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# Homoclinics for singular strong force Lagrangian systems

https://doi.org/10.1515/anona-2020-0018
Received October 6, 2018; accepted January 18, 2019.

**Abstract:** We study the existence of homoclinic solutions for a class of Lagrangian systems $\frac{d}{dt}\left(\nabla\Phi(\dot{u}(t))\right) + \nabla_u V(t, u(t)) = 0$, where $t \in \mathbb{R}$, $\Phi\colon \mathbb{R}^2 \to [0, \infty)$ is a $G$-function in the sense of Trudinger, $V\colon \mathbb{R}\times\left(\mathbb{R}^2 \setminus \{\xi\}\right) \to \mathbb{R}$ is a $C^1$-smooth potential with a single well of infinite depth at a point $\xi \in \mathbb{R}^2 \setminus \{0\}$ and a unique strict global maximum 0 at the origin. Under a strong force condition around the singular point $\xi$, via minimization of an action integral, we will prove the existence of at least two geometrically distinct homoclinic solutions $u^{\pm}\colon \mathbb{R} \to \mathbb{R}^2 \setminus \{\xi\}$.

**Keywords:** homoclinic solution, homotopy class, Lagrangian system, strong force, rotation number (winding number)

**MSC:** Primary: 37J45 46E30 Secondary: 34C37 70H05

## 1 Introduction

In this work we will be concerned with the problem of existence of solutions for a class of Lagrangian systems

$$\begin{cases} \frac{d}{dt}\left(\nabla\Phi(\dot{u}(t))\right) + \nabla_u V(t, u(t)) = 0, \\ \lim\limits_{t\to\pm\infty} u(t) = \lim\limits_{t\to\pm\infty} \dot{u}(t) = 0, \end{cases} \tag{LS}$$

where $t \in \mathbb{R}$, $\Phi\colon \mathbb{R}^n \to [0, \infty)$ is a $G$-function in the sense of Trudinger, and $V\colon \mathbb{R} \times \left(\mathbb{R}^n \setminus \{\xi\}\right) \to \mathbb{R}$ is a $C^1$-smooth potential possessing a single well of infinite depth at a point $\xi \in \mathbb{R}^n\setminus\{0\}$ and a strict global maximum 0 at the origin.

We begin with the notion of $G$-function. Let a $C^1$-function $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ satisfy the following conditions:

- $(G1)$ $\Phi(0) = 0$,
- $(G2)$ $\Phi$ is coercive, i.e. $\lim\limits_{|x|\to\infty} \frac{\Phi(x)}{|x|} = \infty$,
- $(G3)$ $\Phi$ is convex, i.e. $\Phi(ax + (1 - a)y) \leq a\Phi(x) + (1 - a)\Phi(y)$ for each $a \in [0, 1]$ and all $x, y \in \mathbb{R}^n$,
- $(G4)$ $\Phi$ is symmetric, i.e. $\Phi(x) = \Phi(-x)$ for all $x \in \mathbb{R}^n$,
- $(G5)$ $\nabla\Phi \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$.

In particular, $\Phi$ is a $G$-function in the sense of Trudinger (compare [1]). Let us recall that the Fenchel transform $\Phi^\star$ of a $G$-function $\Phi$ is the function $\Phi^\star\colon \mathbb{R}^n \to \mathbb{R}$ defined by

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$$\Phi^\star(y) = \sup_{x\in\mathbb{R}^n} \left((x, y) - \Phi(x)\right),$$

where $(\cdot,\cdot)\colon \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ is the standard inner product in $\mathbb{R}^n$ (c.f. [2, 3]). It is well known that $\Phi^\star$ is continuous and satisfies $(G1) - (G4)$ (c.f. [4]). Furthermore, $\Phi^{\star\star} = \Phi$ (c.f. [5]).

Troughout the paper we will assume that $\Phi$ and $\Phi^\star$ are globally $\Delta_2$-regular [6], i.e. there is a constant $L > 0$ such that for each $x \in \mathbb{R}^n$,

$$\Phi(2x) \le L\Phi(x) \le \frac{1}{2}\Phi(Lx). \tag{$\Delta_2$}$$

Given a function $\Phi$ we define $\phi\colon \mathbb{R} \to \mathbb{R}$ by

$$\phi(r) = \min\{\Phi(x)\ ;\ |x| = r\}$$

and $\phi(-r) = \phi(r)$. Here $|\cdot|\colon \mathbb{R}^n \to [0,\infty)$ is the standard norm. Let us recall that the epigraph of a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is the set

$$\operatorname{epi} f = \{(x, t) \in \mathbb{R}^n \times \mathbb{R}\ ;\ f(x) \le t\}$$

