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Hybrid Reduced Model of Continuous System

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Abstract

The paper introduces an alternative method of modelling and modal reduction of continuous systems. Presented method is a hybrid one. It combines the advantages of modal decomposition method and the rigid finite element method. In the proposed method continuous structure is divided into one-dimensional continuous elements. For each 1D element modal decomposition and reduction is applied. Interactions between substructures are described by lumping techniques. Presented method enables to obtain reduced, low order modal model of considered system. The proposed approach is illustrated by selected examples.

Keywords: modelling, model reduction, modal analysis, mechanical system, dynamic systems, vibration.

1. Introduction

In the static and dynamic analysis of the elastic bodies the Finite Element Method (FEM) is widely used. The conventional discretization (Fig. 1a,c) yields to a set of ordinary differential equations. However, to obtain accurate results it is necessary to apply a great number of finite elements and to solve high order model (a big number of the second order differential equations). To avoid such problem, different methods of model order reduction can be applied. Modal decomposition and reduction is one of them [1]. However, in standard approach to obtain modal reduced order model it is necessary to derive and consider high order model by FEM.

In the paper a new, alternative method of model order reduction is described. It is a hybrid one and combines two well known approaches: modal decomposition method and the rigid finite element method.

In the proposed method the body is divided into strips (for 2D system - Fig. 1b) and prism (for 3D system - Fig. 1c). Each strip or prism represents one-dimensional distributed system and it is described by appropriate second order partial differential equation. However, these equations have also terms related to interactions between strip/prism. Hence, the given system can be described by set of a couplet second order partial differential equations. For each 1D element modal decomposition and reduction is applied whereas interactions between elements are described by lumping technique. In this case no complex FEM model is considered for modal decomposition.

Appropriate mathematical description of 2D system is presented below.

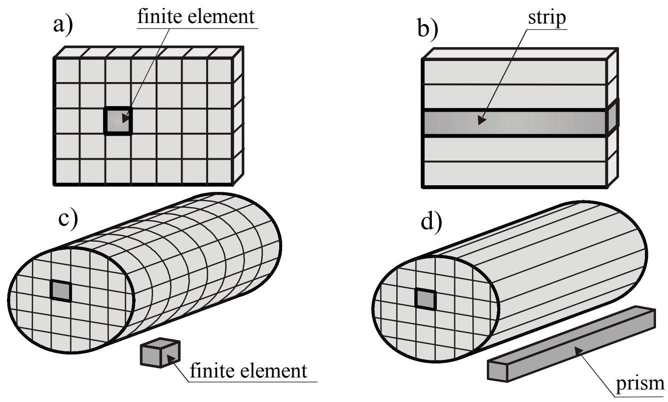

Figure 1. Spatial discretization of 2D and 3D body: a), c) conventional finite element method, b), d) proposed hybrid method

3. Hybride model of 2D body

Applying Rigid Final Element Method to 2D body divided into $n_x \times n_y$ finite elements one obtains appropriate system of ordinary differential equations ($n_x \times n_y$ second order equations) [1]. Such FEM model can be transformed to the continuum representation by letting $dx \rightarrow 0$. In that way small differences divided by dx become derivatives.

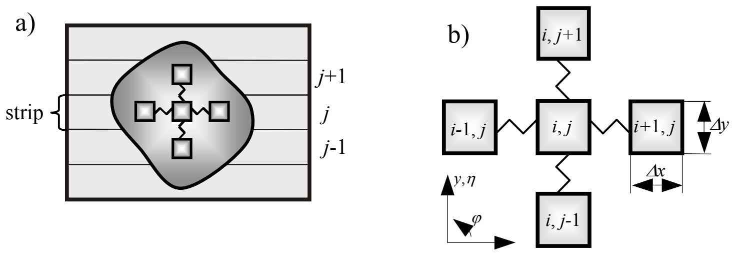

Figure 2. Discrete model of the hybrid 2D structure: a) continuous body, b) elementary

