Implementation of Hermite-Ritz method and Navier’s Technique for Vibration of Functionally Graded Porous Nanobeam Embedded in Winkler-Pasternak Elastic Foundation Using bi-Helmholtz type of nonlocal elasticity

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Abstract

Present study is devoted to investigating the vibration characteristics of Functionally Graded (FG) porous nanobeam embedded in an elastic substrate of Winkler-Pasternak type. Classical beam theory (CBT) or Euler-Bernoulli beam theory (EBT) has been incorporated to address the displacement of the FG nanobeam. Bi-Helmholtz type of nonlocal elasticity is being used to
capture the small scale effect of the FG nanobeam. Further, the nanobeam is assumed to have porosity, distributed evenly along the thickness throughout the cross-section. Young’s modulus and mass density of the nanobeam are considered to vary along the thickness from ceramic to metal constituents in accordance with power-law exponent model. A numerically efficient method, namely the Hermite-Ritz method, is incorporated to compute the natural frequencies of Hinged-Hinged (HH), Clamped-Hinged (CH), and Clamped-Clamped (CC) boundary conditions. A closed-form solution is also obtained for Hinged-Hinged (HH) boundary condition by employing Navier’s technique. The advantages of using Hermite polynomials as shape functions are orthogonality, a large domain that makes the method more computationally efficient and avoids ill-conditioning for higher values of polynomials. Additionally, the present results are validated with other existing results in special cases demonstrating excellent agreement. A comprehensive study has been carried out to justify the effectiveness or convergence of the present model or method. Likewise, impacts of various scaling parameters such as Helmholtz and bi-Helmholtz types of nonlocal elasticity, porosity volume fraction index, power-law exponent, and elastic foundation on frequency parameters have been investigated.

**Keywords**

FG nanobeam; Hermite-Ritz method; Bi-Helmholtz function; Porosity; Winkler-Pasternak elastic foundation; vibration.

1. **Introduction**

Functionally graded materials (FGMs) are inhomogeneous materials consisting of two or more different materials, and the composition or volume of constituents varies continuously along one or more specific dimensions. As a result, their properties and structure will change steadily along the same dimension. This idea was first used by Japanese researchers [Koizumi 1994]. The gradual and continuous changes in these materials have made them very important and useful properties for application in various industries.

The introduction of FGMs to nano-micro technology has led to the development of devices and tools with better properties and capabilities, such as nano-micro-electro-mechanical systems (NEMS/MEMS), thin shape memory alloys, and atomic light microscopy. Nanotechnology is the study of microscopic objects about 1 to 100 nanometers in size and their applicability in various fields of science, such as chemistry, biology, physics, materials science, and engineering.
Recently, due to the special mechanical properties of nanostructures, the application of these structures has been developed in engineering, and researchers have been designing high-performance tools such as nanosensors, nano actuators, nanogenerators, etc. to solve new problems. Nanoscale tools are designed using the properties of nanotubes, nanobeams, nanomembranes, and nanosheets, so the discussion of modeling and analysis of nanobeams has attracted the attention of researchers.

Many experiments and computer simulations (molecular simulation) proved that a nanostructure mechanically has different response while it is analyzed in nanoscale size compared with a macroscale investigation. They showed that size is a crucial factor on nanoscale. Among all tools which aid us to predict mechanical response of these materials, the non-classical continuum elasticity approaches are cost and time-effective methods. Accordingly, it has been observed that classical continuum theories do not provide the right answer in predicting the behavior of these small scale structures. In fact, classical continuum theory is unable to account for size effects. The most popular non-classical continuum mechanic theories are: strain gradient theory [Mindlin 1965], nonlocal elasticity theory [Eringen 2002; Jena et al. 2019a; 2020a; 2020b], stress-driven nonlocal elasticity theory [Barretta et al. 2018; Sedighi and Malikan 2020], nonlocal strain gradient theory [Lim et al. 2015; Jena et al. 2019b; Malikan et al. 2020], modified coupled stress theory [Malikan 2017], surface elasticity theory [Ansari et al. 2013], and bi-Helmholtz nonlocal elasticity theory [Lazar et al. 2006; Koutsoumaris and Eptaimeros 2018]. These aforesaid theories, each in turn has small scale parameters. The small scale parameter makes difference between macro scale and nanoscale. Many research approved that these scale parameters are not material constant and vary with variation in natural features and physical characteristics of the nanomaterial. As an example, boundary and edge conditions affect fundamentally the values of small scale parameter. Moreover, as the nano materials except for being size-dependent, are also temperature-dependent, the thermal environment can significantly affect the value of small scale parameter. Thus, a nanostructure in various boundary conditions and different external temperature requires different values for the small scale parameter to give exact results. That is why all the researchers presented amplitude for numerical values of small scale parameters. There are also further examples for factors that affect the amount of a small scale parameter, such as crack specifications in cracked nanomaterials, arrangement of atoms in atomic lattice
into some special nanomaterials like graphene and nanotubes with changeable arrangement (chirality effect), etc.