(c.f. [2]). We define the supporting function $\varphi\colon \mathbb{R} \to \mathbb{R}$ for $\Phi$ by the formula:

$$\varphi = \operatorname{conv} \phi,$$

which means that $\operatorname{epi} \varphi = \overline{\operatorname{conv}\,(\operatorname{epi} \phi)}$. Obviously,

$$\Phi(x) \ge \varphi(|x|) \text{ for } x \in \mathbb{R}^n. \tag{1}$$

One can easily check that

- $\varphi$ is continuous and satisfies $(G1) - (G4)$, i.e. $\varphi$ is a $G$-function;
- $\varphi$ satisfies the $(\Delta_2)$-condition, i.e. $\varphi$ and $\varphi^\star$ are globally $\Delta_2$-regular.

Our intention is to generalize the following result by Paul H. Rabinowitz from [7] to the Lagrangian systems (LS).

**Theorem 1.1.** *Assume that*

$(V_1)$ $V\colon \mathbb{R}\times(\mathbb{R}^2\setminus\{\xi\}) \to \mathbb{R}$, *where* $\xi \in \mathbb{R}^2\setminus\{0\}$, *is a* $C^1$*-smooth potential,* $1$*-periodic in* $t \in \mathbb{R}$ *and*

$$\lim_{x\to\xi} V(t, x) = -\infty$$

*uniformly in the time variable* $t$,

$(V_2)$ *for all* $t \in \mathbb{R}$, $x \in \mathbb{R}^2\setminus\{0\}$, $V(t, x) \le 0$ *and* $V(t, x) = 0$ *iff* $x = 0$,

$(V_3)$ *there is a negative constant* $V_0$ *such that for all* $t \in \mathbb{R}$,

$$\limsup_{|x|\to\infty} V(t, x) \le V_0,$$

$(V_4)$ *there are a neighbourhood* $\mathcal{N} \subset \mathbb{R}^2$ *of the singular point* $\xi$ *and a function* $U \in C^1(\mathcal{N}\setminus\{\xi\}, \mathbb{R})$ *such that* $|U(x)| \to \infty$ *as* $x \to \xi$, *and for all* $x \in \mathcal{N}\setminus\{\xi\}$ *and* $t \in \mathbb{R}$,

$$|\nabla U(x)|^2 \le -V(t, x).$$

*Then the problem*

$$\begin{cases} \ddot{u}(t) + \nabla_u V(t, u(t)) = 0, \\ \lim\limits_{t\to\pm\infty} u(t) = \lim\limits_{t\to\pm\infty} \dot{u}(t) = 0 \end{cases} \tag{HS}$$

*has at least two solutions* $u^{\pm}\colon \mathbb{R} \to \mathbb{R}^2 \setminus \{\xi\}$, *which wind around* $\xi$ *in opposite directions.*

The proof of Theorem 1.1 in [7] is of variational nature. The basic idea is to take the Lagrangian action corresponding to the problem (HS), defined on the subset of all the functions of the Sobolev space $W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ omitting the singularity at a finite time and to minimize this functional both over the subset of functions with a positive winding number around $\xi$ and the subset of functions possessing a negative rotation.

We are thus led to the following strengthening of Theorem 1.1.

**Theorem 1.2.** *Let* $\Phi\colon \mathbb{R}^2 \to [0, \infty)$ *satisfy* $(G1)$ – $(G5)$ *and* $(\Delta_2)$. *Assume also that the potential* $V\colon \mathbb{R} \times (\mathbb{R}^2 \setminus \{\xi\}) \to \mathbb{R}$ *satisfies* $(V_1)$ – $(V_3)$, *and moreover,*

$(V_4')$ *there are a neighbourhood* $\mathcal{N} \subset \mathbb{R}^2$ *of the point* $\xi$ *and a function* $U \in C^1(\mathcal{N}\setminus\{\xi\}, \mathbb{R})$ *such that* $|U(x)| \to \infty$ *as* $x \to \xi$, *and for all* $x \in \mathcal{N} \setminus \{\xi\}$ *and* $t \in \mathbb{R}$,

$$\varphi^{\star}(|\nabla U(x)|) \le -V(t, x).$$

*Then there exist at least two classical solutions* $u^{\pm}\colon \mathbb{R} \to \mathbb{R}^2 \setminus \{\xi\}$ *of the problem* (LS) *winding around* $\xi$ *in opposite directions.*

