# Homoclinic solutions for a class of the second order Hamiltonian systems 

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#### Abstract

We study the existence of homoclinic orbits for the second order Hamiltonian system $\ddot{q}+$ $V_{q}(t, q)=f(t)$, where $q \in \mathbb{R}^{n}$ and $V \in C^{1}\left(\mathbb{R} \times \mathbb{R}^{n}, \mathbb{R}\right), V(t, q)=-K(t, q)+W(t, q)$ is $T$-periodic in $t$. A map $K$ satisfies the "pinching" condition $b_{1}|q|^{2} \leqslant K(t, q) \leqslant b_{2}|q|^{2}, W$ is superlinear at the infinity and $f$ is sufficiently small in $L^{2}\left(\mathbb{R}, \mathbb{R}^{n}\right)$. A homoclinic orbit is obtained as a limit of $2 k T$-periodic solutions of a certain sequence of the second order differential equations. © 2005 Elsevier Inc. All rights reserved.


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## 1. Introduction

In this paper, we shall be concerned with the existence of homoclinic orbits for the second order Hamiltonian system:

$$
\begin{equation*}
\ddot{q}+V_{q}(t, q)=f(t), \tag{HS}
\end{equation*}
$$

[^0]where $t \in \mathbb{R}, q \in \mathbb{R}^{n}$ and functions $V: \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}$ and $f: \mathbb{R} \rightarrow \mathbb{R}^{n}$ satisfy:
$\left(\mathrm{H}_{1}\right) V(t, q)=-K(t, q)+W(t, q)$, where $K, W: \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}$ are $C^{1}$-maps, $T$-periodic with respect to $t, T>0$,
$\left(\mathrm{H}_{2}\right)$ there are constants $b_{1}, b_{2}>0$ such that for all $(t, q) \in \mathbb{R} \times \mathbb{R}^{n}$
$$
b_{1}|q|^{2} \leqslant K(t, q) \leqslant b_{2}|q|^{2}
$$
$\left(\mathrm{H}_{3}\right)$ for all $(t, q) \in \mathbb{R} \times \mathbb{R}^{n}, K(t, q) \leqslant\left(q, K_{q}(t, q)\right) \leqslant 2 K(t, q)$,
$\left(\mathrm{H}_{4}\right) W_{q}(t, q)=o(|q|)$, as $|q| \rightarrow 0$ uniformly with respect to $t$,
$\left(\mathrm{H}_{5}\right)$ there is a constant $\mu>2$ such that for every $t \in \mathbb{R}$ and $q \in \mathbb{R}^{n} \backslash\{0\}$
$$
0<\mu W(t, q) \leqslant\left(q, W_{q}(t, q)\right)
$$
$\left(\mathrm{H}_{6}\right) f: \mathbb{R} \rightarrow \mathbb{R}^{n}$ is a continuous and bounded function.
Here and subsequently, $(\cdot, \cdot): \mathbb{R}^{n} \times \mathbb{R}^{n} \rightarrow \mathbb{R}$ denotes the standard inner product in $\mathbb{R}^{n}$ and $|\cdot|$ is the induced norm.

We will say that a solution $q$ of (HS) is homoclinic (to 0 ) if $q(t) \rightarrow 0$ as $t \rightarrow \pm \infty$. In addition, if $q \not \equiv 0$ then $q$ is called a nontrivial homoclinic solution.

For each $k \in \mathbb{N}$, let $E_{k}:=W_{2 k T}^{1,2}\left(\mathbb{R}, \mathbb{R}^{n}\right)$, the Hilbert space of $2 k T$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^{n}$ under the norm

$$
\|q\|_{E_{k}}:=\left(\int_{-k T}^{k T}\left(|\dot{q}(t)|^{2}+|q(t)|^{2}\right) d t\right)^{1 / 2}
$$

Furthermore, let $L_{2 k T}^{\infty}\left(\mathbb{R}, \mathbb{R}^{n}\right)$ denote a space of $2 k T$-periodic essentially bounded (measurable) functions from $\mathbb{R}$ into $\mathbb{R}^{n}$ equipped with the norm

$$
\|q\|_{L_{2 k T}^{\infty}}:=e s s \sup \{|q(t)|: t \in[-k T, k T]\} .
$$

We begin with a result which is a direct consequence of estimations made by Rabinowitz in [12].

Proposition 1.1. There is a positive constant $C$ such that for each $k \in N$ and $q \in E_{k}$ the following inequality holds:

$$
\begin{equation*}
\|q\|_{L_{2 k T}^{\infty}} \leqslant C\|q\|_{E_{k}} . \tag{1}
\end{equation*}
$$

One can easily show that the inequality (1) holds true with constant $C=\sqrt{2}$ if $T \geqslant \frac{1}{2}$ (see Fact 2.8).

Set $M:=\sup \{W(t, q): t \in[0, T],|q|=1\}, \bar{b}_{1}:=\min \left\{1,2 b_{1}\right\}, \bar{b}_{2}:=\max \left\{1,2 b_{2}\right\}$ and suppose that:
$\left(\mathrm{H}_{7}\right) 2 M<\bar{b}_{1}$ and $\left(\int_{\mathbb{R}}|f(t)|^{2} d t\right)^{1 / 2} \leqslant \frac{\beta}{2 C}$, where $0<\beta<\bar{b}_{1}-2 M$ and $C$ is a constant from Proposition 1.1.

We will prove the following theorem:
Theorem 1.2. If the conditions $\left(\mathrm{H}_{1}\right)-\left(\mathrm{H}_{7}\right)$ are satisfied then the system (HS) possesses a nontrivial homoclinic solution $q \in W^{1,2}\left(\mathbb{R}, \mathbb{R}^{n}\right)$ such that $\dot{q}(t) \rightarrow 0$ as $t \rightarrow \pm \infty$.