The mechanical behavior of FG nanomaterials with different geometries and various loading and boundary conditions has been extensively investigated by researchers in the current decade. Beams are of great importance due to their wide use in engineering. To date, numerous articles have been written on the study of the dynamics of FGM nanobeams. Eltaher et al. [2012] based on the finite element method (FEM) analyzed natural frequencies of a nanoscale FGM beam by considering nonlocal continuum mechanics. The beam was modeled according to the Euler-Bernoulli beam theory (EBT) approach. The numerical outputs were calculated for a variety of boundary conditions. Sharabiania and Yazdi [2013] depicted a nonlinear frequency analysis on FGM nanosize beams in the framework of EBT while the size dependency was investigated on the basis of surface effects. The results were shown for some different edge conditions. Esmaeili and Tadi Beni [2019] investigated buckling and vibration characteristics of flexoelectric smart nanobeam composed of functionally graded materials. Nazemnezhad and Hosseini-Hashemi [2014] studied nonlocal effects within the framework of nonlinear analysis of vibrations for FGM nanoscale beams. The immovable ends, such as fixed and pinned conditions were assumed when EBT was employed to give the constitutive equations of frequency. Hashemi et al. [2014] considered analytically effects of surface and stress nonlocality for pivot-pivot EBT-FGM nanobeam models. Ansari et al. [2015] examined the excited frequencies nonlinearly for an FGM nanobeam in the body of an exact solution. The influences of the environment, such as temperature differential, were measured as well. The nanosize into the EBT model was investigated utilizing surface elasticity theory, and the Galerkin technique helped to solve the attained equations. Zeighampour and Tadi Beni [2015] developed FGM nanobeams by considering the variation of diameter in the length direction. The strain gradient theory, EBT, and Visco-Pasternak foundation model were combined, which led to the governing equations. The obtained equations were discretized using differential quadrature method (DQM) for pined-pined and clamped-clamped supported and then were solved by eigenvalue solver. Their best results proved the considerable effect of diameter variation on the dynamics behavior of FGM nanobeams. Ebrahimi and Salari [2015] studied the nonlocal effect on the FGM nanobeams by considering EBT beam model with the presence and absence of the influences of the thermal environment utilizing analytical method based on the Navier method.
Simsek [2016] discussed the free vibration of an FGM nanoscale beam based on the nonlinear strains and derived mathematical relation by presenting a new Hamiltonian in combining with EBT and nonlocal strain gradient elasticity. Shafiei et al. [2016] modeled a non-uniform FGM beam taking nanosize effects based on the nonlocal theory of elasticity. The natural frequencies were captured for the beam with the contribution of nonlinear terms. The procedure for solving of the harvested equations was generalized differential quadrature (GDQ) method and Homotopy perturbation method for fixed-fixed, pinned-pinned, and fixed-pinned boundary conditions. Hosseini and Rahmani [2016] combined thermos-elastic relations to study vibrations of an FGM nanoscale beam when the beam is geometrically curved. The nonlocal elasticity provides the size-dependent behavior, and the numerical results were obtained by analytical solutions. Khorshidi and Shariati [2016] investigated the vibration characteristics of the sigmoid-type of FGM nanobeams by using the modified couple stress theory. A variety of beam hypotheses such as EBT, first-order shear deformation theory (FSDT) and some higher-order shear deformation theory (HSDT) were investigated with the help of GDQ. Vosoughi [2016] applied nonlinearity to study free vibration of a FGM nanosize beam embedded on a nonlinear elastic medium. The use of FSDT and nonlocal elasticity addressed the desired equations that were discretized by DQM. Hamed et al. [2016] compared sigmoid with a nonlinear symmetric power varied along the thickness of EB-FGM nonlocal beams in a vibration study. Saffari et al. [2017] inspected the stability of an FGM nonlocal FSDT beam by taking surface effects in a dynamical situation. Thermal effects and foundation influences were implemented as well. Arefi and Zenkour [2017] explored the nonlocal vibration of a Timoshenko FGM nanobeam by taking the Visco-Pasternak matrix into account. Vu-Bac et al. [2016] carried out sensitivity analysis for quantifying the influence of uncertain input parameters by using probability density function on uncertain model outputs. The dynamics of three-dimensional inhomogeneities of FGM nanoscale beams was investigated by Hadi et al. [2018]. Jouneghani et al. [2018] modeled porosity into the material gradation of FGM nanobeams and examined the structural behavior of the system subjected to variation of environmental parameters such as temperature and humidity. Mirjavadi et al. [2018] focused on the nonlinear behavior of FGM nanosize beams considering porosities with respect to the EBT and second stress gradient of Eringen. Different end conditions were taken into consideration by the assistance of GDQM and an iterative technique. Simsek [2019] performed different closed-
form approaches to study a variety of analyses on the FGM nanobeams, namely forced and free vibrations, static bending, and buckling. The nonlocal strain gradient theory was implemented to capture the size dependency influence. Various loading cases were demonstrated in the dynamic analysis of the graded EBT model.

Aria and Friswell [2019] indicated a finite element analysis in the form of nonlocality to consider free vibration and stability of FGM nanobeams. Uzun and Yayli [2019] investigated the free vibration of functionally graded nanobeam for hinged-hinged and clamped-clamped boundary conditions with the help of the finite element model. The nonlocal effect of FG nanobeam was handled by the Eringen’s nonlocal theory. Karami and Janghorban [2019] showed a new shaped function into the higher-order shear deformation theory to study analytically natural frequencies of a FGM nonlocal isotropic/anisotropic beam. Thickness stretching influence was also evaluated by the shape function. The nonlocal strain gradient model determined the nanoscale behavior. Khaniki [2019] studied vibrations of FGM nanoscale beams based on the two phases nonlocal-local models, and then functionality gradation was derived along length. GDQ helped to obtain numerical results. Chen et al. [2020] studied thermal buckling behavior of Euler-Bernoulli beam made up of FG material. The transformed-section method was used to investigate the buckling characteristics analytically. Uzun and Yayli [2020] in a pioneering work studied free vibration of functionally graded nanobeam for Simply Supported boundary condition using Euler-Bernoulli beam theory and Eringen’s nonlocal elasticity by utilizing FEM.

Previous studies, as mentioned above illustrate the fact that the studies involving non-classical theories have rarely used the Bi-Helmholtz nonlocal elasticity theory; and have never used the advanced yet simple Hermit-Ritz method for this purpose. In this study, the Euler-Bernoulli theory is applied to find the numerical response of free vibration of FG nanobeams. The numerical solution of the free vibration is obtained, and the response of the rectangular nanobeam is calculated for the bi-Helmholtz nonlocal parameter by employing the Hermit-Ritz method for HH, CH, and CC boundary conditions while closed-form solution is obtained for HH boundary condition by utilizing the Navier’s technique. The beam is also embedded on the Winkler-Pasternak elastic bed. Due to the importance of porosity in the structure of functionally graded materials, this argument is included in the present analysis as well. The results of the theory presented are compared with those reported by previous researchers, and a good agreement is observed between the results. A parametric analysis is also carried out to
investigate the effect of various scaling parameters such as Helmholtz and bi-Helmholtz types of nonlocal elasticity, porosity volume fraction index, power-law exponent, and elastic foundation on the frequency response of the FG nanobeam.

2. Reviews of Helmholtz and bi-Helmholtz types of nonlocal operators

The bi- Helmholtz type nonlocal modulus, which is the Green’s function of bi Helmholtz operator may be stated as [Lazar et al. 2006; Koutsoumaris and Eptaimeros 2018]

\[ A^{bh}(|x-x'|) = \frac{1}{2} \frac{1}{\xi_1^2 - \xi_2^2} \left\{ \xi_1 e^{-\xi_1} - \xi_2 e^{-\xi_2} \right\} \]  

The corresponding bi-Helmholtz operator may be expressed as [Eringen 2002; Lazar et al. 2006; Koutsoumaris and Eptaimeros 2018]

\[ \ell^{bh} = \left( 1 - \frac{\xi_1^2}{d^2} \right) \left( 1 - \frac{\xi_2^2}{d^2} \right) = 1 - \left( \frac{\xi_1^2 + \xi_2^2}{d^2} \right) \frac{d^2}{dx^2} + \frac{\xi_1^2 \xi_2^2}{d^4} \frac{d^4}{dx^4} = 1 - \varepsilon^2 \frac{d^2}{dx^2} + \gamma^4 \frac{d^4}{dx^4}, \]  

where \( \varepsilon^2 = \xi_1^2 + \xi_2^2 = (e_0 a)^2 \), \( \gamma^4 = \frac{\xi_1^2 \xi_2^2}{\xi_1^2 + \xi_2^2} \) and the constants \( \xi_1 \) and \( \xi_2 \) are demonstrated as [Eringen 2002; Lazar et al. 2006; Koutsoumaris and Eptaimeros 2018]

\[ \xi_1^2 = \frac{\varepsilon^2}{2} \left( 1 + \sqrt{1 - 4 \frac{\gamma^4}{\varepsilon^4}} \right) \]  

\[ \xi_2^2 = \frac{\varepsilon^2}{2} \left( 1 - \sqrt{1 - 4 \frac{\gamma^4}{\varepsilon^4}} \right) \]  

Here the discriminant \( 1 - 4 \frac{\gamma^4}{\varepsilon^4} \geq 0 \), i.e., \( \varepsilon \geq \sqrt{2} \gamma \). Considering \( \varepsilon = \sqrt{2} \gamma \), we will have \( \xi_1 = \xi_2 \), where \( \xi_1, \xi_2 \in R \) and the parameter \( \gamma \) triumphs over \( \varepsilon \). For any other case, i.e., \( \varepsilon > \sqrt{2} \gamma \), the effect of \( \varepsilon \) prevails over \( \gamma \). From Lazar et al. [2006], Koutsoumaris and Eptaimeros [2018], it is evident that that \( \ell^{bh} \) operator matched the Born Karman’s model at the end of the Brillouin zone, when \( \xi_1 = \xi_2 \). Now, the nonlocal modulus is given in Eq. (1) maybe stated as [Lazar et al. 2006; Koutsoumaris and Eptaimeros 2018]
\[ A^{BH}(x-x',\gamma) = \frac{1}{2} \frac{1}{2\gamma^2} (\gamma + |x-x'|) e^{-\frac{|x-x'|}{\gamma}} \]  