Thus, 2D body can be described by the following, n_y partial differential equations (after Laplace transform with respect to time):

$$f_{x\,j}\Delta y = \underline{\Delta y b \rho \xi_j s^2 - Eb\Delta y \xi''_j} + \frac{2\kappa Gb}{\Delta y}\xi_j - \frac{\kappa Gb}{\Delta y}\xi_{j+1} - \frac{\kappa Gb}{\Delta y}\xi_{j-1} + \frac{\kappa Gb}{2}\varphi_{j+1} - \frac{\kappa Gb}{2}\varphi_{j-1}\,, \qquad (1)$$

$$f_{y\,j}\Delta y = \underline{\Delta y b \rho \eta_j s^2 - \kappa Gb\Delta y \eta''_j} + \frac{2Eb}{\Delta y}\eta_j - \frac{Eb}{\Delta y}\eta_{j-1} - \frac{Eb}{\Delta y}\eta_{j+1} + \kappa Gb\Delta y \varphi'\,, \qquad (2)$$

$$\tau_j \Delta y = \underline{\rho I_y \varphi_j s^2 - EI_y \varphi_j''} - \kappa Gb\Delta y \eta_j' + \frac{\kappa Gb}{2}(\xi_{j-1} - \xi_{j+1}) + \frac{3}{2}\kappa Gb\Delta y \varphi_j + \\ + \frac{\kappa Gb\Delta y}{4}(\varphi_{j-1} + \varphi_{j+1}), \qquad j = 1, 2, 3, \ldots, n_y, \tag{3}$$

where: E – Young's modulus, G – shear modulus, I – area moment of inertia, A – cross section area, κ – numerical shape factor of cross section, ρ – mass per unit volume, ξ, η – transverse displacements, φ – rotation (angular displacement), f – distributed force (excitation), τ – distributed torque moment (excitation), i=1,2,…,n_x, j=1,2,…,n_y.

Solution of these equations with appropriate boundary conditions gives accurate prediction of static and dynamic response (displacement, strain, stresses etc.) for many 2D elastic body. Applying modal decomposition for underlined parts of equations (1, 2, 3) and applying FEM for remained parts one can obtain discrete model of the considered system written in the form:

$$\boldsymbol{M}_{xj}\ddot{\boldsymbol{q}}_{xj} + \boldsymbol{K}_{xj}\boldsymbol{q}_{xj} = \boldsymbol{\Phi}_{xj}^T \left[\frac{\kappa Gb}{\Delta y}\boldsymbol{\Phi}_{x,j+1}\boldsymbol{q}_{x,j+1} - \frac{2\kappa Gb}{\Delta y}\boldsymbol{\Phi}_{xj}\boldsymbol{q}_{xj} + \frac{\kappa Gb}{\Delta y}\boldsymbol{\Phi}_{x,j-1}\boldsymbol{q}_{x,j-1} + \\ - \frac{\kappa Gb}{2}\boldsymbol{\Phi}_{\varphi,j+1}\boldsymbol{q}_{\varphi,j+1} + \frac{\kappa Gb}{2}\boldsymbol{\Phi}_{\varphi,j-1}\boldsymbol{q}_{x,j-1} \right]\Delta x + \boldsymbol{\Phi}_{xj}^T \boldsymbol{f}_{xj}, \tag{4}$$

$$\boldsymbol{M}_{yj}\ddot{\boldsymbol{q}}_{yj} + \boldsymbol{K}_{yj}\boldsymbol{q}_{yj} = \boldsymbol{\Phi}_{yj}^T \left[\frac{Eb}{\Delta y}\boldsymbol{\Phi}_{y,j+1}\boldsymbol{q}_{y,j+1} - \frac{2Eb}{\Delta y}\boldsymbol{\Phi}_{yj}\boldsymbol{q}_{yj} + \\ + \frac{Eb}{\Delta y}\boldsymbol{\Phi}_{y,j-1}\boldsymbol{q}_{y,j-1} - \kappa Gb\boldsymbol{\Phi}_{\varphi j}' q_{\varphi j} \right]\Delta x + \boldsymbol{\Phi}_{yj}^T \boldsymbol{f}_{yj}, \tag{5}$$