In recent years several authors studied homoclinic orbits for Hamiltonian systems via critical point theory. In particular, the second order systems were considered in [1,3,5-7,11-13,16], and those of the first order in [4,8-10,14,15]. Our study is motivated by a paper of Rabinowitz [12] in which the existence of a nontrivial homoclinic solution for the second order Hamiltonian system

$$
\ddot{q}+V_{q}(t, q)=0
$$

was proved. The function $V$ considered by the author is of the form

$$
\begin{equation*}
V(t, q)=-\frac{1}{2}(L(t) q, q)+\bar{W}(t, q) \tag{2}
\end{equation*}
$$

where $L$ is a continuous $T$-periodic matrix valued function such that $L(t)$ is positive definite and symmetric for all $t \in[0, T], \bar{W}$ satisfies $\left(\mathrm{H}_{4}\right)$ and $\left(\mathrm{H}_{5}\right)$. Let us note that conditions $\left(\mathrm{H}_{2}\right)$ and $\left(\mathrm{H}_{3}\right)$ are satisfied if $K(t, q)=\frac{1}{2}(L(t) q, q)$. On the other hand, one can easily check that if

$$
K(t, x)= \begin{cases}\left(1+\frac{1}{1+x^{2}}\right) x^{2} & \text { for } x \geqslant 0 \\ \left(1+\frac{2}{1+x^{2}}\right) x^{2} & \text { for } x<0\end{cases}
$$

and $W(t, x)=x^{4}$, where $t, x \in \mathbb{R}$, then $V(t, x)=-K(t, x)+W(t, x)$ cannot be represented in the form (2) with $\bar{W}$ satisfying $\left(\mathrm{H}_{4}\right)$, $\left(\mathrm{H}_{5}\right)$ while $V$ satisfies conditions $\left(\mathrm{H}_{1}\right)-\left(\mathrm{H}_{5}\right)$. Hence, our theorem extends the result from [12] even if $f(t)=0$. It follows from our assumptions that $q(t)=0$ is a solution of (HS) only if $f(t)=0$. Therefore, if $f$ is a nonzero function the existence of a homoclinic solution of (HS) implies its nontriviality.

Similarly to [12] a homoclinic solution of (HS) is obtained as a limit, as $k \rightarrow+\infty$, of a certain sequence of functions $q_{k} \in E_{k}$. However, in our approach, we consider a sequence of systems of differential equations:

$$
\begin{equation*}
\ddot{q}+V_{q}(t, q)=f_{k}(t) \tag{k}
\end{equation*}
$$

where for each $k \in \mathbb{N}, f_{k}: \mathbb{R} \rightarrow \mathbb{R}^{n}$ is a $2 k T$-periodic extension of the restriction of $f$ to the interval $[-k T, k T]$ and $q_{k}$ is a $2 k T$-periodic solution of $\left(\mathrm{HS}_{k}\right)$ obtained via the Mountain Pass Theorem.

Part of the difficulty in treating (HS) is caused by the fact that in order to get appropriate convergence of the sequence of approximative functions $\left\{q_{k}\right\}_{k \in \mathbb{N}}$ we need the constants $\rho$ and $\alpha$ appearing in the condition (iii) of the Mountain Pass Theorem (see Theorem 2.5) to be independent of $k$.

## 2. Proof of Theorem 1.2

At first let us recall some properties of the function $W(t, q)$ from [12]. They all are necessary to the proof of Theorem 1.2.

Fact 2.1. For every $t \in[0, T]$ the following inequalities hold:

$$
\begin{gather*}
W(t, q) \leqslant W\left(t, \frac{q}{|q|}\right)|q|^{\mu} \quad \text { if } 0<|q| \leqslant 1,  \tag{3}\\
W(t, q) \geqslant W\left(t, \frac{q}{|q|}\right)|q|^{\mu} \quad \text { if }|q| \geqslant 1 . \tag{4}
\end{gather*}
$$

To prove this fact it suffices to show that for every $q \neq 0$ and $t \in[0, T]$ the function $(0,+\infty) \ni \zeta \rightarrow W\left(t, \zeta^{-1} q\right) \zeta^{\mu}$ is nonincreasing. It is an immediate consequence of $\left(\mathrm{H}_{5}\right)$.

Fact 2.2. Set $m:=\inf \{W(t, q): t \in[0, T],|q|=1\}$. Then for every $\zeta \in \mathbb{R} \backslash\{0\}$ and $q \in E_{k} \backslash\{0\}$ we have

$$
\begin{equation*}
\int_{-k T}^{k T} W(t, \zeta q(t)) d t \geqslant m|\zeta|^{\mu} \int_{-k T}^{k T}|q(t)|^{\mu} d t-2 k T m \tag{5}
\end{equation*}
$$

Proof. Fix $\zeta \in \mathbb{R} \backslash\{0\}$ and $q \in E_{k} \backslash\{0\}$. Set $A_{k}=\{t \in[-k T, k T]:|\zeta q(t)| \leqslant 1\}$, and $B_{k}=\{t \in[-k T, k T]:|\zeta q(t)| \geqslant 1\}$. From (4) we obtain

$$
\begin{aligned}
\int_{-k T}^{k T} W(t, \zeta q(t)) d t & \geqslant \int_{B_{k}} W(t, \zeta q(t)) d t \geqslant \int_{B_{k}} W\left(t, \frac{\zeta q(t)}{|\zeta q(t)|}\right)|\zeta q(t)|^{\mu} d t \\
& \geqslant m \int_{B_{k}}|\zeta q(t)|^{\mu} d t \\
& \geqslant m \int_{-k T}^{k T}|\zeta q(t)|^{\mu} d t-m \int_{A_{k}}|\zeta q(t)|^{\mu} d t \\
& \geqslant m|\zeta|^{\mu} \int_{-k T}^{k T}|q(t)|^{\mu} d t-2 k T m .
\end{aligned}
$$

Fact 2.3. Let $Y:[0,+\infty) \rightarrow[0,+\infty)$ be given as follows: $Y(0)=0$ and

$$
\begin{equation*}
Y(s)=\max _{\substack{t \in[0, T] \\ 0<|q| \leqslant s}} \frac{\left(q, W_{q}(t, q)\right)}{|q|^{2}} \tag{6}
\end{equation*}
$$

for $s>0$. Then $Y$ is continuous, nondecreasing, $Y(s)>0$ for $s>0$ and $Y(s) \rightarrow+\infty$ as $s \rightarrow+\infty$.