Let us remark that if we substitute $\Phi(x) = \frac{1}{2}|x|^2$, $x \in \mathbb{R}^2$, into (LS) then we obtain (HS). What is more, for $\Phi(x) = \frac{1}{p}|x|^p$, $x \in \mathbb{R}^2$, $p > 1$, we have

$$\frac{d}{dt}\left(\nabla\Phi(\dot{u}(t))\right) = \frac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right),$$

i.e. the $p$-Laplacian, and for $\Phi(x) = \chi(|x|)$, where $\chi\colon \mathbb{R} \to \mathbb{R}$ is a so-called $N$-function (a $G$-function of one variable with extra growth conditions, c.f. [8]) we obtain a $\chi$-Laplacian. Let us note that $\varphi^{\star}$ in the condition $(V_4')$ is the Fenchel transform of the supporting function $\varphi$ for $\Phi$. Thus $\varphi^{\star}$ depends on $\Phi$. Let us briefly discuss now our assumptions in Theorem 1.2.

Condition $(V_4)$ was introduced by W.B. Gordon in [9] and in the literature it is known as the strong force condition or Gordon's condition. It governs the rate at which $V(x) \to -\infty$ as $x \to \xi$ and holds, for example, if $\alpha \ge 2$ for $V(x) = -|x-\xi|^{-\alpha}$ nearby $\xi$. Gordon's condition excludes the gravitational case and leads to the disclosure between the behaviour of strong force systems and gravitational ones. Condition $(V_4')$ is an extension of $(V_4)$ to the Lagrangian system (LS). Following Gordon, if $V\colon \mathbb{R} \times (\mathbb{R}^2 \setminus \{\xi\}) \to \mathbb{R}$ satisfies $(V_4')$ then $\nabla_u V\colon \mathbb{R} \times (\mathbb{R}^2 \setminus \{\xi\}) \to \mathbb{R}^2$ will be called a strong force. Moreover, (LS) is said to be a strong force Lagrangian system. $(V_4')$ implies that the system (LS) does not possess solutions in the Orlicz-Sobolev space associated with $\varphi$, entering the singular point $\xi$ in a finite time. Condition $(V_3)$ can be replaced by a somewhat weaker assumption, namely,

$(V_3')$ $\lim\limits_{|x|\to\infty} |x|^2 V(x) = -\infty.$

During the past thirty years, there has been made a great deal of progress in the use of variational methods to investigate homoclinic solutions for Lagrangian systems. Some basic material on variational methods can be found in [2, 10–13]. Since homoclinics are global in time, it is natural to use global methods to study their existence. Both minimization and minimax arguments have been employed to obtain homoclinic solutions (see [7, 14–18]). The variational formulation for Lagrangian systems leads to action functionals. Although there may be a natural class of curves or functions to work with, there is not always an easy choice of an associated norm or metric. Choosing a good setting in which to formulate the variational problem is often a great difficulty.

To study homoclinic solutions of the problem (LS), in Section 2 a technical framework will be introduced to treat a corresponding action functional in an appropriate Sobolev-Orlicz space. Section 3 contains the proof of our main result. The basic idea of the proof of Theorem 1.2 is to find two minimizers of the action functional winding around the singularity in opposite directions.

## 2 Preliminaries

From now on, we assume that $\Phi\colon \mathbb{R}^n \to [0,\infty)$ satisfy $(G1)-(G5)$ and $(\Delta_2)$.

Let $\Omega \subset \mathbb{R}$ be a domain. Following Trudinger [1] we define the space

$$L_\Phi(\Omega) = \left\{ u\colon \Omega \to \mathbb{R}^n \;:\; u \text{ is Lebesgue measurable and } \int_\Omega \Phi(u)dt < \infty \right\}.$$

This space equipped with the Luxemburg norm

$$\|u\|_\Phi = \inf\left\{ \nu > 0 \;:\; \int_\Omega \Phi\left(\frac{u}{\nu}\right) dt \leq 1 \right\} \tag{2}$$

is a Banach space. Since $\Phi$ is $\Delta_2$-regular, $L_\Phi(\Omega)$ is also a separable space (c.f. Rem. 8.22 in [8]). Furthermore, $L_\Phi(\Omega)$ is reflexive if and only if $(\Delta_2)$ is satisfied (c.f. Thm. 8.20 in [8]).