(4)

Substituting \( \varepsilon = \sqrt{2}\gamma \) or \( e_0 a = \sqrt{2}\gamma \), Eq. (4) can be expressed as [Lazar et al. 2006; Koutsoumaris and Eptaimeros 2018]

\[ A^{BH}(x-x', \frac{e_0 a}{\sqrt{2}}) = \frac{1}{2(e_0 a)^2} \frac{\sqrt{2}|x-x'|}{e_0 a} \]  

(5)

Thus, the bi-Helmholtz operator in differential form may be given as [Lazar et al. 2006; Koutsoumaris and Eptaimeros 2018]

\[ \ell^{BH} = 1 - (e_0 a)^2 \frac{d^2}{dx^2} + \frac{(e_0 a)^4}{4} \frac{d^4}{dx^4} \]  

(6)

Assuming \( \xi_1 = e_0 a \) and \( \xi_2 = 0 \) in Eq. (1), the Helmholtz-type nonlocal modulus is given as [Eringen 2002; Koutsoumaris and Eptaimeros 2018]

\[ A^{H}(x-x', e_0 a) = \frac{1}{2(e_0 a)^2} e^{-\frac{|x-x'|}{e_0 a}} \]  

(7)

and the corresponding Helmholtz operator in differential form is given as [Eringen 2002]

\[ \ell^H = 1 - (e_0 a)^2 \frac{d^2}{dx^2} \]  

(8)

3. Mathematical formulation of the proposed model

In this study, a functionally graded porous nanobeam with length \( (L) \), breadth \( (b) \), thickness \( (h) \), and porosity volume fraction \( \vartheta, (\vartheta < 1) \) is taken into consideration, as depicted in Fig. 1.a. The material composition at the top surface \( (z = h/2) \) is assumed to be ceramic-rich while the bottom surface \( (z = -h/2) \) is considered to be metal-rich, and the gradation along thickness from the ceramic-rich surface to metal-rich surface is governed by power-law variation model. The porosity in the nanobeam is assumed as evenly distributed throughout the metal and ceramic constituents, as illustrated in Fig. 1.b.
Fig 1.a Schematic diagram of rectangular FG nanobeam embedded in the Winkler-Pasternak elastic foundation.

Fig 1.b Graphical representation of the rectangular cross-section of the FG nanobeam with evenly distributed porosity

Thus, according to the modified rule of the mixture [Wattanasakulpong and Ungbhakorn 2014; Shamsavari et al. 2018]

\[ P = P_U V_U + P_L V_L - \frac{9}{2} (P_U + P_L) \]  

(9)
Here $P$ denotes the material property FG nanobeam. $P_U, V_U$ is the material property and volume fraction for the ceramic constituent, whereas $P_L, V_L$ symbolize the material property and volume fraction of the metal constituent.

As per the power law variation model, the volume fractions of the ceramic and metal components are expressed as [Wattanasakulpong and Ungbhakorn 2014; Shahsavari et al. 2018]

$$V_U = \left(\frac{z}{h} + \frac{1}{2}\right)^k$$  \hspace{1cm} (10)

$$V_L = 1 - \left(\frac{z}{h} + \frac{1}{2}\right)^k$$  \hspace{1cm} (11)

Where $k$ is the non-negative parameter, namely power-law exponent, that regulates the distribution of material along the thickness of the nanobeam and $z$ denotes the distance from the mid-plane of the FG nanobeam. Using Eq. (9), Eq. (10), and Eq. (11), the material properties of the FG nanobeam with porosity may be given as [Wattanasakulpong and Ungbhakorn 2014; Shahsavari et al. 2018]

$$P = \left(P_U - P_L\right) \left(\frac{z}{h} + \frac{1}{2}\right)^k + P_L - \frac{9}{2} \left(P_U + P_L\right)$$  \hspace{1cm} (12)

The Young’s modulus $E(z)$, and material density $\rho(z)$ of the FG nanobeam can be demonstrated graphically in Figs. (2-3) and mathematically as [Wattanasakulpong and Ungbhakorn 2014; Shahsavari et al. 2018]

$$E(z) = \left(E_U - E_L\right) \left(\frac{z}{h} + \frac{1}{2}\right)^k + E_L - \frac{9}{2} \left(E_U + E_L\right)$$  \hspace{1cm} (13.a)

$$\rho(z) = \left(\rho_U - \rho_L\right) \left(\frac{z}{h} + \frac{1}{2}\right)^k + \rho_L - \frac{9}{2} \left(\rho_U + \rho_L\right)$$  \hspace{1cm} (13.b)
Fig. 2 Power-law variation of Young’s modulus for FG nanobeam composed of alpha-beta titanium alloy (Ti-6AL-4V) and zirconia (ZrO₂)

Fig. 3 Power-law variation of Mass density for FG nanobeam composed of alpha-beta titanium alloy (Ti-6AL-4V) and zirconia (ZrO₂)
According to the classical beam theory or Euler-Bernoulli beam theory, the displacement field can be given as [Reddy 2007]

\[
\begin{align*}
    u_1(x, z, t) &= u(x, t) - z \left( \frac{\partial w}{\partial x} \right) \\
    u_2(x, z, t) &= 0 \\
    u_3(x, z, t) &= w(x, t)
\end{align*}
\]

(14.a - 14.c)

Where \(u(x, t)\) and \(w(x, t)\) represent the axial and transverse displacements on the mid-plane of the FG nanobeam, respectively.

The strain-displacement relation of the FG nanobeam is stated as

\[
\varepsilon_{xx} = \frac{\partial u_1(x, z, t)}{\partial x} = \frac{\partial u(x, t)}{\partial x} - z \frac{\partial^2 w(x, t)}{\partial x^2}
\]

(15)

The stress component of the FG nanobeam as generalized Hooke’s law may be given as [Pradhan and Chakraverty 2014]

\[
\sigma_{xx} = Q_{11} \varepsilon_{xx} = \left( \frac{E(z)}{1 - \nu^2} \right) \varepsilon_{xx}
\]

(16)

### 3.1 Energy form of Equation for Hermite-Ritz method

The strain energy \(S\) of the FG nanobeam is stated as

\[
S = \frac{1}{2} \int_0^L \int_A \left( \sigma_{xx} \varepsilon_{xx} \right) dA dx
\]

\[
= \frac{1}{2} \int_0^L \int_A \left[ \sigma_{xx} \left( \frac{\partial u(x, t)}{\partial x} - z \frac{\partial^2 w(x, t)}{\partial x^2} \right) \right] dA dx
\]

\[
= \frac{1}{2} \int_0^L \int_A \left[ N \left( \frac{\partial u(x, t)}{\partial x} \right) - M \left( \frac{\partial^2 w(x, t)}{\partial x^2} \right) \right] dx
\]

(17)

Where the stress resultants \((N, M) = \int_A \left( \sigma_{xx}, z \sigma_{xx} \right) dA\).