It is easy to verify this fact applying $\left(\mathrm{H}_{4}\right)$, $\left(\mathrm{H}_{5}\right)$ and (4).
Assumptions $\left(\mathrm{H}_{4}\right)$ and $\left(\mathrm{H}_{5}\right)$ imply that $W(t, q)=o\left(|q|^{2}\right)$ as $q \rightarrow 0$ uniformly for $t \in[0, T]$ and $W(t, 0)=0, W_{q}(t, 0)=0$. Moreover, from $\left(\mathrm{H}_{2}\right)$ we conclude that $K(t, 0)=0, K_{q}(t, 0)=0$.

Before we will prove Theorem 1.2, we have to introduce more notation and some necessary definitions. For each $k \in \mathbb{N}$, let $L_{2 k T}^{2}\left(\mathbb{R}, \mathbb{R}^{n}\right)$ denote the Hilbert space of $2 k T$ periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^{n}$ under the norm $\|q\|_{L_{2 k T}^{2}}=\left(\int_{-k T}^{k T}|q(t)|^{2} d t\right)^{1 / 2}$. Let $f_{k}: \mathbb{R} \rightarrow \mathbb{R}^{n}$ be a $2 k T$-periodic extension of $f_{[[-k T, k T]}$ onto $\mathbb{R}$. From $\left(\mathrm{H}_{7}\right)$ it follows that $\left\|f_{k}\right\|_{L_{2 k T}^{2}} \leqslant \beta / 2 C$. Consider the second order Hamiltonian system:

$$
\begin{equation*}
\ddot{q}+V_{q}(t, q)=f_{k}(t) . \tag{k}
\end{equation*}
$$

Let $\eta_{k}: E_{k} \rightarrow[0,+\infty)$ be given by

$$
\begin{equation*}
\eta_{k}(q)=\left(\int_{-k T}^{k T}\left[|\dot{q}(t)|^{2}+2 K(t, q(t))\right] d t\right)^{1 / 2} \tag{7}
\end{equation*}
$$

By $\left(\mathrm{H}_{2}\right)$,

$$
\begin{equation*}
\bar{b}_{1}\|q\|_{E_{k}}^{2} \leqslant \eta_{k}^{2}(q) \leqslant \bar{b}_{2}\|q\|_{E_{k}}^{2} . \tag{8}
\end{equation*}
$$

It is worth pointing out that if the function $K(t, q)$ is of the form $\frac{1}{2}(L(t) q, q)$ with a matrix valued function $L$ satisfying the same conditions as in [12] then $\eta_{k}$ determined by (7) is a norm in $E_{k}$ equivalent to the norm $\|\cdot\|_{E_{k}}$. Let $I_{k}: E_{k} \rightarrow \mathbb{R}$ be defined by

$$
\begin{align*}
I_{k}(q) & =\int_{-k T}^{k T}\left[\frac{1}{2}|\dot{q}(t)|^{2}-V(t, q(t))\right] d t+\int_{-k T}^{k T}\left(f_{k}(t), q(t)\right) d t \\
& =\frac{1}{2} \eta_{k}^{2}(q)-\int_{-k T}^{k T} W(t, q(t)) d t+\int_{-k T}^{k T}\left(f_{k}(t), q(t)\right) d t \tag{9}
\end{align*}
$$

Then $I_{k} \in C^{1}\left(E_{k}, \mathbb{R}\right)$ and it is easy to check that

$$
\begin{equation*}
I_{k}^{\prime}(q) v=\int_{-k T}^{k T}\left[(\dot{q}(t), \dot{v}(t))-\left(V_{q}(t, q(t)), v(t)\right)\right] d t+\int_{-k T}^{k T}\left(f_{k}(t), v(t)\right) d t \tag{10}
\end{equation*}
$$

and

$$
\begin{equation*}
I_{k}^{\prime}(q) q \leqslant \eta_{k}^{2}(q)-\int_{-k T}^{k T}\left(W_{q}(t, q(t)), q(t)\right) d t+\int_{-k T}^{k T}\left(f_{k}(t), q(t)\right) d t \tag{11}
\end{equation*}
$$

Moreover, it is clear that critical points of $I_{k}$ are classical $2 k T$-periodic solutions of ( $\mathrm{HS}_{k}$ ).

We have divided the proof of Theorem 1.2 into a sequence of lemmas.
Lemma 2.4. If $V$ and $f$ satisfy $\left(\mathrm{H}_{1}\right)-\left(\mathrm{H}_{7}\right)$ then for every $k \in \mathbb{N}$ the system $\left(\mathrm{HS}_{k}\right)$ possesses a $2 k T$-periodic solution.

We will obtain a critical point of $I_{k}$ by the use of a standard version of the Mountain Pass Theorem (see [2]). It provides the minimax characterisation for the critical value which is important for what follows. Therefore, we state this theorem precisely.

Theorem 2.5 (see Ambrosetti and Rabinowitz [2]). Let E be a real Banach space and $I: E \rightarrow \mathbb{R}$ be a $C^{1}$-smooth functional. If I satisfies the following conditions:
(i) $I(0)=0$,
(ii) every sequence $\left\{u_{j}\right\}_{j \in \mathbb{N}}$ in $E$ such that $\left\{I\left(u_{j}\right)\right\}_{j \in \mathbb{N}}$ is bounded in $\mathbb{R}$ and $I^{\prime}\left(u_{j}\right) \rightarrow$ 0 in $E^{*}$, as $j \rightarrow+\infty$, contains a convergent subsequence (the Palais-Smale condition),
(iii) there exist constants $\varrho, \alpha>0$ such that $I_{\mid \partial B_{\varrho}(0)} \geqslant \alpha$,
(iv) there exists $e \in E \backslash \bar{B}_{\varrho}(0)$ such that $I(e) \leqslant 0$,
where $B_{\varrho}(0)$ is an open ball in $E$ of radius $\varrho$ centred at 0 , then I possesses a critical value $c \geqslant \alpha$ given by

$$
c=\inf _{g \in \Gamma} \max _{s \in[0,1]} I(g(s))
$$

where

$$
\Gamma=\{g \in C([0,1], E): g(0)=0, g(1)=e\}
$$

Proof of Lemma 2.4. In our case it is clear that $I_{k}(0)=0$. We show that $I_{k}$ satisfies the Palais-Smale condition. Assume that $\left\{u_{j}\right\}_{j \in \mathbb{N}}$ in $E_{k}$ is a sequence such that $\left\{I_{k}\left(u_{j}\right)\right\}_{j \in \mathbb{N}}$ is bounded and $I_{k}^{\prime}\left(u_{j}\right) \rightarrow 0$ as $j \rightarrow+\infty$. Then there exists a constant $C_{k}>0$ such that