$$\boldsymbol{M}_{\varphi j}\ddot{\boldsymbol{q}}_{\varphi j} + \boldsymbol{K}_{\varphi j}\boldsymbol{q}_{\varphi j} = \boldsymbol{\Phi}_{\varphi j}^T \left[\kappa Gb\Delta y \boldsymbol{\Phi}_{yj}'\boldsymbol{q}_{yj} - \frac{\kappa Gb}{2}\boldsymbol{\Phi}_{x,j-1}\boldsymbol{q}_{x,j-1} + \frac{\kappa Gb}{2}\boldsymbol{\Phi}_{x,j+1}\boldsymbol{q}_{x,j+1} + \\ - \frac{3}{2}\kappa Gb\Delta y \boldsymbol{\Phi}_{\varphi j}\boldsymbol{q}_{\varphi j} - \frac{\kappa Gb\Delta y}{4}\boldsymbol{\Phi}_{\varphi,j-1}\boldsymbol{q}_{\varphi,j-1} - \frac{\kappa Gb\Delta y}{4}\boldsymbol{\Phi}_{\varphi,j+1}\boldsymbol{q}_{\varphi,j+1} \right]\Delta x + \boldsymbol{\Phi}_{\varphi j}^T \boldsymbol{f}_{\varphi j}, \tag{6}$$

where: $\boldsymbol{M}_{xj} = diag(m_{xj1} \ \cdots \ m_{xjn})$, $\boldsymbol{M}_{yj} = diag(m_{yj1} \ \cdots \ m_{yjn})$,
$\boldsymbol{M}_{\varphi r} = diag(m_{\varphi j1} \ \cdots \ m_{\varphi jn})$, $\boldsymbol{K}_{xj} = diag(k_{xj1} \ \cdots \ k_{xjn})$,
$\boldsymbol{K}_{yj} = diag(k_{yj1} \ \cdots \ k_{yjn})$, $\boldsymbol{K}_{\varphi j} = diag(k_{\varphi j1} \ \cdots \ k_{\varphi jn})$,
$\boldsymbol{f}_{xj} = \Delta x \cdot \Delta y \cdot col(f_{xj1} \ \cdots \ f_{xjn_x})$, $\boldsymbol{f}_{yj} = \Delta x \cdot \Delta y \cdot col(f_{yj1} \ \cdots \ f_{yjn_x})$,
$\boldsymbol{f}_{\varphi j} = \Delta x \cdot \Delta y \cdot col(\tau_{\varphi j1} \ \cdots \ \tau_{\varphi jn_x})$, $\boldsymbol{q}_{xj} = col(q_{xj1} \ \cdots \ q_{xjn_x})$,
$\boldsymbol{q}_{yj} = col(q_{yj1} \ \cdots \ q_{yjn_x})$, $\boldsymbol{q}_{\varphi j} = col(q_{\varphi j1} \ \cdots \ q_{\varphi jn_x})$,

$$\boldsymbol{\Phi}_{xj} = \begin{bmatrix} Y_{xj1}(x_1) & \cdots & Y_{xjn}(x_1) \\ \vdots & & \vdots \\ Y_{xj1}(x_{n_x}) & \cdots & Y_{xjn}(x_{n_x}) \end{bmatrix}, \quad \boldsymbol{\Phi}_{yj} = \begin{bmatrix} Y_{yj1}(x_1) & \cdots & Y_{yjn}(x_1) \\ \vdots & & \vdots \\ Y_{yj1}(x_{n_x}) & \cdots & Y_{yjn}(x_{n_x}) \end{bmatrix},$$