Now, the variation in strain energy \((\delta S)\) can be given as
\[
\delta S = \int_A \left( \sigma_{xx} \delta e_{xx} \right) dA dx
\]

\[
= \int_A \int_L \left[ \sigma_{xx} \left( \frac{\partial \delta u (x,t)}{\partial x} - z \frac{\partial^2 \delta w (x,t)}{\partial x^2} \right) \right] dA dx
\]

\[
= \int_L \left[ N \left( \frac{\partial \delta u (x,t)}{\partial x} \right) - M \left( \frac{\partial^2 \delta w (x,t)}{\partial x^2} \right) \right] dx
\]

\[
= \int_L \left[ - \left( \frac{\partial N}{\partial x} \right) \delta u - \left( \frac{\partial^2 M}{\partial x^2} \right) \delta w \right] dx
\]

(18)

The kinetic energy \((T)\) of the FG nanobeam can be stated as

\[
T = \frac{1}{2} \int_0^L \rho(z) \left[ \left( \frac{\partial u_1}{\partial t} \right)^2 + \left( \frac{\partial u_2}{\partial t} \right)^2 + \left( \frac{\partial u_3}{\partial t} \right)^2 \right] dA dx
\]

\[
= \frac{1}{2} \int_0^L \rho(z) \left[ \left( \frac{\partial u}{\partial t} - z \left( \frac{\partial^2 w}{\partial x \partial t} \right) \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right] dA dx
\]

(19)

\[
= \frac{1}{2} \int_0^L \left[ I_0 \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right] - 2I_1 \left( \frac{\partial u}{\partial t} \right) \left( \frac{\partial^2 w_b}{\partial x \partial t} \right) + I_2 \left( \frac{\partial^2 w}{\partial x \partial t} \right)^2 \right] dx
\]

In which \((I_0, I_1, I_2) = \int_A \rho(z)(1, z, z^2) dA\) are the mass moment of inertias.

The variation in kinetic energy \((\delta T)\) can be obtained from Eq. (19) as

\[
\delta T = \frac{1}{2} \int_0^L \left[ I_0 \delta \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right] - 2I_1 \delta \left( \frac{\partial u}{\partial t} \right) \left( \frac{\partial^2 w_b}{\partial x \partial t} \right) + I_2 \delta \left( \frac{\partial^2 w}{\partial x \partial t} \right)^2 \right] dx
\]

(20)

\[
= \int_0^L \left[ - I_0 \left( \frac{\partial^2 u}{\partial t^2} \right) \delta u - I_0 \left( \frac{\partial^2 w}{\partial t^2} \right) \delta w + I_1 \left( \frac{\partial^3 w_b}{\partial x \partial t^2} \right) \delta u + I_1 \left( \frac{\partial^3 w}{\partial x \partial t^2} \right) \delta w + I_2 \left( \frac{\partial^4 w}{\partial x^2 \partial t^2} \right) \delta w \right] dx
\]

The work done \((W)\) by the Winkler-Pasternak elastic foundation can be expressed as [Uzun and Yaylι 2020]

\[
W = - \frac{1}{2} \int_0^L k_w w^2 + k_g \left( \frac{\partial w}{\partial x} \right)^2 \right] dx,
\]

(21)

where \(k_w\) and \(k_g\) are Winkler and Pasternak elastic constants, respectively.
The variation in external work done \( (\delta W) \) can be derived from Eq. (21) as

\[
\delta W = -\int_0^T \left[ k_w w + k_x \frac{\partial^2 w}{\partial x^2} \right] \delta W dx
\]  
(22)

Using Eq. (18), Eq. (20), and Eq. (22) in the extended Hamilton’s principle and collecting the co-efficient of \( \delta u \) and \( \delta w \), the governing equations of motion in terms of stress resultants and displacements can be obtained as

\[
\frac{\partial N}{\partial x} = I_0 \left( \frac{\partial^2 u}{\partial t^2} \right) - I_1 \left( \frac{\partial^3 w}{\partial x \partial t^2} \right)
\]  
(23.a)

\[
\frac{\partial^2 M}{\partial x^2} = I_0 \left( \frac{\partial^2 w}{\partial t^2} \right) + I_1 \left( \frac{\partial^3 u}{\partial x \partial t^2} \right) - I_2 \left( \frac{\partial^4 w}{\partial x^2 \partial t^2} \right) + k_w w - k_x \frac{\partial^2 w}{\partial x^2}
\]  
(23.b)

Multiplying Eq. (16) by \( dA \) and \( z dA \) and integrating over the area of cross-section of the FG nanobeam, the local stress resultants can be written as

\[
N = A_{11} \left( \frac{\partial u}{\partial x} \right) - B_{11} \left( \frac{\partial^2 w}{\partial x^2} \right)
\]  
(24.a)

\[
M = B_{11} \left( \frac{\partial u}{\partial x} \right) - D_{11} \left( \frac{\partial^2 w}{\partial x^2} \right)
\]  
(24.b)

where \( (A_{11}, B_{11}, D_{11}) = \int_A Q_{11}(1, z, z^2) dA \), are the stiffness coefficients of FG nanobeam.

Applying bi-Helmholtz operator to Eq. (24) and using Eq. (23), the nonlocal stress resultant resultants of the FG nanobeam can be obtained as

\[
N = A_{11} \left( \frac{\partial u}{\partial x} \right) - B_{11} \left( \frac{\partial^2 w}{\partial x^2} \right) + \left( e_a a \right)^2 \left[ \left\{ I_0 \left( \frac{\partial^3 u}{\partial x \partial t^2} \right) - I_1 \left( \frac{\partial^4 w}{\partial x^2 \partial t^2} \right) \right\} - \left( e_a a \right)^2 \left( \frac{e_a a}{4} \right) \left[ I_0 \left( \frac{\partial^5 u}{\partial x^3 \partial t^2} \right) - I_1 \left( \frac{\partial^6 w}{\partial x^4 \partial t^2} \right) \right] \right]
\]  
(25.a)
Substituting Eq. (25) in Eq. (17), the strain energy, kinetic energy, and work done by elastic foundation for the FG nanobeam can be depicted as

\[
T = \frac{1}{2} \int_0^L \left[ I_0 \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right] - 2I_1 \left( \frac{\partial u}{\partial t} \right) \left( \frac{\partial^2 w}{\partial x \partial t} \right) + I_2 \left( \frac{\partial^2 w}{\partial x^2 \partial t} \right)^2 \, dx
\]

\[
W = -\frac{1}{2} \int_0^L \left[ k_w w^2 + k_g \left( \frac{\partial w}{\partial x} \right)^2 \right] \, dx
\]

\[
S = \frac{1}{2} \int_0^L \left[ I_0 \left( \frac{\partial^2 w}{\partial t^2} \right)^2 + I_1 \left( \frac{\partial^3 u}{\partial x \partial t^2} \right) - I_2 \left( \frac{\partial^4 w}{\partial x^2 \partial t^2} \right)^2 \right] \, dx
\]