$$
\begin{equation*}
\left|I_{k}\left(u_{j}\right)\right| \leqslant C_{k}, \quad\left\|I_{k}^{\prime}\left(u_{j}\right)\right\|_{E_{k}^{*}} \leqslant C_{k} \tag{12}
\end{equation*}
$$

for every $j \in \mathbb{N}$. We first prove that $\left\{u_{j}\right\}_{j \in \mathbb{N}}$ is bounded. By (9) and ( $\mathrm{H}_{5}$ ),

$$
\begin{align*}
\eta_{k}^{2}\left(u_{j}\right) \leqslant & 2 I_{k}\left(u_{j}\right)+\frac{2}{\mu} \int_{-k T}^{k T}\left(W_{q}\left(t, u_{j}(t)\right), u_{j}(t)\right) d t \\
& -2 \int_{-k T}^{k T}\left(f_{k}(t), u_{j}(t)\right) d t \tag{13}
\end{align*}
$$

From (13) and (11) we obtain

$$
\begin{align*}
\left(1-\frac{2}{\mu}\right) \eta_{k}^{2}\left(u_{j}\right) \leqslant & 2 I_{k}\left(u_{j}\right)-\frac{2}{\mu} I_{k}^{\prime}\left(u_{j}\right) u_{j} \\
& -\left(2-\frac{2}{\mu}\right) \int_{-k T}^{k T}\left(f_{k}(t), u_{j}(t)\right) d t \tag{14}
\end{align*}
$$

From (14) and (8) it follows that

$$
\begin{align*}
\left(1-\frac{2}{\mu}\right) \bar{b}_{1}\left\|u_{j}\right\|_{E_{k}}^{2} \leqslant & 2 I_{k}\left(u_{j}\right) \\
& +\left(\frac{2}{\mu}\left\|I_{k}^{\prime}\left(u_{j}\right)\right\|_{E_{k}^{*}}+\left(2-\frac{2}{\mu}\right)\left\|f_{k}\right\|_{L_{2 k T}^{2}}\right)\left\|u_{j}\right\|_{E_{k}} . \tag{15}
\end{align*}
$$

Combining (15) with $\left(\mathrm{H}_{7}\right)$ and (12) we get

$$
\begin{equation*}
\left(1-\frac{2}{\mu}\right) \bar{b}_{1}\left\|u_{j}\right\|_{E_{k}}^{2}-\left(\frac{2 C_{k}}{\mu}+\left(2-\frac{2}{\mu}\right) \frac{\beta}{2 C}\right)\left\|u_{j}\right\|_{E_{k}}-2 C_{k} \leqslant 0 . \tag{16}
\end{equation*}
$$

Since $\mu>2$, (16) shows that $\left\{u_{j}\right\}_{j \in \mathbb{N}}$ is bounded in $E_{k}$. Going if necessary to a subsequence, we can assume that there exists $u \in E_{k}$ such that $u_{j} \rightharpoonup u$, as $j \rightarrow+\infty$, in $E_{k}$, which implies $u_{j} \rightarrow u$ uniformly on [-kT, $k T$ ]. Hence $\left(I_{k}^{\prime}\left(u_{j}\right)-I_{k}^{\prime}(u)\right)\left(u_{j}-u\right) \rightarrow$ $0,\left\|u_{j}-u\right\|_{L_{2 k T}^{2}} \rightarrow 0$ and

$$
\int_{-k T}^{k T}\left(V_{q}\left(t, u_{j}(t)\right)-V_{q}(t, u(t)), u_{j}(t)-u(t)\right) d t \rightarrow 0
$$

as $j \rightarrow+\infty$. Moreover, an easy computation shows that

$$
\begin{aligned}
\left(I_{k}^{\prime}\left(u_{j}\right)-I_{k}^{\prime}(u)\right)\left(u_{j}-u\right)= & \left\|i_{j}-\dot{u}\right\|_{L_{2 k T}^{2}}^{2} \\
& -\int_{-k T}^{k T}\left(V_{q}\left(t, u_{j}(t)\right)-V_{q}(t, u(t)), u_{j}(t)-u(t)\right) d t
\end{aligned}
$$

and so $\left\|\dot{u_{j}}-\dot{u}\right\|_{L_{2 k T}^{2}}^{2} \rightarrow 0$. Consequently, $\left\|u_{j}-u\right\|_{E_{k}} \rightarrow 0$.

We now show that there exist constants $\varrho, \alpha>0$ independent of $k$ such that every $I_{k}$ satisfies the assumption (iii) of Theorem 2.5 with these constants. Assume that $0<\|q\|_{L_{2 k T}^{\infty}} \leqslant 1$. By (3) we have

$$
\begin{aligned}
\int_{-k T}^{k T} W(t, q(t)) d t & \leqslant \int_{-k T}^{k T} W\left(t, \frac{q(t)}{|q(t)|}\right)|q(t)|^{\mu} d t \\
& \leqslant M \int_{-k T}^{k T}|q(t)|^{2} d t \leqslant M\|q\|_{E_{k}}^{2}
\end{aligned}
$$

and, in consequence, combining this with (8) and $\left(\mathrm{H}_{7}\right)$ we obtain

$$
\begin{align*}
I_{k}(q) & \geqslant \frac{1}{2} \bar{b}_{1}\|q\|_{E_{k}}^{2}-M\|q\|_{E_{k}}^{2}-\left\|f_{k}\right\|_{L_{2 k T}^{2}}\|q\|_{L_{2 k T}^{2}} \\
& \geqslant \frac{1}{2} \bar{b}_{1}\|q\|_{E_{k}}^{2}-M\|q\|_{E_{k}}^{2}-\frac{\beta}{2 C}\|q\|_{E_{k}} \\
& =\frac{1}{2}\left(\bar{b}_{1}-\beta-2 M\right)\|q\|_{E_{k}}^{2}+\frac{\beta}{2}\|q\|_{E_{k}}^{2}-\frac{\beta}{2 C}\|q\|_{E_{k}} \tag{17}
\end{align*}
$$

