D$. Since v is not a leaf and v' is the only neighbor of v in $V_G - D$, the set $N_G(v) \cap D$ is non-empty. Choose any vertex $u \in N_G(v) \cap D$. The minimality of D and the fact that v and u are adjacent elements of D imply that the private neighborhoods $PN_G[v, D]$ and $PN_G[u, D]$ are disjoint and non-empty subsets of $V_G - D$. Certainly, $PN_G[v, D] = \{v'\}$ and, therefore, $N_G(x) \cap \{v'\} = \emptyset$ for every $x \in D - \{v\}$. Now, since G is a P_4-free graph, the vertex v' is not a leaf and it is adjacent to every vertex in $PN_G[u, D]$ and, therefore, to every vertex in $\bigcup_{u \in N_G(v) \cap D} PN_G[u, D]$. Again from the fact that G is P_4-free it is easily seen that $N_G(v) \cap D = \{u\}$ and $N_G(u) \cap D = \{v\}$ (otherwise, the set $\{u', v', v, w\}$, where $u' \in PN_G[u, D]$ and $w \in (N_G(v) \cap D) - \{u\}$, or, respectively, the set $\{v', v, u, z\}$, where $z \in (N_G(u) \cap D) - \{v\}$, would induce a 4-vertex path in G, a contradiction). Now let us consider the set $D' = (D - \{v\}) \cup \{v'\}$. It is obvious that D' is a minimum dominating set of G and D' contains no leaf of G. It remains to show that $q(D') \subseteq q(D) - \{v\}$. First, let us observe that $v \notin q(D')$ (as $v \notin D'$) and $\{v', u\} \cap q(D') = \emptyset$ (as $|N_G(x) \cap (V_G - D')| \geq |\{v\} \cup PN_G[u, D]| \geq 2$ if $x \in \{v', u\}$). Now, since $N_G(v) \cap q(D') = \{v', u\} \cap q(D') = \emptyset$, we have $N_G(x) \cap \{v\} = \emptyset$ for every $x \in q(D')$. Consequently, if $x \in q(D')$, then $x \in q(D') - \{v, v', u\}$ and therefore we have

$$\begin{aligned}
1 &= |N_G(x) \cap (V_G - D')| \\
&= |N_G(x) \cap (V_G - ((D - \{v\}) \cup \{v'\}))| \\
&= |N_G(x) \cap (((V_G - D) - \{v'\}) \cup \{v\})| \\
&= |N_G(x) \cap ((V_G - D) - \{v'\}) \cup (N_G(x) \cap \{v\})| \\
&= |((N_G(x) \cap (V_G - D)) - (N_G(x) \cap \{v'\})) \cup (N_G(x) \cap \{v\})| \\
&= |N_G(x) \cap (V_G - D)|,
\end{aligned}$$

as $N_G(x) \cap \{v'\} = \emptyset$ and $N_G(x) \cap \{v\} = \emptyset$. This shows that $q(D') \subseteq q(D) - \{v\}$, a contradiction, which completes the proof. $\square$

Theorem 2.7 immediately implies the following characterization of the $\gamma\gamma_{\mathrm{cer}}$-perfect graphs.

Corollary 2.8. *A graph G is a $\gamma\gamma_{\mathrm{cer}}$-perfect graph if and only if G is a P_4-free graph.*

Proof. The "only if" part of the theorem follows from the fact that $\gamma(P_4) = 2 < 4 = \gamma_{\mathrm{cer}}(P_4)$. The "if" part follows from Theorem 2.7. $\square$