(25.b)
Assuming the motion of the FG nanobeam as sinusoidal i.e., plugging $u(x,t) = U(x)\cos(\omega t)$ and $w(x,t) = W(x)\cos(\omega t)$, the maximum strain energy ($S_{\text{max}}$), kinetic energy ($T_{\text{max}}$), and work done by elastic foundation ($W_{\text{max}}$) for the FG nanobeam can be obtained as

$$S_{\text{max}} = \frac{1}{2} \int_0^L \left[ A_{11} \left( \frac{dU}{dx} \right)^2 - 2B_{11} \left( \frac{dU}{dx} \right) \left( \frac{d^2W}{dx^2} \right) + D_{11} \left( \frac{d^2W}{dx^2} \right)^2 - (\omega)^2 (e_0a)^2 I_0 \left( \frac{dU}{dx} \right)^2 + 2(\omega)^2 (e_0a)^2 I_1 \left( \frac{dU}{dx} \right) \left( \frac{d^2W}{dx^2} \right) + (\omega)^2 (e_0a)^4 I_2 \left( \frac{d^2W}{dx^2} \right)^2 \right] \, dx$$

$$T_{\text{max}} = \frac{\omega^2}{2} \int_0^L \left[ I_0 \left( U^2 + W^2 \right) - 2I_1 \left( U \frac{dW}{dx} \right) + I_2 \left( \frac{dW}{dx} \right)^2 \right] \, dx$$

$$W_{\text{max}} = -\frac{1}{2} \int_0^L \left[ k_w W^2 + k_g \left( \frac{dW}{dx} \right)^2 \right] \, dx$$

Substituting Eqs. (29-31) into Lagrangian energy function $\Pi = S_{\text{max}} - W_{\text{max}} - T_{\text{max}}$ and setting $\Pi = 0$, one may get
\[
\left[ A_{11}\left(\frac{dU}{dx}\right)^2 - 2B_{11}\left(\frac{d^2W}{dx^2}\right)\left(\frac{dU}{dx}\right) + D_{11}\left(\frac{d^2W}{dx^2}\right)^2 - \left(e_0a\right)^2 k_e(W)\left(\frac{d^2W}{dx^2}\right) + \left(e_0a\right)^2 \right] \right]
\]
\[
= \omega^2 \int_0^L \left[ \left( \frac{dU}{dx} \right)^2 - \left(\frac{d^2W}{dx^2}\right) + \left(\frac{d^4W}{dx^4}\right) \right] \, dx
\]

Where;

\[ A_{11} = \frac{bh}{1-u^2} \left[ \frac{(E_U-E_L)}{k+1} + E_L - \left(\frac{g}{2}\right)(E_U+E_L) \right] \]

\[ B_{11} = \frac{bh^2k}{1-u^2} \left[ \frac{(E_U-E_L)}{2(k+1)(k+2)} \right] \]

\[ D_{11} = \frac{bh^3}{1-u^2} \left[ \frac{(E_U-E_L)(k^2+k+2)}{4(k+1)(k+2)(k+3)} + \frac{E_L}{12} - \left(\frac{g}{24}\right)(E_U+E_L) \right] \]

\[ I_0 = bh \left[ \frac{(\rho_U-\rho_L)}{k+1} + \rho_L - \left(\frac{g}{2}\right)(\rho_U+\rho_L) \right] \]

\[ I_1 = bh^2k \left[ \frac{(\rho_U-\rho_L)}{2(k+1)(k+2)} \right] \]

\[ I_2 = bh^3 \left[ \frac{(\rho_U-\rho_L)(k^2+k+2)}{4(k+1)(k+2)(k+3)} + \frac{\rho_L}{12} - \left(\frac{g}{24}\right)(\rho_U+\rho_L) \right] \]

Eq. (32) is the Lagrangian energy function of FG porous nanobeam with bi-Helmholtz type of nonlocal elasticity. The Lagrangian energy function for the Helmholtz type of nonlocal elasticity can be obtained from Eq. (32) as
\[
\begin{align*}
&\left[ A_{II}\left(\frac{dU}{dx}\right)^2 - 2B_{II}\left(\frac{d^2W}{dx^2}\right)\left(\frac{dU}{dx}\right) + D_{II}\left(\frac{d^2W}{dx^2}\right)^2 - (e_o a)^2 k_w \right] \\
&= \omega^2 \int_0^L \left[ I_0 \left( U^2 + W^2 \right) - 2I_1(\frac{dU}{dx}) + I_2 \left( \frac{dW}{dx} \right)^2 + (e_o a)^2 I_0 \left( \frac{dU}{dx} \right)^2 - (e_o a)^2 I_1 \left( \frac{dU}{dx} \right) \right] dx \\
&= \omega^2 \int_0^L \left[ \left( \frac{d^2W}{dx^2} \right) - (e_o a)^2 I_0(\frac{dW}{dx}) - (e_o a)^2 I_1 \left( \frac{d^2W}{dx^2} \right) \frac{dU}{dx} + (e_o a)^2 I_2 \left( \frac{d^2W}{dx^2} \right)^2 \right] dx 
\end{align*}
\]

(33)

3.2 Equation of motion for Navier’s technique

Substituting Eq. (25) into Eq. (23), the governing equations of motion in terms of displacement can be obtained as

\[
\begin{align*}
&\left\{ A_{II}\left(\frac{\partial^2 u}{\partial x^2}\right) - B_{II}\left(\frac{\partial^3 w}{\partial x^3}\right) = I_0 \left( \frac{\partial^2 u}{\partial t^2} \right) - I_1 \left( \frac{\partial^3 w}{\partial x \partial t^2} \right) - (e_o a)^2 \right\} \\
&\left\{ I_0 \left( \frac{\partial^3 u}{\partial x^3 \partial t^2} \right) - I_1 \left( \frac{\partial^5 w}{\partial x^5 \partial t^2} \right) \right\} + \left( e_o a \right)^4 \left\{ I_0 \left( \frac{\partial^6 u}{\partial x^6 \partial t^2} \right) - I_1 \left( \frac{\partial^7 w}{\partial x^7 \partial t^2} \right) \right\} 
\end{align*}
\]

(34.a)

\[
\begin{align*}
&\left\{ B_{II}\left(\frac{\partial^2 u}{\partial x^3}\right) - D_{II}\left(\frac{\partial^4 w}{\partial x^4}\right) = I_0 \left( \frac{\partial^2 u}{\partial t^2} \right) + I_1 \left( \frac{\partial^3 u}{\partial x \partial t^2} \right) - I_2 \left( \frac{\partial^4 w}{\partial x^2 \partial t^2} \right) + k_w w - k_s \left( \frac{\partial^2 w}{\partial x^2} \right) \right\} \\
&\left\{ \left( e_o a \right)^4 \left\{ I_0 \left( \frac{\partial^4 w}{\partial x^2 \partial t^2} \right) + I_1 \left( \frac{\partial^5 u}{\partial x^3 \partial t^2} \right) - I_2 \right\} + \left( e_o a \right)^4 \left\{ I_0 \left( \frac{\partial^6 u}{\partial x^6 \partial t^2} \right) + I_1 \left( \frac{\partial^7 u}{\partial x^5 \partial t^2} \right) - I_2 \right\} \right\} 
\end{align*}
\]

(34.b)

The governing equations of motion for the Helmholtz type of nonlocal elasticity can be obtained from Eq. (34) as

\[
\begin{align*}
&A_{II}\left(\frac{\partial^2 u}{\partial x^2}\right) - B_{II}\left(\frac{\partial^3 w}{\partial x^3}\right) = I_0 \left( \frac{\partial^2 u}{\partial t^2} \right) - I_1 \left( \frac{\partial^3 w}{\partial x \partial t^2} \right) - (e_o a)^2 \left\{ I_0 \left( \frac{\partial^4 u}{\partial x^2 \partial t^2} \right) - I_1 \left( \frac{\partial^5 w}{\partial x^3 \partial t^2} \right) \right\}
\end{align*}
\]