Note that $\left(\mathrm{H}_{7}\right)$ implies $\bar{b}_{1}-\beta-2 M>0$. Set

$$
\varrho=\frac{1}{C}, \quad \alpha=\frac{\bar{b}_{1}-\beta-2 M}{2 C^{2}} .
$$

By (1), if $\|q\|_{E_{k}}=\varrho$ then $0<\|q\|_{L_{2 k T}^{\infty}} \leqslant 1$ and (17) gives $I_{k}(q) \geqslant \alpha$.
It remains to prove that for every $k \in \mathbb{N}$ there exists $e_{k} \in E_{k}$ such that $\left\|e_{k}\right\|_{E_{k}}>\varrho$ and $I_{k}\left(e_{k}\right) \leqslant 0$. By the use of (5), (9) and (8) we have that for every $\zeta \in \mathbb{R} \backslash\{0\}$ and $q \in E_{k} \backslash\{0\}$ the following inequality holds:

$$
\begin{align*}
I_{k}(\zeta q) \leqslant & \frac{\bar{b}_{2} \zeta^{2}}{2}\|q\|_{E_{k}}^{2}-m|\zeta|^{\mu} \int_{-k T}^{k T}|q(t)|^{\mu} d t \\
& +|\zeta| \cdot\left\|f_{k}\right\|_{L_{2 k T}^{2}}\|q\|_{L_{2 k T}^{2}}+2 k T m \tag{18}
\end{align*}
$$

Take $Q \in E_{1}$ such that $Q( \pm T)=0$. Since $\mu>2$ and $m>0$, (18) implies that there exists $\xi \in \mathbb{R} \backslash\{0\}$ such that $\|\xi Q\|_{E_{1}}>\varrho$ and $I_{1}(\xi Q)<0$. Set $e_{1}(t)=\xi Q(t)$ and

$$
e_{k}(t)= \begin{cases}e_{1}(t) & \text { for }|t| \leqslant T,  \tag{19}\\ 0 & \text { for } T<|t| \leqslant k T\end{cases}
$$

for $k>0$. Then $e_{k} \in E_{k},\left\|e_{k}\right\|_{E_{k}}=\left\|e_{1}\right\|_{E_{1}}>\varrho$ and $I_{k}\left(e_{k}\right)=I_{1}\left(e_{1}\right)<0$ for every $k \in \mathbb{N}$. By Theorem 2.5, $I_{k}$ possesses a critical value $c_{k} \geqslant \alpha$ given by

$$
\begin{equation*}
c_{k}=\inf _{g \in \Gamma_{k}} \max _{s \in[0,1]} I_{k}(g(s)), \tag{20}
\end{equation*}
$$

where

$$
\Gamma_{k}=\left\{g \in C\left([0,1], E_{k}\right): g(0)=0, g(1)=e_{k}\right\}
$$

Hence, for every $k \in \mathbb{N}$, there is $q_{k} \in E_{k}$ such that

$$
\begin{equation*}
I_{k}\left(q_{k}\right)=c_{k}, \quad I_{k}^{\prime}\left(q_{k}\right)=0 \tag{21}
\end{equation*}
$$

The function $q_{k}$ is a desired classical $2 k T$-periodic solution of $\left(\mathrm{HS}_{k}\right)$. Since $c_{k}>0$, $q_{k}$ is a nontrivial solution even if $f_{k}(t)=0$.

Let $C_{\text {loc }}^{p}\left(\mathbb{R}, \mathbb{R}^{n}\right)$, where $p \in \mathbb{N} \cup\{0\}$, denote the space of $C^{p}$ functions on $\mathbb{R}$ with values in $\mathbb{R}^{n}$ under the topology of almost uniformly convergence of functions and all derivatives up to the order $p$. Using the Arzelà-Ascoli theorem we prove what follows.

Lemma 2.6. Let $\left\{q_{k}\right\}_{k \in \mathbb{N}}$ be the sequence given by (21). There exist an increasing function $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ and a $C^{1}$ function $q_{0}: \mathbb{R} \rightarrow \mathbb{R}^{n}$ such that $q_{\varphi(k)} \rightarrow q_{0}$, as $k \rightarrow$ $+\infty$, in $C_{\text {loc }}^{1}\left(\mathbb{R}, \mathbb{R}^{n}\right)$.

Proof. The first step in the proof is to show that the sequences $\left\{c_{k}\right\}_{k \in \mathbb{N}}$ and $\left\{\left\|q_{k}\right\|_{E_{k}}\right\}_{k \in \mathbb{N}}$ are bounded. For every $k \in \mathbb{N}$, let $g_{k}:[0,1] \rightarrow E_{k}$ be a curve given by $g_{k}(s)=s e_{k}$, where $e_{k}$ is determined by (19). Then $g_{k} \in \Gamma_{k}$ and $I_{k}\left(g_{k}(s)\right)=I_{1}\left(g_{1}(s)\right)$ for all $k \in \mathbb{N}$ and $s \in[0,1]$. Therefore, by (20),

$$
\begin{equation*}
c_{k} \leqslant \max _{s \in[0,1]} I_{1}\left(g_{1}(s)\right) \equiv M_{0} \tag{22}
\end{equation*}
$$

independently of $k \in \mathbb{N}$. As $I_{k}^{\prime}\left(q_{k}\right)=0$, we receive from (9), (11) and $\left(\mathrm{H}_{5}\right)$ that

$$
\begin{aligned}
c_{k} & =I_{k}\left(q_{k}\right)-\frac{1}{2} I_{k}^{\prime}\left(q_{k}\right) q_{k} \\
& \geqslant\left(\frac{\mu}{2}-1\right) \int_{-k T}^{k T} W\left(t, q_{k}(t)\right) d t+\frac{1}{2} \int_{-k T}^{k T}\left(f_{k}(t), q_{k}(t)\right) d t
\end{aligned}
$$

and hence

$$
\int_{-k T}^{k T} W\left(t, q_{k}(t)\right) d t \leqslant \frac{1}{\mu-2}\left(2 c_{k}-\int_{-k T}^{k T}\left(f_{k}(t), q_{k}(t)\right) d t\right)
$$