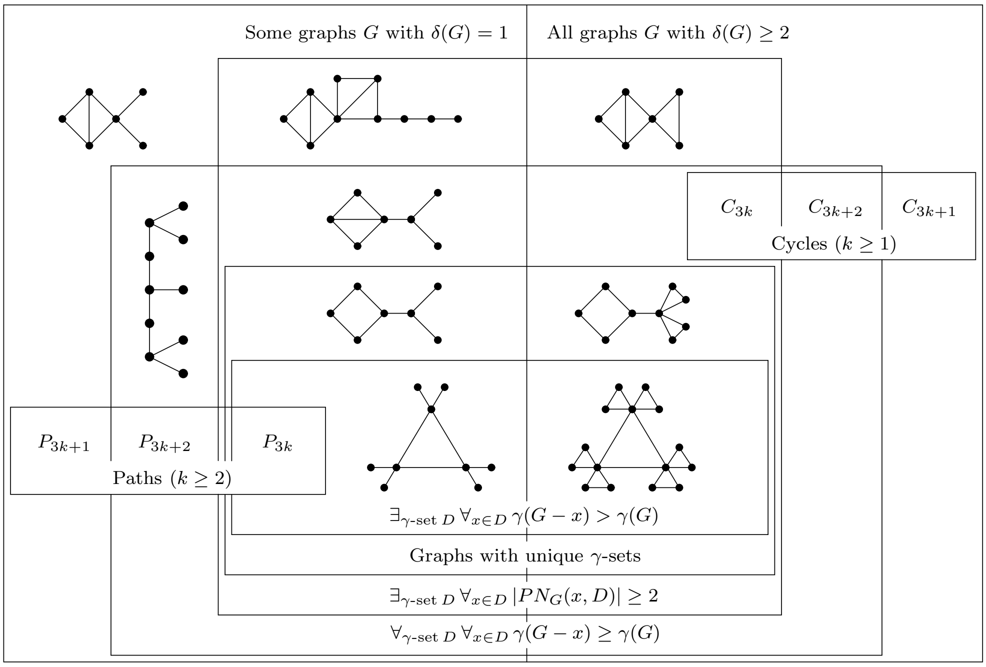

Fig. 2. The variety of relations between the classes of graphs G with $\gamma(G) = \gamma_{\text{cer}}(G)$

3. PROPERTIES OF UPPER CERTIFIED DOMINATION NUMBER

In this section we study main properties of upper certified domination number Γ_{cer}. We give a characterization of all graphs with $\Gamma_{\text{cer}} = n$ and $\Gamma_{\text{cer}} = n - 2$, respectively. In addition, we focus on the relation between upper domination number and upper certified domination number of a graph. We start with the following useful lemma.

Lemma 3.1. *Let G be a connected graph of order at least two. If D is a minimal certified dominating set of G and v is a vertex such that $N_G[v] \subseteq D$, then $v \in L_G \cup S_G^1$, that is, v is a leaf or a weak support vertex of G. In addition, the induced subgraph $G[\{v \in D\colon N_G[v] \subseteq D\}]$ is a corona.*

Proof. The result is obvious if $G = K_2$. Thus assume that G is a graph of order at least three. Suppose, contrary to our claim, that the set

$$D' = \{v \in D\colon N_G[v] \subseteq D\} - (L_G \cup S_G^1)$$

is non-empty. Let F denote the subgraph $G[D']$. Let I be a maximal set of independent vertices of degree at least two in F. The set I is a proper (possibly empty) subset of D' and, if I is non-empty, then every vertex in I has at least two neighbors in $D' - I$. Now let I' denote the set $D' - N_F[I]$. We claim that I' is dominated by the set $D - (L_G \cup D')$.

This is trivially true if $I' = \emptyset$. Thus assume that $I' \neq \emptyset$ and $v_0 \in I'$. Then $d_G(v_0) \geq 2$ (as $v_0 \notin L_G$) but $d_F(v_0) \leq 1$ (otherwise $I \cup \{v_0\}$ would be a larger set of independent vertices of degree at least two in F) and therefore $N_G(v_0) - V_F$ is a non-empty subset of $D - (L_G \cup D')$. This proves our claim. Consequently, $D - (L_G \cup D')$ dominates I' whereas I is a certified dominating set of $F - I'$. This implies that $D - (D' - I)$ is a dominating set of G. In addition, it is no problem to observe that $D - ((D' - I) \cup L')$ is a dominating set of G for every subset L' of L_G (as every $x \in L_G$ is dominated by its only neighbor in $N_G(L_G)$ and $N_G(L_G) \subseteq D$).