$$\boldsymbol{\Phi}_{\varphi j} = \begin{bmatrix} Y_{\varphi j1}(x_1) & \cdots & Y_{\varphi jn}(x_1) \\ \vdots & & \vdots \\ Y_{\varphi j1}(x_{n_x}) & \cdots & Y_{\varphi jn}(x_{n_x}) \end{bmatrix},$$

wheras: q – modal coordinates, m – modal coefficients of inertia, k – modal coefficients of stiffness, Y – eigenfunction, n – number of retained modes, n_x – number of ports for lumped interactions, j=1,…, n_y, n_y – number of strips, f – generalized external force, $\boldsymbol{\Phi}' = d\boldsymbol{\Phi}/dx$, subscripts x, y, φ are related to translations in x, y directions and rotation respectively.

It is very easy to construct the modal models because eigenvalues and eigenfunctions related to one-dimensional second order systems are known. Fig.3 presents general concept of developed hybrid model. Proposed approach can be applied for modeling of 2D, 3D and 1D continuous systems. Of course, in the case of 1D system, there are not interactions between strips/prisms. In this case the method can be applied for modelling of discrete-distributed systems with non-self-adjoined components – see illustrative example 2 and [2, 3, 4].

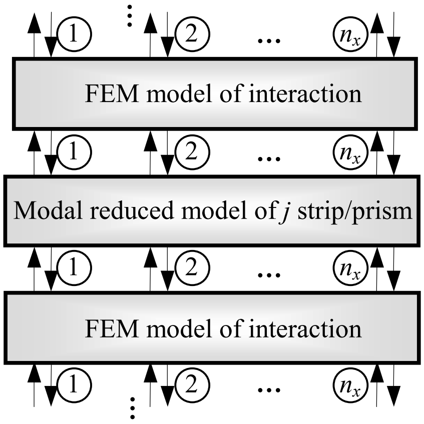

Figure 3. General block diagram of hybrid model

3.1. Illustrative Example 1

As an simple example let us consider one-strip system - the Timoshenko beam model (Fig. 4) which is described by the following equations (they can be obtain as the special case of equations (1÷3)):

$$f - \kappa AG\eta' = \underline{F = \rho As^2\eta - \kappa AG\eta''}\,, \tag{7}$$

$$\tau - \kappa AG\varphi + \kappa AG\eta' = \underline{T = \rho Is^2\varphi - EJ\varphi''}\,. \tag{8}$$

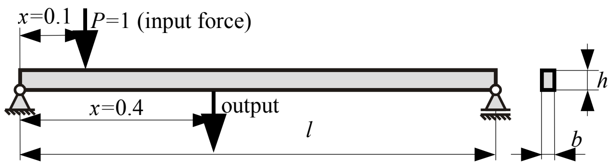

Figure 4. Simply supported beam with the following parameters: $E = 2 \cdot 10^{11}$, $G = 7.93 \cdot 10^{10}$, $\rho = 8000$, b=0.05, h=0.1, $\kappa = 1.2$, l=1.

The results are presented in Fig. 5. Frequency characteristics of the beam are obtained for the hybrid models with 6 retained modes and with 12 finite elements. From these one can see that in the range of frequency related to a number of retained modes frequency responses for reduced models have the same shape as for the reference continuous one.

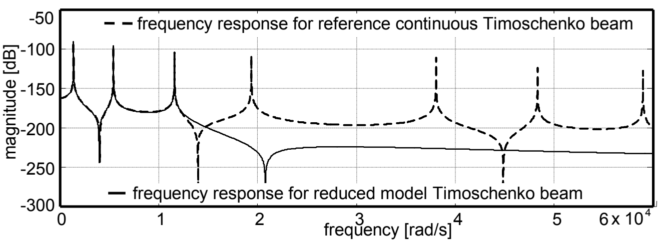

Figure 5. Verification of the reduced Timoshenko beam model

3.2. Illustrative Example 2

As the second example let us consider the rotor presented in Fig. 6a.
The difficulties in modal analysis of rotor system arise from the non-self-adjointness. To avoid that problem the following approach is proposed. Modal reduced model is built up for the system without gyroscopic effect.