Set $\psi = \varphi \circ |\cdot|$, i.e. $\psi(x) = \varphi(|x|)$ for each $x \in \mathbb{R}^n$. As a consequence of (1), the space $L_\Phi(\Omega)$ is continuously imbedded in $L_\psi(\Omega)$ (c.f. Thm. 8.12 in [8]),

$$L_\Phi(\Omega) \subset L_\psi(\Omega).$$

Note that $\|u\|_\psi = \||u|\|_\varphi$.

For simplicity of notation, we write $L_\Phi$ instead of $L_\Phi(\mathbb{R})$. Although the norm formula (2) depends on the domain $\Omega$, we use the same notation $\|\cdot\|_\Phi$ for different subsets of $\mathbb{R}$. It will be clear from the context what $\Omega$ is.

Let $AC_{loc}(\mathbb{R},\mathbb{R}^n)$ be the space of locally absolutely continuous functions on $\mathbb{R}$ with values in $\mathbb{R}^n$. Finally, let $E$ denote the Orlicz-Sobolev space

$$E = \left\{ u \in AC_{loc}(\mathbb{R},\mathbb{R}^n)\colon \dot{u} \in L_\Phi(\mathbb{R},\mathbb{R}^n)\right\}$$

with the norm

$$\|u\| = \|\dot{u}\|_\Phi + |u(0)|.$$

We note for later reference that $E$ is a separable reflexive Banach space (see [19]).

For every $T > 0$ we define the Banach space $E_T$ consisting of restrictions of $u \in E$ to the interval $[0, T]$ with the induced norm,

$$\|u\|_{E_T} = |u(0)| + \|\dot{u}\|_\Phi.$$

Let $C([0,T],\mathbb{R}^n)$ denote the space of continuous functions from $[0,T]$ into $\mathbb{R}^n$ with the standard norm.

**Proposition 2.1.** *The inclusion map $E_T \to C([0,T],\mathbb{R}^n)$ is continuous, i.e. there is $C_T > 0$ such that for each $u \in E_T$ one has*

$$\max_{t\in[0,T]} |u(t)| \leq C_T \|u\|_{E_T}.$$

*Proof.* One has

$$\begin{aligned}|u(t)| &= \left|u(0) + \int_0^t \dot{u}(s)ds\right| \le |u(0)| + \int_0^t |\dot{u}(s)|ds \\ &\le |u(0)| + \int_0^T |\dot{u}(s)|ds \le |u(0)| + 2\|1\|_{\varphi^\star}\||\dot{u}|\|_\varphi \\ &\le (1 + 2\|1\|_{\varphi^\star})\left(|u(0)| + \||\dot{u}|\|_\varphi\right) \\ &\le C_T\left(|u(0)| + \|\dot{u}\|_\Phi\right) = C_T\|u\|_{E_T}.\end{aligned}$$

□

**Proposition 2.2.** *If a sequence $\{u_k\}_{k\in\mathbb{N}} \subset E_T$ converges weakly to $u_0 \in E_T$ then it converges uniformly to $u_0$ in $C([0,T],\mathbb{R}^n)$.*

*Proof.* Since $\{u_k\}_{k\in\mathbb{N}}$ converges to $u_0$ weakly in $E_T$ then, by Proposition 2.1, it also converges to $u_0$ weakly in $C([0,T],\mathbb{R}^n)$. Furthermore, $\|u_k\|_{E_T} \le M$ for some $M > 0$ and every $k \in \mathbb{N}$.

Let $0 \le s \le t \le T$. Then

$$\begin{aligned}|u_k(t) - u_k(s)| &= \left|\int_s^t \dot{u}_k(\tau)d\tau\right| \le \int_s^t |\dot{u}_k(\tau)|d\tau \\ &\le 2\|1\|_{\varphi^\star}\||\dot{u}_k|\|_\varphi \le 2\|1\|_{\varphi^\star}\|u_k\|_{E_T} \\ &\le 2M\left((\varphi^\star)^{-1}\left(\frac{1}{t-s}\right)\right)^{-1}.\end{aligned}$$

Thus $\{u_k\}_{k\in\mathbb{N}}$ is a sequence of equicontinuous functions. By the Arzela-Ascoli Theorem, every sequence $\{u_{k_i}\}_{i\in\mathbb{N}}$ contains a subsequence converging to a certain $\hat{u}$ in $C([0,T],\mathbb{R}^n)$. By the uniqueness of the weak limit, $\hat{u} = u_0$, which completes the proof. □

In what follows, $\Phi\colon \mathbb{R}^2 \to \mathbb{R}$ and $V\colon \mathbb{R}\times(\mathbb{R}^2 \setminus \{\xi\}) \to \mathbb{R}$ satisfy the assumptions of Theorem 1.2.