Let us consider the function $s\colon L_G \to S_G$, where $s(x)$ is the only neighbor of a leaf x, i.e. $s(x)$ is the only element of the set $N_G(x)$. This function is not necessarily an injection, but, since D is a minimal certified dominating set of G, the restriction of s to $L_G \cap D$ is indeed an injection and, in addition, the set $N_G[s(x)]$ is a subset of D for every $x \in L_G \cap D$ (since otherwise, as both x and $s(x)$ belong to D, the set $D - \{x\}$ would be a smaller certified dominating set, a contradiction with the minimality of D). Moreover, the mapping $s\colon L_G \cap D \to S_G$ is a bijection between $L_G \cap D$ and $\{y \in S_G \colon N_G[y] \subseteq D\}$ $(= N_G(L_G \cap D))$. Let S_0 and S_1 be the sets $\{x \in N_G(L_G \cap D) : N_G(x) \cap (D' - I) = \emptyset\}$ and $\{x \in N_G(L_G \cap D) \colon N_G(x) \cap (D' - I) \neq \emptyset\}$, respectively. Now let $L_0 = N_G(S_0) \cap L_G$ $(= s^{-1}(S_0))$ and $L_1 = N_G(S_1) \cap L_G$ $(= s^{-1}(S_1))$. Observe that $S_0 \cap S_1 = \emptyset$ and $L_0 \cap L_1 = \emptyset$, whereas $S_0 \cup S_1 = N_G(L_G \cap D) \subseteq S_G^1$ and $L_0 \cup L_1 = L_G \cap D$.

We now prove that the set $D'' = D - ((D' - I) \cup L_1)$ is a certified dominating set of G. Since D'' is a dominating set of G, it suffices to show that no vertex belonging to D'' has exactly one neighbor in $V_G - D''$. To show this, let us take a vertex $x \in D''$. Since $D'' \subseteq D$ and D is a certified dominating set of G, we have $|N_G(x) \cap (V_G - D)| \geq 2$ or $|N_G(x) \cap (V_G - D)| = 0$. In the first case $|N_G(x) \cap (V_G - D'')| \geq 2$, as $V_G - D \subseteq V_G - D''$. Thus assume that $|N_G(x) \cap (V_G - D)| = 0$. Then $N_G[x] \subseteq D$ and, since $x \notin (D' - I) \cup L_1$, x is an element of the set $I \cup S_1 \cup (L_0 \cup S_0)$. If $x \in I$, then $|N_F(x) \cap (D' - I)| \geq 2$ (by the definition of I) and therefore $|N_G(x) \cap (V_G - D'')| \geq 2$ (as $N_F(x) \subseteq N_G(x)$ and $D' - I \subseteq V_G - D''$). If $x \in S_1$ $(= N_G(L_1))$, then $N_G(x) \cap L_1 \neq \emptyset$ and $N_G(x) \cap (D' - I) \neq \emptyset$, and therefore $|N_G(x) \cap (V_G - D'')| \geq 2$ as L_1 and $D' - I$ are disjoint subsets of $V_G - D''$. It remains to show that $N_G(x) \cap (V_G - D'') = \emptyset$ (or, equivalently, that $N_G(x) \subseteq D''$) if $x \in L_0 \cup S_0$. Since $D'' = D - ((D' - I) \cup L_1)$, it suffices to show that $N_G(x) \subseteq D$, $N_G(x) \cap (D' - I) = \emptyset$ and $N_G(x) \cap L_1 = \emptyset$ if $x \in L_0 \cup S_0$. We know already that $N_G[S_0] \subseteq D$, and hence

$$N_G(L_0 \cup S_0) = N_G(L_0) \cup N_G(S_0) = S_0 \cup N_G(S_0) = N_G[S_0] \subseteq D,$$

which proves that $N_G(x) \subseteq D$ if $x \in L_0 \cup S_0$. It follows from the definition of the set S_0 that $N_G(S_0) \cap (D' - I) = \emptyset$ and therefore $N_G[S_0] \cap (D' - I) = \emptyset$ (as S_0 and D' are disjoint). Consequently $N_G(L_0 \cup S_0) \cap (D' - I) = N_G[S_0] \cap (D' - I) = \emptyset$ and this proves that $N_G(x) \cap (D' - I) = \emptyset$ if $x \in L_0 \cup S_0$. Finally, it follows from the properties of the sets S_0, S_1, L_0 and L_1 that $N_G(L_1) \cap (L_0 \cup S_0) = S_1 \cap (L_0 \cup S_0) = \emptyset$ and consequently $N_G(x) \cap L_1 = \emptyset$ if $x \in L_0 \cup S_0$. This completes the proof of the fact that D'' is a certified dominating set of G, which, however, contradicts the minimality of

D (as D'' is a proper subset of D). This proves that the set $\{v \in D \colon N_G[v] \subseteq D\}$ is a subset of $L_G \cup S_G^1$, and therefore the induced subgraph $G[\{v \in D \colon N_G[v] \subseteq D\}]$ is a corona. □

Now we give a characterization of the graphs for which the upper certified domination number is equal to their order.