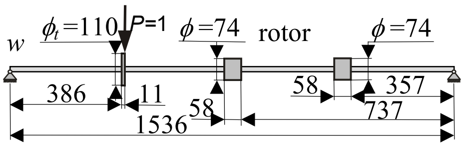

Figure 6. Rotor: $E = 1.9995 \cdot 10^{11}$ [Pa], $\rho = 7834.6$ [kg/m^3], $G = 7.5842 \cdot 10^{10}$ [Pa], $\kappa = 32/27$ (shape factor of cross section)

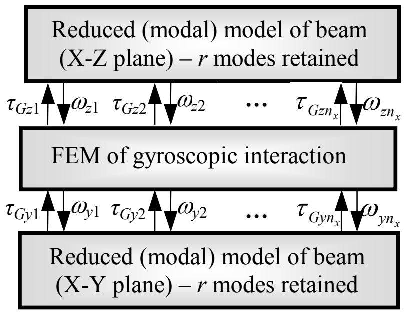

Figure 7. General block diagram of hybrid model of rotor

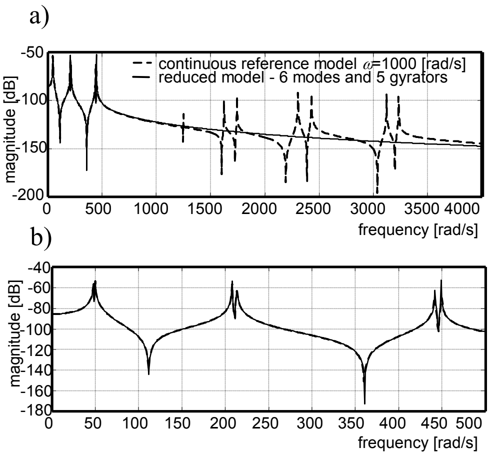

Figure 8. Frequency response with gyroscopic phenomena for reduced and non-reduced model Ω=1000 [rad/s] (a) and scaled up (b)

Gyroscopic moments are then modeled by application of rigid finite element method. Because of above reduced modal model must contain an appropriate number of inputs and outputs needed to connect lumped elements related to gyroscopic interactions between beams vibrating in X-Z and Y-Z planes.

Frequency characteristics of the rotor (Fig. 8) are obtained for the unit step force input signal acting at the left disk (Fig. 6) and the displacement output signal observed at the same point. From these one can see that in the range of frequency related to a number of retained modes frequency responses for reduced models have the same shape as for the reference model.

4. Conclusions

In this paper model reduction of continuous systems is presented. Two techniques: modal decomposition and finite element approach are applied simultaneously. The final reduced model consists of two parts - the reduced modal model and the finite element model. General idea of such approach has been presented in simple illustrative examples. The proposed approach enables to obtain accurate low order lumped parameter model representation of considered system. Computer simulations and numerical calculations proved that the proposed method is efficient and can be applied for others, more complex systems.

Acknowledgments

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Hybrydowe modele zredukowane układów ciągłych

W artykule przedstawiono alternatywną metodę modelowania i modalnej redukcji układów ciągłych. Zaprezentowana metoda jest metodą hybrydową. Łączy zalety metod dekompozycji modalnej i sztywnych elementów skończonych. W proponowanej metodzie układ ciągły dzielony jest na jednowymiarowe podukłady ciągłe. Dla każdego podukładu jednowymiarowego budowany jest modalny model zredukowany. Poszczególne modele zredukowane wiąże się ze sobą poprzez oddziaływania między nimi modelowane za pomocą metody sztywnych elementów skończonych. Zaprezentowana metoda umożliwia otrzymanie zredukowanego modelu modalnego niskiego rzędu. Proponowane podejście jest zilustrowane prostymi przykładami.