For each $u \in E$, we define a functional $I$ by setting

$$I(u) = \int_{-\infty}^{\infty} \left(\Phi(\dot{u}(t)) - V(t,u(t))\right) dt. \tag{3}$$

Let

$$\alpha_\varepsilon = \inf\{-V(t,x)\colon x \notin B_\varepsilon(0)\}, \tag{4}$$

where $0 < \varepsilon \le \frac{1}{2}|\xi|$ and $B_\varepsilon(0)$ denotes the ball of radius $\varepsilon$ centered at the origin. By $(V_1)-(V_3)$ we have $\alpha_\varepsilon > 0$.

**Lemma 2.3.** *Suppose that $u \in E$ and $u(t) \notin B_\varepsilon(0)$ for each $t \in [a,b]$. Then, there is $C > 0$ such that*

$$(I(u)+1)^2 \ge C \cdot \mathit{length}\,(u_{|[a,b]}) \ge C|u(b) - u(a)|. \tag{5}$$

*Proof.* One has

$$|u(b) - u(a)| = \left|\int_a^b \dot{u}(t)dt\right| \le \int_a^b |\dot{u}(t)|dt \le 2\||\dot{u}|\|_\varphi\|1\|_{\varphi^\star}.$$

The last estimation follows from Hölder's inequality in Orlicz spaces (c.f. [5], Par. 8.11). Directly from the definition, one has

$$\|1\|_{\varphi^\star} = \left[(\varphi^\star)^{-1}\left(\frac{1}{b-a}\right)\right]^{-1}.$$

Set $\delta = \text{length}\,(u_{|[a,b]})$ and $\tau = b-a$. Then

$$|||\dot{u}|||_\varphi \geq \frac{1}{2}\delta\|1\|_{\varphi^\star}^{-1} = \frac{1}{2}\delta\cdot(\varphi^\star)^{-1}\left(\frac{1}{\tau}\right).$$

Consequently,

$$\begin{aligned} I(u) &\geq \int_a^b \big(\Phi(\dot{u}(t)) - V(t,u(t))\big)\,dt = \int_a^b \Phi(\dot{u}(t))dt + \int_a^b -V(t,u(t))dt \\ &\geq \int_a^b \varphi(|\dot{u}(t)|)dt + \alpha_\varepsilon\tau \geq |||\dot{u}|||_\varphi - 1 + \alpha_\varepsilon\tau \\ &\geq \frac{1}{2}\delta\cdot(\varphi^\star)^{-1}\left(\frac{1}{\tau}\right) - 1 + \alpha_\varepsilon\tau. \end{aligned} \tag{6}$$

Hence

$$I(u) + 1 \geq \frac{1}{2}\delta\cdot(\varphi^\star)^{-1}\left(\frac{1}{\tau}\right) + \alpha_\varepsilon\tau \geq \frac{1}{2}\frac{\delta}{\tau k}\cdot(\varphi^\star)^{-1}(k) + \alpha_\varepsilon\tau,$$

where the natural number $k$ satisfies $\tau k \geq 1$ and the last inequality follows from the fact that $(\varphi^\star)^{-1}$ is concave. We choose the smallest $k$ with the property $\tau k \geq 1$. In particular, we set $k = 1$ if $\tau \geq 1$. Now, if $\tau \geq 1$ then

$$f(\tau) = \frac{1}{2}\frac{\delta}{\tau}\cdot(\varphi^\star)^{-1}(1) + \alpha_\varepsilon\tau$$

achieves its minimum at the point

$$\tau_{\min} = \left(\frac{\delta\cdot(\varphi^\star)^{-1}(1)}{2\alpha_\varepsilon}\right)^{\frac{1}{2}},$$

which is equal to $f_{\min} = (2\delta\alpha_\varepsilon(\varphi^\star)^{-1}(1))^{\frac{1}{2}}$. If $\tau < 1$ then

$$\frac{1}{2}\frac{\delta}{\tau k}\cdot(\varphi^\star)^{-1}(k) + \alpha_\varepsilon\tau \geq \frac{1}{4}\delta\cdot(\varphi^\star)^{-1}(k) + \alpha_\varepsilon\tau \geq \frac{1}{4}\delta\cdot(\varphi^\star)^{-1}(1).$$