Theorem 3.2. *Let G be a graph of order n. Then the following statements are equivalent:*

(1) *every non-trivial component of G is a corona;*
(2) $\gamma_{\text{cer}}(G) = n$*;*
(3) $\Gamma_{\text{cer}}(G) = n$*.*

Proof. The equivalence of (1) and (2) has been proved in [2]. It remains to prove the equivalence of (2) and (3). If $\gamma_{\text{cer}}(G) = n$, then $n = \gamma_{\text{cer}}(G) \leq \Gamma_{\text{cer}}(G) \leq n$ and, therefore, $\Gamma_{\text{cer}}(G) = n$. Assume now that $\Gamma_{\text{cer}}(G) = n$. Then V_G is a minimal certified dominating set of G, and, consequently, no proper subset of V_G is a certified dominating set of G. This implies that it cannot be $\gamma_{\text{cer}}(G) < n$, and, thus, it must be $\gamma_{\text{cer}}(G) = n$. □

It follows from the definition of a certified dominating set that if G is a graph of order n, then no $n-1$ vertices form a certified dominating set of G. Consequently, either $\Gamma_{\text{cer}}(G) = n$ or $\Gamma_{\text{cer}}(G) \leq n-2$. It is natural then to characterize all graphs G of order n for which $\Gamma_{\text{cer}}(G) = n-2$. We need the following definitions and notation. A *simple diadem* is a graph obtained from a corona by adding one new vertex and joining it to exactly one support vertex of the corona, whereas a *diadem* is a graph obtained from a corona by adding one new vertex and joining it to one leaf and its neighbor in the corona. Finally, by $G + H$ we denote the join of graphs G and H.

Theorem 3.3. *Let G be a connected graph of order $n \geq 3$. Then $\Gamma_{\text{cer}}(G) = n-2$ if and only if G is a simple diadem, a diadem, or one of the graphs $K_2 + \overline{K}_{n-2}$ and $\overline{K}_2 + \overline{K}_{n-2}$.*

Proof. It is a simple matter to observe that if a connected graph G of order $n \geq 3$ is a simple diadem, a diadem, $K_2 + \overline{K}_{n-2}$ or $\overline{K}_2 + \overline{K}_{n-2}$, then $\Gamma_{\text{cer}}(G) = n-2$.

Assume now that G is a connected graph of order $n \geq 3$ such that $\Gamma_{\text{cer}}(G) = n-2$. Let D be a Γ_{cer}-set of G, and let x and y be the only vertices in $V_G - D$. Let D_0 and D_2 be the sets $\{v \in D \colon N_G[v] \subseteq D\}$ and $\{v \in D \colon \{x, y\} \subseteq N_G(v)\}$, respectively.

Assume first that $D_0 = \emptyset$. Then $D_2 = D = V_G - \{x, y\}$ and the minimality of D easily implies the independence of D. Thus $G[D] = \overline{K}_{n-2}$ and now, from the fact that every vertex in D is adjacent to both x and y, it follows that G is one of the graphs $K_2 + \overline{K}_{n-2}$ and $\overline{K}_2 + \overline{K}_{n-2}$ (depending on whether or not x and y are adjacent).

Assume now that $D_0 \neq \emptyset$. It follows from Lemma 3.1 that every vertex in D_0 is a leaf or a weak support vertex of G and the induced subgraph $G[D_0]$ of G is a corona. We consider two subcases $|D_2| = 1$ and $|D_2| \geq 2$ separately.

First assume that $|D_2| = 1$ and let z be the only vertex in D_2. Let F be a connected component of $G[D_0]$. It follows from the connectivity of G that z is adjacent to at least one support vertex of F if F is of order at least four. If F is of order two, then z is adjacent to exactly one vertex of F (for otherwise z would be adjacent to both vertices of F and then the proper subset $D - V_F$ of D would be a certified dominating set of G, contrary to the minimality of D). From the above and from the fact that z is adjacent to both x and y it follows that G is a diadem or a simple diadem (depending on whether or not x and y are adjacent).