(35.a)
\[ B_{11} \left( \frac{\partial^3 u}{\partial x^3} \right) - D_{11} \left( \frac{\partial^4 w}{\partial x^4} \right) = I_0 \left( \frac{\partial^2 w}{\partial t^2} \right) + I_1 \left( \frac{\partial^3 u}{\partial x \partial t^2} \right) - I_2 \left( \frac{\partial^5 w}{\partial x^5 \partial t^2} \right) + k_u w - k_g \left( \frac{\partial^2 w}{\partial x^2} \right) \]

\[-(e_0a)^2 \left\{ I_0 \left( \frac{\partial^2 w}{\partial x^2 \partial t^2} \right) + I_1 \left( \frac{\partial^3 u}{\partial x^3 \partial t^2} \right) - I_2 \left( \frac{\partial^5 w}{\partial x^5 \partial t^2} \right) + k_u \left( \frac{\partial^2 w}{\partial x^2} \right) - k_g \left( \frac{\partial^4 w}{\partial x^4} \right) \right\} \]

(35.b)

4. Solution procedures

In the upcoming subsections, the Hermite-Ritz method and Navier’s technique have been described to solve the governing equations of motion for the proposed model.

4.1 Application of Hermite-Ritz method

Hermite polynomials [Bayın and Bayin 2006] \( H(n, x) \) are set of orthogonal polynomials with respect to the weight function \( e^{-x^2} \) defined over the domain \((-\infty, \infty)\), i.e.,

\[
\int_{-\infty}^{\infty} e^{-x^2} H(m, x)H(n, x)dx = \begin{cases} \sqrt{\pi} \frac{2^n n!}{2}, & n = m \\ 0, & n \neq m \end{cases}
\]

(36)

First five terms of Hermite polynomials with recurrence relations can be expressed as [Bayın and Bayin 2006]

\[ H(0, x) = 1 \]

\[ H(1, x) = 2x \]

\[ H(2, x) = 4x^2 - 2 \]

\[ H(3, x) = 8x^3 - 12x \]

\[ H(4, x) = 16x^4 - 48x^2 + 12 \]

\[ H(n, x) = 2x H(n-1, x) - 2(n-1)H(n-2, x) \text{ and } H'(n, x) = 2n H(n-1, x) \]

(37)

In this investigation, Hermite polynomials are taken as shape functions, i.e., both the axial and transverse displacements of the FG nanobeam are expressed in terms of Hermite polynomials. The main reasons behind choosing the Hermite polynomials as shape functions are:

- Hermite polynomials are the orthogonal polynomials which reduce the computation time.
• Unlike other orthogonal polynomials such as Chebyshev polynomials, Legendre polynomials, etc., the domain is \((-\infty, \infty)\) that offers flexibility in the limit of the Lagrangian energy function.

• It helps to restrict ill-conditioning of the matrix for higher values of polynomials.

The axial displacement \(U(X)\), and transverse displacement \(W(X)\) can be now expressed as [Pradhan and Chakraverty 2014]

\[
U(X) = X^\eta (R - X)^\kappa \sum_{i=1}^{n} c_i H(i - 1, X) \quad (38.a)
\]

\[
W(X) = X^\eta (R - X)^\kappa \sum_{i=1}^{n} d_i H(i - 1, X) \quad (38.b)
\]

Here \(c_i\)'s , and \(d_i\)'s are unknown coefficients, \(H(n, X)\) is the nth term of Hermite polynomial which is used shape function, \(X^\eta (R - X)^\kappa\) is the admissible functions with exponents \(\eta\), and \(\kappa\). For different boundary conditions \(\eta\), and \(\kappa\) possess different values, as shown in Table 1.

**Table 1** \(\eta\), and \(\kappa\) for different boundary conditions [Pradhan and Chakraverty 2014].

<table>
<thead>
<tr>
<th>B.C.</th>
<th>(\eta)</th>
<th>(\kappa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>H-H</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>C-H</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>C-C</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Substituting Eq. (38) into the Lagrangian energy function of bi-Helmholtz and Helmholtz types of nonlocal elasticity, i.e., Eq.(32) and Eq.(33) and minimizing \(\Omega^2\) with respect to the unknown coefficients \(c_i\)'s , and \(d_i\)'s , \(i = 1, 2, 3...n\) , give rise to the generalized eigenvalue problem as

\[
[K]\{X\} = \Omega^2 [M]\{X\} \quad (39)
\]

where \(\{X\}=[c_1,c_2,c_3,...c_n,d_1,d_2,d_3,...d_n]^T\), \([K]\) represents the stiffness matrix, and \([M]\) denotes the mass matrix.

**4.2 Application of Navier’s technique**
As per the Navier’s technique the axial displacement \( u(x,t) \) and transverse the displacement \( w(x,t) \) can be expanded in terms of sine and cosine series as [Bekhadda et al. 2019];

\[
\begin{align*}
    u(x,t) &= \sum_{m=1}^{\infty} u_m \cos\left(\frac{m\pi}{L} x\right) e^{j\omega t} \\
    w(x,t) &= \sum_{m=1}^{\infty} w_m \sin\left(\frac{m\pi}{L} x\right) e^{j\omega t}
\end{align*}
\]  

(40.1)

(40.2)

where \( u_m \) and \( w_m \) are arbitrary parameters and \( \omega \) is the natural frequency of vibration. Plugging Eq. (40) into the Eq. (34), and Eq. (35), generalized Eigenvalue problem for free vibration of FG nanobeam for bi- Helmholtz and Helmholtz types nonlocal elasticity, respectively, will be obtained as

\[
\begin{align*}
    \left[ \mathbf{K}^{bH} \right] \mathbf{x} &= \omega^2 \left[ \mathbf{M}^{bH} \right] \mathbf{x} \quad \text{(41.1)} \\
    \left[ \mathbf{K}^{H} \right] \mathbf{x} &= \omega^2 \left[ \mathbf{M}^{H} \right] \mathbf{x} \quad \text{(41.2)}
\end{align*}
\]

Here \( \left[ \mathbf{K}^{bH} \right] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \), \( \left[ \mathbf{M}^{bH} \right] = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \), \( \left[ \mathbf{K}^{H} \right] = \begin{bmatrix} k_{11} & \tilde{k}_{12} \\ \tilde{k}_{21} & k_{22} \end{bmatrix} \), \( \left[ \mathbf{M}^{H} \right] = \begin{bmatrix} \tilde{m}_{11} & \tilde{m}_{12} \\ \tilde{m}_{21} & \tilde{m}_{22} \end{bmatrix} \) and \( \mathbf{x} = \begin{bmatrix} u_m \\ w_m \end{bmatrix} \),

where;