Finally, set

$$C = \min\left\{2\alpha_\varepsilon(\varphi^\star)^{-1}(1),\ \frac{1}{4}(\varphi^\star)^{-1}(1)\right\}.$$

□

**Remark 2.4.** *In the above lemma the interval* $[a,b]$ *can be replaced by a finite sum of disjoint intervals.*

We will denote by $L^\infty(\mathbb{R},\mathbb{R}^2)$ the space of Lebesgue measurable essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^2$ with the norm

$$\|u\|_\infty = \text{ess}\ \sup|u(t)|.$$

**Corollary 2.5.** *If* $u \in E$ *and* $I(u) < \infty$ *then* $u \in L^\infty(\mathbb{R},\mathbb{R}^2)$.

*Proof.* Assume that $u \notin L^\infty(\mathbb{R},\mathbb{R}^2)$. Then for every $n \in \mathbb{N}$ there exists $t_n \in \mathbb{R}$ such that $|u(t_n)| > n$. Consequently, by Lemma 2.3 we get

$$(I(u)+1)^2 \geq C|u(t_n)-u(t_1)| \geq C(|u(t_n)|-|u(t_1)|) \geq C(n-|u(t_1)|)$$

for $n \in \mathbb{N}$, contrary to $I(u) < \infty$. □

**Lemma 2.6.** *If $u \in E$ and $I(u) < \infty$ then $\lim_{t\to\pm\infty} u(t) = 0$.*

Lemma 2.6 is analogous to Proposition 3.11 of [20] and Lemma 2.4 of [21]. In spite of different assumptions on the potential $V$, the claims are similar.

*Proof.* Let $A(u)$ denote the set of limit points of $u(t)$, as $t \to -\infty$. From Corollary 2.5 we conclude that $A(u) \neq \emptyset$. Assume that there are $\varepsilon > 0$ and $\rho \in \mathbb{R}$ such that if $t < \rho$ then $u(t) \notin B_\varepsilon(0)$. By (4) we obtain,

$$I(u) \geq \int_{-\infty}^{\rho} -V(t, u(t))dt = \infty,$$

a contradiction. Thus $A(u)$ contains 0. It is sufficient to note that $A(u)$ consists of a point. If not, there is $\varepsilon > 0$ such that $u(t)$ intersects $\partial B_{\frac{\varepsilon}{2}}(0)$ and $\partial B_\varepsilon(0)$ infinitely many times. Let $\tau_0 \geq 0$ be the smallest number such that

$$I(u)+1 \geq \frac{1}{2}\frac{\varepsilon}{2} \cdot (\varphi^\star)^{-1}\left(\frac{1}{\tau_0}\right) + \alpha_{\frac{\varepsilon}{2}}\tau_0.$$

Since $\lim_{\tau\to\infty}(\varphi^\star)^{-1}(\tau) = \infty$, one has $\tau_0 > 0$. By Remark 2.4, we obtain

$$I(u)+1 \geq n\alpha_{\frac{\varepsilon}{2}}\tau_0$$

for each $n \in \mathbb{N}$, and hence $I(u) = \infty$, a contradiction.

In the same manner we can see that $\lim_{t\to\infty} u(t) = 0$. □

**Lemma 2.7.** *If $[a, b]$ is an interval such that $u([a, b]) \subset \mathcal{N} \setminus \{\xi\}$ then it holds*

$$|U(u(b))| - |U(u(a))| \leq 2(I(u)+1)^2. \tag{7}$$

*Proof.* We first note that

$$\begin{aligned} |U(u(b))| &\leq |U(u(a))| + \left|\int_a^b \frac{d}{dt}U(u(t))dt\right| \\ &\leq |U(u(a))| + \left|\int_a^b \left(\nabla U(u(t)), \dot{u}(t)\right) dt\right| \\ &\leq |U(u(a))| + \int_a^b |\nabla U(u(t))||\dot{u}(t)|dt \\ &\leq |U(u(a))| + 2\||\nabla U(u)|\|_{\varphi^\star}\||\dot{u}|\|_\varphi \end{aligned}$$

Since

$$\||\nabla U(u)|\|_{\varphi^\star} \leq 1 + \int_a^b \varphi^\star(|\nabla U(u(t))|)dt \leq 1 + \int_a^b -V(t, u(t))dt$$

and

$$|||\dot{u}|||_\varphi \leq 1 + \int_a^b \varphi(|\dot{u}(t)|)dt$$

we obtain

$$|U(u(b))| \leq |U(u(a))| + 2(I(u) + 1)^2.$$

□

As an immediate consequence of (7) one has that $u(t) \neq \xi$ for $t \in \mathbb{R}$ provided that $I(u) < \infty$ (c.f. [7], Eq. (2.21)). In fact, we obtain the following