Finally assume that $|D_2| \geq 2$. We shall prove that this case is impossible. The connectivity of G implies that there exists a vertex $u \in D_2$ adjacent to a support belonging to D_0. Let us consider the sets $L = \{t \in L_G\colon d_G(t,u) = 2\}$ and $D' = D - (\{u\} \cup L)$. Now we claim that the set D' is a certified dominating set of G. It is obvious that $L \subseteq L_G \subseteq D_0$ and therefore $D' = (D_2 - \{u\}) \cup (D_0 - L)$ and $V_G - D' = \{x, y, u\} \cup L$. The vertices x and y are dominated by every vertex belonging to the non-empty set $D_2 - \{u\}$. If $t \in L$, then its only neighbor is in $D_0 - L$ and it dominates both t and u. This proves that D' is a dominating set of G. Thus it remains to observe that no vertex belonging to D' has exactly one neighbor in $V_G - D'$. This is obvious for every vertex t in $D_2 - \{u\}$, since x and y are two neighbors of t in $V_G - D'$. Thus assume that t is in $D_0 - L$. Then, since

$$D_0 - L = (S_G \cup L_G) - L = N_G(L) \cup N_G[L_G - L],$$

either $t \in N_G(L)$ or $t \in N_G[L_G - L] = (L_G - L) \cup N_G(L_G - L)$. If $t \in N_G(L)$, then u and the only vertex in $N_G(t) \cap L$ are two neighbors of t in $V_G - D'$. If $t \in L_G - L$, then the only element of $N_G(t)$ belongs to $N_G(L_G - L)$ and so $N_G(t) \cap (V_G - D') = \emptyset$ (since $N_G(L_G - L) \subseteq N_G[L_G - L] \subseteq D'$). Finally, if $t \in N_G(L_G - L)$, then $t \in N_G(L_G) - N_G(L)$ and $N_G(t) \cap (V_G - D') = \emptyset$ (because $V_G - D' = \{x, y, u\} \cup L$, $N_G(\{x, y, u\} \cup L) \subseteq \{x, y\} \cup D_2 \cup N_G(L)$, and $t \notin \{x, y\} \cup D_2 \cup N_G(L)$). This proves that the proper subset $D' = D - (\{u\} \cup L)$ of D is a certified dominating set of G, contrary to the minimality of D. Therefore the case $|D_2| \geq 2$ is impossible, which completes the proof. □

Next, we study the relation between upper domination number and upper certified domination number of a graph. The equality $\gamma(G) = \gamma_{\rm cer}(G)$ for any graph G with $\delta(G) \geq 2$ (see Corollary 2.3) could suggest that the analogous equality holds for the parameters Γ and $\Gamma_{\rm cer}$. Graph G of Figure 3 shows that this is not the case.

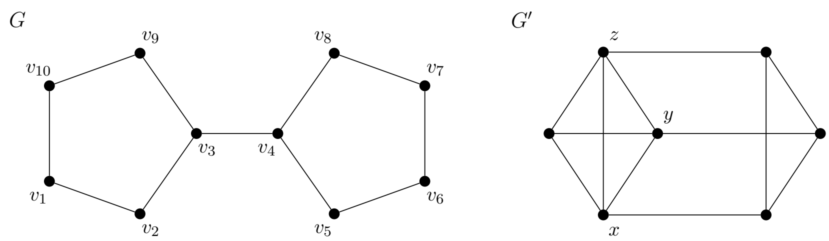

Fig. 3. Graphs G and G'

For this graph we have $\delta(G) = 2$, $\Gamma(G) = 5$ and $\Gamma_{\mathrm{cer}}(G) = 4$, and $\{v_2, v_3, v_6, v_7, v_9\}$ and $\{v_2, v_4, v_6, v_{10}\}$ are examples of Γ- and Γ_{cer}-sets of G, respectively. For graphs G with $\delta(G) \geq 2$, in fact, we always have $\Gamma_{\mathrm{cer}}(G) \leq \Gamma(G)$.