Combining the above with (8), (9) and (22) we have

$$
\bar{b}_{1}\left\|q_{k}\right\|_{E_{k}}^{2} \leqslant \frac{2 \mu M_{0}}{\mu-2}+\frac{2 \mu-2}{\mu-2}\left\|f_{k}\right\|_{L_{2 k T}^{2}}\left\|q_{k}\right\|_{L_{2 k T}^{2}}
$$

and, in consequence, by $\left(\mathrm{H}_{7}\right)$

$$
\begin{equation*}
\bar{b}_{1}\left\|q_{k}\right\|_{E_{k}}^{2}-\frac{\beta(\mu-1)}{C(\mu-2)}\left\|q_{k}\right\|_{E_{k}}-\frac{2 \mu M_{0}}{\mu-2} \leqslant 0 \tag{23}
\end{equation*}
$$

Since $\bar{b}_{1}>0$ and all coefficients of (23) are independent of $k$, we see that there is $M_{1}>0$ independent of $k$ such that

$$
\begin{equation*}
\left\|q_{k}\right\|_{E_{k}} \leqslant M_{1} \tag{24}
\end{equation*}
$$

We now observe that the sequences $\left\{q_{k}\right\}_{k \in \mathbb{N}},\left\{\dot{q}_{k}\right\}_{k \in \mathbb{N}}$ and $\left\{\ddot{q}_{k}\right\}_{k \in \mathbb{N}}$ are uniformly bounded. By (1),

$$
\begin{equation*}
\left\|q_{k}\right\|_{L_{2 k T}^{\infty}}^{\infty} \leqslant C M_{1} \equiv M_{2} \tag{25}
\end{equation*}
$$

for every $k \in \mathbb{N}$. Since $q_{k}$ satisfies $\left(\mathrm{HS}_{k}\right)$, if $t \in[-k T, k T]$ we have

$$
\left|\ddot{q}_{k}(t)\right| \leqslant\left|f_{k}(t)\right|+\left|V_{q}\left(t, q_{k}(t)\right)\right|=|f(t)|+\left|V_{q}\left(t, q_{k}(t)\right)\right| .
$$

Therefore $(25),\left(\mathrm{H}_{1}\right)$ and $\left(\mathrm{H}_{6}\right)$ imply that there is $M_{3}>0$ independent of $k$ such that

$$
\begin{equation*}
\left\|\ddot{q}_{k}\right\|_{L_{2 k T}^{\infty}} \leqslant M_{3} . \tag{26}
\end{equation*}
$$

From the Mean Value Theorem it follows that for every $k \in \mathbb{N}$ and $t \in \mathbb{R}$ there exists $\tau_{k} \in[t-1, t]$ such that

$$
\dot{q}_{k}\left(\tau_{k}\right)=\int_{t-1}^{t} \dot{q}_{k}(s) d s=q_{k}(t)-q_{k}(t-1) .
$$

In consequence, combining the above with (25) and (26)

$$
\begin{aligned}
\left|\dot{q}_{k}(t)\right| & =\left|\int_{\tau_{k}}^{t} \ddot{q}_{k}(s) d s+\dot{q}_{k}\left(\tau_{k}\right)\right| \\
& \leqslant \int_{t-1}^{t}\left|\ddot{q}_{k}(s)\right| d s+\left|q_{k}(t)-q_{k}(t-1)\right| \leqslant M_{3}+2 M_{2} \equiv M_{4}
\end{aligned}
$$

and hence for every $k \in \mathbb{N}$

$$
\begin{equation*}
\left\|\dot{q}_{k}\right\|_{L_{2 k T}}^{\infty} \leqslant M_{4} . \tag{27}
\end{equation*}
$$

The task is now to show that $\left\{q_{k}\right\}_{k \in \mathbb{N}}$ and $\left\{\dot{q}_{k}\right\}_{k \in \mathbb{N}}$ are equicontinuous. Of course, it suffices to prove that both sequences satisfy the Lipschitz condition with some constants independent of $k$. Let $k \in \mathbb{N}$ and $t, t_{0} \in \mathbb{R}$. Then

$$
\left|q_{k}(t)-q_{k}\left(t_{0}\right)\right|=\left|\int_{t_{0}}^{t} \dot{q}_{k}(s) d s\right| \leqslant\left|\int_{t_{0}}^{t}\right| \dot{q}_{k}(s)|d s| \leqslant M_{4}\left|t-t_{0}\right|
$$

by (27), and analogously,

$$
\left|\dot{q}_{k}(t)-\dot{q}_{k}\left(t_{0}\right)\right| \leqslant M_{3}\left|t-t_{0}\right|
$$

by (26). Since $\left\{q_{k}\right\}_{k \in \mathbb{N}}$ and $\left\{\dot{q}_{k}\right\}_{k \in \mathbb{N}}$ are bounded in $L_{2 k T}^{\infty}\left(\mathbb{R}, \mathbb{R}^{n}\right)$ and equicontinuous, we obtain the existence of a subsequence $\left\{q_{\varphi(k)}\right\}_{k \in \mathbb{N}}$ convergent to a certain $q_{0}$ in $C_{\text {loc }}^{1}\left(\mathbb{R}, \mathbb{R}^{n}\right)$ by using the Arzelà-Ascoli theorem.

Our next goal is to show that $q_{0}$ is the desired homoclinic solution of (HS). For this purpose, we need the following observations.