\[
\begin{align*}
    k_{11} &= -A_{11} \left( \frac{m\pi}{L} \right)^2, \\
    k_{12} &= k_{21} = B_{11} \left( \frac{m\pi}{L} \right)^3, \\
    k_{22} &= -D_{11} \left( \frac{m\pi}{L} \right)^4 - (k_w - (k_g - (e_0 a)^2 (k_w) (m\pi/L)^2) \\
    &\quad - (e_0 a)^3 (k_g) (m\pi/L)^4 - \frac{(e_0 a)^4}{4} (k_w) (m\pi/L)^6, \\
    m_{11} &= I_0 - (e_0 a)^2 (I_0) \left( \frac{m\pi}{L} \right)^2 - \frac{(e_0 a)^4}{4} (I_0) \left( \frac{m\pi}{L} \right)^4, \\
    m_{12} &= m_{21} = I_1 \left( \frac{m\pi}{L} \right) + (e_0 a)^2 (I_1) \left( \frac{m\pi}{L} \right)^3 + \frac{(e_0 a)^4}{4} (I_1) \left( \frac{m\pi}{L} \right)^5.
\end{align*}
\]
\[ m_{22} = I_0 - I_2 \left( \frac{m\pi}{L} \right)^2 - I_0 \left( e_0a \right)^2 \left( \frac{m\pi}{L} \right)^2 - I_2 \left( e_0a \right)^2 \]

\[ \left( \frac{m\pi}{L} \right)^4 - I_0 \left( \frac{e_0a}{4} \right)^4 \left( \frac{m\pi}{L} \right)^4 - I_2 \left( \frac{e_0a}{4} \right)^4 \left( \frac{m\pi}{L} \right)^6, \]

\[ \tilde{k}_{11} = -A_{11} \left( \frac{m\pi}{L} \right)^2, \quad \tilde{k}_{12} = \tilde{k}_{21} = B_{11} \left( \frac{m\pi}{L} \right)^3, \]

\[ \tilde{k}_{22} = -D_{11} \left( \frac{m\pi}{L} \right)^2 \left(-k_w \right)-e_0a \left( \frac{m\pi}{L} \right)^2 - e_0a \left( k_w \right) \left( \frac{m\pi}{L} \right)^2 - e_0a \left( k_w \right) \left( \frac{m\pi}{L} \right)^4, \]

\[ \bar{m}_{11} = -I_0 - e_0a \left( I_0 \right) \left( \frac{m\pi}{L} \right)^2, \quad \bar{m}_{12} = \bar{m}_{21} = I_1 \left( \frac{m\pi}{L} \right) + e_0a \left( I_1 \right) \left( \frac{m\pi}{L} \right)^3, \]

\[ \bar{m}_{22} = -I_0 - I_2 \left( \frac{m\pi}{L} \right)^2 - I_0 \left( e_0a \right)^2 \left( \frac{m\pi}{L} \right)^2 - I_2 \left( e_0a \right)^2 \left( \frac{m\pi}{L} \right)^4. \]

By solving the Eigenvalue problem mentioned in Eq. (41), the natural frequencies for the proposed model will be obtained for Hinged-Hinged (HH) boundary condition.

### 5. Numerical results and discussions

In this investigation, the FG nanobeam is considered to be composed of metal constituents as alpha-beta titanium alloy or titanium (Ti-6AL-4V) and ceramic constituent as zirconia or zirconium dioxide (ZrO₂). The geometrical properties or dimension of the specimen is taken from [Uzun and Yaylı 2020] as width \( (b) = 400nm \), thickness \( (h) = 100nm \), and length \( (L) = 8000nm \), whereas the mechanical properties [Uzun and Yaylı 2020] are given as:

- zirconia or zirconium dioxide (ZrO₂): \( E_u = 151GPa \), \( \rho_u = 3000 Kg.m^{-3} \), and \( \nu_u = 0.3 \)
- titanium (Ti-6AL-4V): \( E_L = 105.7GPa \), \( \rho_L = 4429 Kg.m^{-3} \), and \( \nu_L = 0.298 \).

The Young’s modulus and mass density are assumed to vary through the thickness in accordance with the power-law exponent model. At the same time, for the sake of convenience, the Poisson’s ratio is taken constant throughout the thickness of the FG nanobeam, which is \( \nu = 0.3 \).

#### 5.1 Validation
In this subsection, the validation of the present model has been conducted with other existing results in special cases. In this regard, the first three natural frequencies of the functionally graded nanobeam of HH boundary condition has been compared with [Uzun and Yaylı 2020], by neglecting the porosity and assuming Helmholtz nonlocal operator. The numerical results are computed for HH boundary condition by using both the Navier’s technique (NT) and the Hermite-Ritz method (H-RM), which is demonstrated in Table 2. Likewise, the fundamental frequency parameter \( \lambda = \frac{\omega L^2}{h} \sqrt{\frac{\rho}{E_L}} \) for HH boundary condition has been compared with [Pradhan and Chakraverty 2014; Aydogdu and Taskin 2007] by neglecting the porosity, nonlocal effect, and elastic foundation. Here the material is considered as Alumina (Al₂O₃), and Aluminum (Al), and the gradation is taken along Young’s modulus only with \( E_L = 70 \text{GPa}, \quad E_U = 380 \text{GPa} \) and \( \nu = 0.3 \). The tabular result is depicted in Table 3, with various power-law exponent and aspect ratio. From these results, it is evident that the present model is accurate and copes well with the existing results in special cases.

Table 2 Comparison of natural frequencies (in MHz) obtained by present study with Uzun and Yaylı [2020], in special cases

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>k</th>
<th>( \omega_1 )</th>
<th></th>
<th>( \omega_2 )</th>
<th></th>
<th>( \omega_3 )</th>
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<tbody>
<tr>
<td></td>
<td></td>
<td>([\text{A}] ) Present (NT)</td>
<td>Present (H-RM)</td>
<td>([\text{A}] ) Present (NT)</td>
<td>Present (H-RM)</td>
<td>([\text{A}] ) Present (NT)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>10.2084</td>
<td>10.3295</td>
<td>10.3294</td>
<td>26.4247</td>
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<tr>
<td>2</td>
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<td>8.7835</td>
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<td>22.2756</td>
</tr>
<tr>
<td>4</td>
<td>8.2208</td>
<td>8.2772</td>
<td>8.2816</td>
<td>17.4076</td>
<td>17.6348</td>
<td>17.6350</td>
</tr>
<tr>
<td>6</td>
<td>8.1285</td>
<td>8.1851</td>
<td>8.1880</td>
<td>17.1877</td>
<td>17.4160</td>
<td>17.4162</td>
</tr>
<tr>
<td>4</td>
<td>7.9245</td>
<td>7.9559</td>
<td>7.9584</td>
<td>15.6414</td>
<td>15.7285</td>
<td>15.7287</td>
</tr>
</tbody>
</table>
Table 3 Comparison of frequency parameters obtained by present study with Pradhan and Chakraverty [2014], Aydogdu and Taskin [2007], in special cases

<table>
<thead>
<tr>
<th>(\frac{L}{h})</th>
<th>(k)</th>
<th>0</th>
<th>0.1</th>
<th>1</th>
<th>2</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present</td>
<td>6.8470</td>
<td>6.5120</td>
<td>5.1764</td>
<td>4.7518</td>
<td>3.9597</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>6.9516</td>
<td>6.6115</td>
<td>5.2562</td>
<td>4.8258</td>
<td>4.0208</td>
</tr>
</tbody>
</table>

B: Pradhan and Chakraverty [2014]
C: Aydogdu and Taskin [2007]

5.2 Convergence

Through this subsection, the convergence of the present model has been carried out for first four natural frequencies of FG nanobeam by considering the power-law exponent \(k = 1\), porosity volume fraction \(\vartheta = 0.1\), non-dimensional nonlocal parameter \(\alpha = \left(\frac{e_0 a}{L}\right) = 0.1\), non-dimensional Winkler elastic constant \(K_w = \left(\frac{k_w L^4}{E_l I}\right) = 40\), and non-dimensional Pasternak elastic constant \(K_g = \left(\frac{k_g L^2}{E_l I}\right) = 40\). Variations of the first four natural frequencies have studied with no. of terms of the Hermite polynomial for HH, CH, and CC boundary conditions,
which are depicted in Fig. (4), Fig. (5), and Fig. (6), respectively. Natural frequencies have also been computed from the closed-form solution by using Navier’s technique (NT) for HH boundary condition and compared with the results of the Hermite-Ritz method (HRM) showing good agreement as illustrated in Fig. 4. From these graphical results, it is quite evident that the first four natural frequencies of all the boundary conditions are attaining the convergence on or after no. of terms \( n = 6 \). Also, it may be observed that the CC boundary condition is approaching convergence faster than HH and CH boundary conditions.