Lemma 3.4. *If G is a graph with $\delta(G) \geq 2$, then $\Gamma_{\mathrm{cer}}(G) \leq \Gamma(G)$.*

Proof. Let D be a Γ_{cer}-set of G. Since $\delta(G) \geq 2$, it follows from Lemma 3.1 that the set $\{v \in V_G : N_G[v] \subseteq D\}$ is empty. Hence D is also a minimal dominating set of G which implies that $\Gamma_{\mathrm{cer}}(G) \leq \Gamma(G)$. □

Taking into account the above lemma, it is natural then to characterize all graphs G with $\delta(G) \geq 2$ for which $\Gamma(G) = \Gamma_{\mathrm{cer}}(G)$. Here we have the following theorem.

Theorem 3.5. *Let G be a connected graph with $\delta(G) \geq 2$. If G has an independent Γ-set, then $\Gamma(G) = \Gamma_{\mathrm{cer}}(G)$.*

Proof. Let D be an independent Γ-set of G. If $v \in D$, then $N_G(v) \subseteq V_G - D$ (as D is independent), $|N_G(v)| \geq 2$ (as $\delta(G) \geq 2$), and therefore $|N_G(v) \cap (V_G - D)| \geq 2$. This proves that D is a minimal certified dominating set of G and implies the inequality $\Gamma(G) \leq \Gamma_{\mathrm{cer}}(G)$. The last inequality and Lemma 3.4 yield the equality $\Gamma(G) = \Gamma_{\mathrm{cer}}(G)$. □

As we have just seen, for graphs G with $\delta(G) \geq 2$ the equality $\Gamma(G) = \Gamma_{\mathrm{cer}}(G)$ holds, if G has an independent Γ-set. It should be noted that the converse implication, however, is not true. For example, one can check that for the graph G' shown in Figure 3 is $\delta(G') \geq 2$ and $\Gamma(G') = \Gamma_{\mathrm{cer}}(G') = 3$, but the only Γ-set $\{x, y, z\}$ of G' is not independent.

The *independence number* $\beta_0(G)$ of a graph G is the cardinality of a largest independent set of vertices of G. It is well-known that $\beta_0(G) \leq \Gamma(G)$ for every graph G. Therefore, by Theorem 3.5, we immediately have our final corollary.

Corollary 3.6. *If G is a graph with $\delta(G) \geq 2$ and $\beta_0(G) = \Gamma(G)$, then also $\beta_0(G) = \Gamma(G) = \Gamma_{\mathrm{cer}}(G)$.*

The equality of the parameters $\beta_0(G)$ and $\Gamma(G)$ has been studied by a number of authors (see for instance [1] and [7, pp. 80–84], and the references there) for well-known families of graphs, including strongly perfect graphs and their different subclasses: bipartite graphs, chordal graphs, and circular arc graphs, just to name a few. For all such graphs G, the equality $\Gamma(G) = \Gamma_{\mathrm{cer}}(G)$ is true if $\delta(G) \geq 2$.

4. CLOSING OPEN PROBLEMS

We close with the following list of open problems that we have yet to settle.

Problem 4.1. *Determine the class of graphs G for which $\gamma_{\mathrm{cer}}(G) = \Gamma_{\mathrm{cer}}(G)$.*

Problem 4.2. *Determine all the trees T for which $\gamma_{\mathrm{cer}}(T) = \gamma(T)$.*

Problem 4.3. *Let a, b, c, d be positive integers with $a \leq b \leq c \leq d$. Find necessary and sufficient conditions on a, b, c, d such that there exists a graph G with $\gamma(G) = a$, $\Gamma(G) = b$, $\gamma_{\mathrm{cer}}(G) = c$ and $\Gamma_{\mathrm{cer}}(G) = d$. Similarly, find necessary and sufficient conditions on a, b, c, d such that there exists a graph G with $\gamma(G) = a$, $\gamma_{\mathrm{cer}}(G) = b$, $\Gamma(G) = c$ and $\Gamma_{\mathrm{cer}}(G) = d$. Finally, find necessary and sufficient conditions on a, b, c, d such that there exists a graph G with $\gamma(G) = a$, $\gamma_{\mathrm{cer}}(G) = b$, $\Gamma_{\mathrm{cer}}(G) = c$ and $\Gamma(G) = d$.*

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