**Corollary 2.8.** *(c.f. [17]) If the action functional $I$ is bounded on some set $W \subset E$, say $I(W) \subset [0, \beta]$ then there is $\rho > 0$ depending on $\beta$ such that for every $u \in W$ and $t \in \mathbb{R}$ one has $|u(t) - \xi| \geq \rho$.*

Set

$$\Lambda = \left\{ u \in E \colon \lim_{t \to \pm\infty} u(t) = 0,\ u(\mathbb{R}) \subset \mathbb{R}^2 \setminus \{\xi\} \right\}.$$

If $I(u) < \infty$ then $u \in \Lambda$. Consequently, $u$ describes a closed curve in $\mathbb{R}^2 \setminus \{\xi\}$ that starts and ends at 0. Hence its homotopy class $[u]$ represents an element of the fundamental group $\pi_1(\mathbb{R}^2 \setminus \{\xi\})$.

Let us remind that two functions $u_0, u_1 \in \Lambda$ are homotopic if and only if there exists a continuous map $h\colon [0, 1] \to \Lambda$ such that $h(0) = u_0$ and $h(1) = u_1$. The rotation number (or winding number) $rot_\xi(u)$ of $u$ around $\xi$ is constant on every connected component of $\Lambda$ and induces an isomorphism $rot_\star\colon \pi_1(\mathbb{R}^2 \setminus \{\xi\}) \to \mathbb{Z}$,

$$rot_\star([u]) = rot_\xi(u).$$

Equivalently, $\Lambda$ is a sum of its path connected components labeled by the integers.

Similarly to [17] one can prove the following result.

**Proposition 2.9.** *Let $W \subset \Lambda$ be a set such that the functional $I$ restricted to $W$ is bounded. Then there exists $D \in \mathbb{N}$ such that $|rot_\xi(u)| \leq D$ for all $u \in W$.*

Let

$$\Lambda^\pm = \{u \in \Lambda \colon \pm rot_\xi(u) > 0\},$$

and

$$\lambda^\pm = \inf_{u \in \Lambda^\pm} I(u). \tag{8}$$

Our main result is an immediate consequence of the following.

**Theorem 2.10.** *If the assumptions of Theorem 1.2 are satisfied then there exists $u^\pm \in \Lambda^\pm$ such that $I(u^\pm) = \lambda^\pm > 0$. Moreover, $u^\pm$ is a classical homoclinic solution of* (LS).

# 3 Proof of Theorem 2.10

The proof will be carried out for the "+" case. The proof for the "-" case is similar. We set $\lambda = \lambda^+$. Let $\{u_n\}_{n=1}^\infty$ be a minimizing sequence for (8). With no loss of generality we assume that for every $n \in \mathbb{N}$,

$$\lambda \leq I(u_n) \leq \lambda + 1,$$

and by Proposition 2.9, for some $d \in \mathbb{N}$,

$$rot_\xi(u_n) = d.$$

Since $d > 0$, there are $v_n$ and $\theta_n > 1$ such that $u_n(v_n) = \theta_n \cdot \xi$. In particular, by Corollary 2.8

$$\|u_n\|_\infty > |\xi|.$$

Furthermore, there are $\sigma_n, \mu_n$ and $\tau_n \in [\sigma_n, \mu_n]$ such that:

**a.** $u_n([\sigma_n, \mu_n]) \subset \mathbb{R}^2 \setminus B_{\frac{|\xi|}{2}}(0)$,
**b.** $|u_n(\sigma_n)| = |u_n(\mu_n)| = \frac{1}{2}|\xi|$,
**c.** $|u_n(\tau_n)| = \|u_n\|_\infty$