Fact 2.7. Let $q: \mathbb{R} \rightarrow \mathbb{R}^{n}$ be a continuous mapping. If a weak derivative $\dot{q}: \mathbb{R} \rightarrow \mathbb{R}^{n}$ is continuous at $t_{0}$, then $q$ is differentiable at $t_{0}$ and

$$
\lim _{t \rightarrow t_{0}} \frac{q(t)-q\left(t_{0}\right)}{t-t_{0}}=\dot{q}\left(t_{0}\right)
$$

Proof. Fix $\varepsilon>0$. By the assumption, there exists $\delta>0$ such that for every $t \in \mathbb{R}$, if $\left|t-t_{0}\right|<\delta$ then $\left|\dot{q}(t)-\dot{q}\left(t_{0}\right)\right|<\varepsilon$. Hence

$$
\left|\frac{q(t)-q\left(t_{0}\right)}{t-t_{0}}-\dot{q}\left(t_{0}\right)\right|=\left|\frac{\int_{t_{0}}^{t}\left(\dot{q}(s)-\dot{q}\left(t_{0}\right)\right) d s}{t-t_{0}}\right| \leqslant \frac{\int_{t_{0}}^{t}\left|\dot{q}(s)-\dot{q}\left(t_{0}\right)\right| d s}{\left|t-t_{0}\right|} \leqslant \varepsilon
$$

provided that $0<\left|t-t_{0}\right|<\delta$.
Let $L_{\mathrm{loc}}^{2}\left(\mathbb{R}, \mathbb{R}^{n}\right)$ denote the space of functions on $\mathbb{R}$ with values in $\mathbb{R}^{n}$ locally square integrable.

Fact 2.8. Let $q: \mathbb{R} \rightarrow \mathbb{R}^{n}$ be a continuous mapping such that $\dot{q} \in L_{\mathrm{loc}}^{2}\left(\mathbb{R}, \mathbb{R}^{n}\right)$. For every $t \in \mathbb{R}$ the following inequality holds:

$$
\begin{equation*}
|q(t)| \leqslant \sqrt{2}\left(\int_{t-1 / 2}^{t+1 / 2}\left(|q(s)|^{2}+|\dot{q}(s)|^{2}\right) d s\right)^{1 / 2} \tag{28}
\end{equation*}
$$

Proof. Fix $t \in \mathbb{R}$. For every $\tau \in \mathbb{R}$,

$$
\begin{equation*}
|q(t)| \leqslant|q(\tau)|+\left|\int_{\tau}^{t} \dot{q}(s) d s\right| \tag{29}
\end{equation*}
$$

Integrating (29) over [ $t-\frac{1}{2}, t+\frac{1}{2}$ ] and using the Hölder inequality we obtain

$$
\begin{aligned}
|q(t)| & \leqslant \int_{t-1 / 2}^{t+1 / 2}\left(|q(\tau)|+\left|\int_{\tau}^{t} \dot{q}(s) d s\right|\right) d \tau \\
& \leqslant\left(\int_{t-1 / 2}^{t+1 / 2}\left(|q(\tau)|+\left|\int_{\tau}^{t} \dot{q}(s) d s\right|\right)^{2} d \tau\right)^{1 / 2} \\
& \leqslant\left(2 \int_{t-1 / 2}^{t+1 / 2}\left(|q(\tau)|^{2}+\left|\int_{\tau}^{t} \dot{q}(s) d s\right|^{2}\right) d \tau\right)^{1 / 2} \\
& \leqslant \sqrt{2}\left(\int_{t-1 / 2}^{t+1 / 2}|q(\tau)|^{2} d \tau+\int_{t-1 / 2}^{t+1 / 2}|\dot{q}(s)|^{2} d s\right)^{1 / 2}
\end{aligned}
$$

Lemma 2.9. The function $q_{0}$ determined by Lemma 2.6 is the desired homoclinic solution of (HS).

Proof. The proof will be divided into four steps.
Step 1: We show that $q_{0}$ is a solution of (HS). For every $k \in \mathbb{N}$ and $t \in \mathbb{R}$ we have

$$
\begin{equation*}
\ddot{q}_{\varphi(k)}(t)=f_{\varphi(k)}(t)-V_{q}\left(t, q_{\varphi(k)}(t)\right) . \tag{30}
\end{equation*}
$$

Since $q_{\varphi(k)} \rightarrow q_{0}$ and $f_{\varphi(k)} \rightarrow f$ almost uniformly on $\mathbb{R}$, we obtain that $\ddot{q}_{\varphi(k)} \rightarrow w$ almost uniformly on $\mathbb{R}$, where $w(t)=f(t)-V_{q}\left(t, q_{0}(t)\right)$. Fix $a, b \in \mathbb{R}$ such that $a<b$. There is $k_{0} \in \mathbb{N}$ such that for every $k \geqslant k_{0}$ and $t \in[a, b]$, (30) becomes

$$
\ddot{q}_{\varphi(k)}(t)=f(t)-V_{q}\left(t, q_{\varphi(k)}(t)\right) .
$$

Hence, if $k \geqslant k_{0}$ then the restriction of $\ddot{q}_{\varphi(k)}$ onto $[a, b]$ is continuous. From Fact 2.7 it follows that $\ddot{q}_{\varphi(k)}$ is a derivative of $\dot{q}_{\varphi(k)}$ in $(a, b)$ for every $k \geqslant k_{0}$. Since $\ddot{q}_{\varphi(k)} \rightarrow w$ and $\dot{q}_{\varphi(k)} \rightarrow \dot{q}_{0}$ almost uniformly on $\mathbb{R}$, we have $w=\ddot{q}_{0}$ in $(a, b)$. By the above, we conclude that $w=\ddot{q}_{0}$ in $\mathbb{R}$ and $q_{0}$ satisfies (HS). Moreover, note that we have actually proved that $\left\{q_{\varphi(k)}\right\}_{k \in \mathbb{N}}$ converges to $q_{0}$ in the topology of $C_{\text {loc }}^{2}\left(\mathbb{R}, \mathbb{R}^{n}\right)$.