Hence, by Lemma 2.3,

$$\begin{aligned}(\lambda + 2)^2 &\geq (I(u_n) + 1)^2 \geq C \cdot \text{length } (u_{n|[\sigma_n,\mu_n]}) \\ &> C(2\|u_n\|_\infty - |\xi|) > C\|u_n\|_\infty,\end{aligned}$$

and thus the sequence $\{\|u_n\|_\infty\}_{n\in\mathbb{N}}$ is bounded. Furthermore, since by (6)

$$\lambda + 2 \geq I(u_n) + 1 \geq \frac{1}{2}\delta \cdot (\varphi^\star)^{-1}\left(\frac{1}{\tau}\right) + \alpha_\varepsilon \tau$$

with $\delta \geq |\xi|$, there are $M > m > 0$ such that $m < \tau < M$. In particular, $\mu_n - \sigma_n > m$ for each $n \in \mathbb{N}$. Consequently, $\lambda = \inf\{I(u_n);\ n \in \mathbb{N}\} \geq \alpha_\varepsilon m > 0$. From $(G3)$ we obtain

$$\int_{\mathbb{R}} \Phi(A \cdot \omega(t))dt \leq A \int_{\mathbb{R}} \Phi(\omega(t))dt$$

for $0 \leq A \leq 1$ and $\omega \in L_\Phi$. If we let $A = (\lambda + 1)^{-1}$ then

$$\int_{\mathbb{R}} \Phi((\lambda + 1)^{-1}\dot{u}_n(t))dt \leq (\lambda + 1)^{-1} \int_{\mathbb{R}} \Phi(\dot{u}_n(t))dt \leq (\lambda + 1)^{-1} I(u_n) \leq 1,$$

which implies that $\|\dot{u}_n\|_\Phi \leq \lambda + 1$. In consequence, $\{u_n\}_{n=1}^\infty$ is bounded in $E$.

Now, let $C_0^\infty(\mathbb{R}, \mathbb{R}^2)$ denote the space of smooth functions from $\mathbb{R}$ into $\mathbb{R}^2$ with compact supports.

We say that a set $Z \subset \Lambda$ has the perturbation property and write $Z \in \mathcal{P}$ if for each $u \in Z$ and for each $v \in C_0^\infty(\mathbb{R}, \mathbb{R}^2)$ there exists $\delta > 0$ such that if $s \in (-\delta, \delta)$ then $u + sv \in Z$.

Let us remark that if $u$ is a minimizer of $I$ on a set $Z \in \mathcal{P}$ then

$$\frac{d}{ds} I(u + sv)_{|s=0} = 0 = \int_{-\infty}^{\infty} ((\nabla \Phi(\dot{u}(t)), \dot{v}(t)) - (\nabla V(t, u(t)), v(t)))dt,$$

and consequently, $u$ is a weak solution of (LS). A similar argument as in the proof of Proposition 3.18 in [20] shows that $u$ is a classical solution of (LS). Finally, using (LS), $(V_1)$ and $(V_2)$ as in [18] gives $\dot{u}(\pm\infty) = 0$.

Of course $\Lambda^\pm \in \mathcal{P}$. We expect that minimizing $I$ over $\Lambda^+$ and $\Lambda^-$ gives two solutions.

Let $L^\infty_{loc}(\mathbb{R}, \mathbb{R}^2)$ be the space of Lebesgue measurable functions from $\mathbb{R}$ into $\mathbb{R}^2$ that are essentially bounded on each compact subset of $\mathbb{R}$.

Since $E$ is reflexive, the sequence $\{u_n\}_{n=1}^\infty$ converges along a subsequence to $Q \in E$ weakly in $E$ and, by Proposition 2.2, strongly in $L^\infty_{loc}(\mathbb{R}, \mathbb{R}^2)$. It follows from Fatou's Lemma that $I(Q) \leq \lambda$. Thus $Q \in \Lambda$. Finally, we apply the following version of the shadowing chain lemma

**Lemma 3.1.** *Let $Z \in \mathcal{P}$ be an arbitrary set all of whose elements have the same rotation number $d \in \mathbb{Z}$. Set*

$$z = \inf\{I(q) \colon q \in Z\}.$$

*Under the conditions of Thm.1.2, there are a finite number of homoclinic solutions:* $Q_1, Q_2, \dots, Q_l \in \Lambda$ *of* (LS) *such that*

$$z = I(Q_1) + I(Q_2) + \dots + I(Q_l)$$

*and*

$$d = rot_\xi(Q_1) + rot_\xi(Q_2) + \dots + rot_\xi(Q_l).$$

The proof is analogous to that of Lemma 3.2 in [17].

Since $d > 0$ there is at least one $Q_i$ with $rot_\xi(Q_i) > 0$. In fact, this nontrivial solution is unique. If $Q_j$ is another nontrivial solution then $I(Q_j) > 0$. Thus $I(Q_i) < \lambda$, which is a contradiction.

**Acknowledgement:** The first two authors are supported by Grant BEETHOVEN2 of the National Science Centre, Poland, no. 2016/23/G/ST1/04081.

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