Step 2: We prove that $q_{0}(t) \rightarrow 0$, as $t \rightarrow \pm \infty$. We have

$$
\begin{aligned}
\int_{-\infty}^{+\infty}\left(\left|q_{0}(t)\right|^{2}+\left|\dot{q}_{0}(t)\right|^{2}\right) d t & =\lim _{i \rightarrow+\infty} \int_{-i T}^{i T}\left(\left|q_{0}(t)\right|^{2}+\left|\dot{q}_{0}(t)\right|^{2}\right) d t \\
& =\lim _{i \rightarrow+\infty} \lim _{k \rightarrow+\infty} \int_{-i T}^{i T}\left(\left|q_{\varphi(k)}(t)\right|^{2}+\left|\dot{q}_{\varphi(k)}(t)\right|^{2}\right) d t
\end{aligned}
$$

Clearly, for every $i \in \mathbb{N}$ there exists $k_{i} \in \mathbb{N}$ such that for all $k \geqslant k_{i}$ we have

$$
\int_{-i T}^{i T}\left(\left|q_{\varphi(k)}(t)\right|^{2}+\left|\dot{q}_{\varphi(k)}(t)\right|^{2}\right) d t \leqslant\left\|q_{\varphi(k)}\right\|_{E_{\varphi(k)}}^{2} \leqslant M_{1}^{2}
$$

by (24). Letting $k \rightarrow+\infty$, we get

$$
\int_{-i T}^{i T}\left(\left|q_{0}(t)\right|^{2}+\left|\dot{q}_{0}(t)\right|^{2}\right) d t \leqslant M_{1}^{2}
$$

and now, letting $i \rightarrow+\infty$, we have

$$
\int_{-\infty}^{+\infty}\left(\left|q_{0}(t)\right|^{2}+\left|\dot{q}_{0}(t)\right|^{2}\right) d t \leqslant M_{1}^{2}
$$

and so

$$
\begin{equation*}
\int_{|t| \geqslant r}\left(\left|q_{0}(t)\right|^{2}+\left|\dot{q}_{0}(t)\right|^{2}\right) d t \rightarrow 0 \tag{31}
\end{equation*}
$$

as $r \rightarrow+\infty$. Combining (31) with (28) we receive our claim.
Step 3: We now show that $\dot{q}_{0}(t) \rightarrow 0$, as $t \rightarrow \pm \infty$. To do this, observe that

$$
\begin{equation*}
\left|\dot{q}_{0}(t)\right|^{2} \leqslant 2 \int_{t-1 / 2}^{t+1 / 2}\left(\left|q_{0}(s)\right|^{2}+\left|\dot{q}_{0}(s)\right|^{2}\right) d s+2 \int_{t-1 / 2}^{t+1 / 2}\left|\ddot{q}_{0}(s)\right|^{2} d s \tag{32}
\end{equation*}
$$

by (28). Since we have (31) and (32) it suffices to prove that

$$
\begin{equation*}
\int_{r}^{r+1}\left|\ddot{q}_{0}(s)\right|^{2} d s \rightarrow 0 \tag{33}
\end{equation*}
$$

as $r \rightarrow \pm \infty$. By (HS) we obtain

$$
\begin{aligned}
\int_{r}^{r+1}\left|\ddot{q}_{0}(s)\right|^{2} d s= & \int_{r}^{r+1}\left(\left|V_{q}\left(s, q_{0}(s)\right)\right|^{2}+|f(s)|^{2}\right) d s \\
& -2 \int_{r}^{r+1}\left(V_{q}\left(s, q_{0}(s)\right), f(s)\right) d s
\end{aligned}
$$

Since $V_{q}(t, 0)=0$ for all $t \in \mathbb{R}, q_{0}(t) \rightarrow 0$, as $t \rightarrow \pm \infty$ and $\int_{r}^{r+1}|f(s)|^{2} d s \rightarrow 0$, as $r \rightarrow \pm \infty$, (33) follows.

Step 4: In the end, we have to show that if $f \equiv 0$ then $q_{0} \not \equiv 0$. For this purpose, as Rabinowitz we use the properties of $Y$ given by (6). The definition of $Y$ implies

$$
\begin{equation*}
\int_{-k T}^{k T}\left(W_{q}\left(t, q_{k}(t)\right), q_{k}(t)\right) d t \leqslant Y\left(\left\|q_{k}\right\|_{L_{2 k T}}\right)\left\|q_{k}\right\|_{E_{k}}^{2} \tag{34}
\end{equation*}
$$

for every $k \in \mathbb{N}$. Since $I_{k}^{\prime}\left(q_{k}\right) q_{k}=0$, (10) gives

$$
\begin{equation*}
\int_{-k T}^{k T}\left(W_{q}\left(t, q_{k}(t)\right), q_{k}(t)\right) d t=\int_{-k T}^{k T}\left|\dot{q}_{k}(t)\right|^{2} d t+\int_{-k T}^{k T}\left(K_{q}\left(t, q_{k}(t)\right), q_{k}(t)\right) d t \tag{35}
\end{equation*}
$$

Substituting (35) into (34), and next applying $\left(\mathrm{H}_{3}\right)$ and $\left(\mathrm{H}_{2}\right)$ we obtain

$$
Y\left(\left\|q_{k}\right\|_{L_{2 k T}^{\infty}}^{\infty}\right)\left\|q_{k}\right\|_{E_{k}}^{2} \geqslant \min \left\{1, b_{1}\right\}\left\|q_{k}\right\|_{E_{k}}^{2}
$$

and hence

$$
\begin{equation*}
Y\left(\left\|q_{k}\right\|_{L_{2 k T}^{\infty}}\right) \geqslant \min \left\{1, b_{1}\right\}>0 . \tag{36}
\end{equation*}
$$

The remainder of the proof is the same as in [12]. If $\left\|q_{k}\right\|_{L_{2 k T}} \rightarrow 0$, as $k \rightarrow+\infty$, we would have $Y(0) \geqslant \min \left\{1, b_{1}\right\}>0$, a contradiction. Thus there is $\gamma>0$ such that

$$
\begin{equation*}
\left\|q_{k}\right\|_{L_{2 k T}^{\infty}} \geqslant \gamma \tag{37}
\end{equation*}
$$

for every $k \in \mathbb{N}$. Clearly, $q_{k}(t+j T)$ is a $2 k T$-periodic solution of $\left(\mathrm{HS}_{k}\right)$ for every $j \in \mathbb{Z}$. By replacing earlier, if necessary, $q_{k}$ by $q_{k}(t+j T)$ for some $j \in[-k, k] \cap \mathbb{Z}$, one can assume that the maximum of $q_{k}$ occurs in $[-T, T]$. Suppose, contrary to our claim, that $q_{0} \equiv 0$. Then, by Lemma 2.6,

$$
\left\|q_{\varphi(k)}\right\|_{L_{2 \varphi(k) T}^{\infty}}^{\infty}=\max _{t \in[-T, T]}\left|q_{\varphi(k)}(t)\right| \rightarrow 0
$$

which contradicts (37).

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