**Fig. 4** Variation of first four natural frequencies \( (\omega) \) with no. of terms \( (n) \) and comparison with analytical results for HH boundary condition.
Fig. 5 Variation of first four natural frequencies ($\omega$) with no. of terms ($n$) for CH boundary condition

Fig. 6 Variation of first four natural frequencies ($\omega$) with no. of terms ($n$) for CC boundary condition
5.3 Effect of bi-Helmholtz nonlocal elasticity

In this subsection, the influence of the bi-Helmholtz operator has been studied on natural frequencies of HH, CH, and CC boundary conditions as compared with the Helmholtz operator. For the computational purpose, power-law exponent \( (k) = 1 \), porosity volume fraction \( (\vartheta) = 0.1 \), non-dimensional Winkler elastic constant \( (K_w) = 40 \), and non-dimensional Winkler elastic constant \( (K_g) = 40 \) are taken into consideration. The graphical results, i.e., Fig. 7.a, and Fig. 7.b represents the variation of first four natural frequencies with respect to the nonlocal parameters \( (\alpha) \) for bi-Helmholtz and Helmholtz operators, respectively, for HH boundary condition, and these results are computed by employing Navier’s technique. Likewise, Fig. 8 and Fig. 9 illustrate the graphical results for CH and CC boundary conditions, respectively, which are computed using the Hermite-Ritz method. Here, the nonlocal parameters are assumed to vary from 0 to 0.5 with an increment of 0.1. From these graphical results, it may clear that the natural frequencies for all modes and all boundary conditions are decreasing with increase in nonlocal parameters except for the first and second modes of CH and CC boundary conditions with respect to bi-Helmholtz operator. Also, this decrease is very significant in the case of higher modes.

![Graphs showing variation of natural frequencies]( figures/7.png)

**Fig. 7** Variation of first four natural frequencies \( (\omega) \) with nonlocal parameter \( (\alpha) \) for HH boundary condition

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Fig. 8 Variation of first four natural frequencies ($\omega$) with nonlocal parameter ($\alpha$) for CH boundary condition

Fig. 9 Variation of first four natural frequencies ($\omega$) with nonlocal parameter ($\alpha$) for CC boundary condition
5.4 Effect of porosity or porosity volume fraction index

This subsection is dedicated to investigating the effect of porosity or porosity vol. fraction index \( \vartheta \) on natural frequencies of FG porous nanobeam. Here the porosity vol. fraction \( \vartheta \) is varied from 0 to 0.5 with an increase of 0.1, and other scaling parameters are taken as; power-law exponent \( k = 1 \), non-dimensional nonlocal parameter \( \alpha = 0.1 \) non-dimensional Winkler elastic foundation \( K_w = 40 \), and non-dimensional Pasternak elastic foundation \( K_g = 40 \). In this regard, graphical and tabular results are given in Table 4 and Figs. 10-12. Natural frequencies of the FG nanobeam increase with the rise in porosity index, which is applicable for all modes, all boundary conditions, and in the case of both the bi-Helmholtz and Helmholtz operators. This is because although with more value of porosity parameter the stiffness of beam becomes lesser and also its cross-sectional moment of inertia reduces, the reduction rate of inertia is more than that of the stiffness in the beam. It should be noted that other types of porosity may have opposite result. Also, it may be noted that the increase in natural frequencies is more significant in higher modes. The results obtained by both the bi-Helmholtz and Helmholtz operators are almost equal in lower modes where it can be clearly distinguished for higher modes, Helmholtz operator possesses more natural frequencies than bi-Helmholtz, and this trend is valid in all the boundary conditions.

Table 4 Natural frequencies (MHz) for Helmholtz operator (Ho) and Bi-Helmholtz operators (B-Ho) with respect to porosity volume fraction index.

(a) Hinged-Hinged (HH) boundary condition

<table>
<thead>
<tr>
<th>Porosity ( \vartheta )</th>
<th>( \omega_1 )</th>
<th>( \omega_2 )</th>
<th>( \omega_3 )</th>
<th>( \omega_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ho</td>
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<td>Ho</td>
<td>B-Ho</td>
</tr>
<tr>
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<td>8.9954</td>
<td>8.9933</td>
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<td>21.1592</td>
</tr>
<tr>
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<td>9.3766</td>
<td>21.8957</td>
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<tr>
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<td>22.4899</td>
</tr>
<tr>
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<td>10.3930</td>
<td>23.5144</td>
<td>23.3904</td>
</tr>
<tr>
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<td>24.6526</td>
<td>24.5353</td>
</tr>
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</table>

(b) Clamped-Hinged (CH) boundary condition
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<th>( \omega_2 )</th>
<th>( \omega_3 )</th>
<th>( \omega_4 )</th>
</tr>
</thead>
<tbody>
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<td>Ho</td>
<td>B-Ho</td>
</tr>
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</tr>
<tr>
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<td>11.7097</td>
<td>26.3055</td>
<td>25.8800</td>
</tr>
<tr>
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<td>27.2248</td>
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<td>13.8634</td>
<td>29.9541</td>
<td>29.3777</td>
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</table>

(c) Clamped-Clamped (CC) boundary condition

<table>
<thead>
<tr>
<th>Porosity ( \vartheta )</th>
<th>( \omega_1 )</th>
<th>( \omega_2 )</th>
<th>( \omega_3 )</th>
<th>( \omega_4 )</th>
</tr>
</thead>
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<td>Ho</td>
<td>B-Ho</td>
</tr>
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<td>16.0379</td>
<td>34.3065</td>
<td>30.7424</td>
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</table>

**Fig. 10** Variation of first four natural frequencies \( (\omega) \) with porosity vol. fraction index \( (\vartheta) \) for HH boundary condition
Fig. 11 Variation of first four natural frequencies ($\omega$) with porosity vol. fraction index ($\varphi$) for CH boundary condition

Fig. 12 Variation of first four natural frequencies ($\omega$) with porosity vol. fraction index ($\varphi$) for CC boundary condition
5.5 Effect of Power-law exponent

In this subsection, the influence of the power-law exponent \( k \) has been studied on the natural frequencies of FG nanobeam. The power-law exponent \( k \) is taken as 0, 0.2, 0.5, 1, 2, 3, 5, with porosity volume fraction \( (\vartheta) = 0.1 \), non-dimensional parameter \( (\alpha) = 0.1 \), non-dimensional Winkler elastic constant \( (K_w) = 40 \), and non-dimensional Pasternak elastic constant \( (K_p) = 40 \).

Table 5(a-c) and Figs. (13-15) represent the tabular and graphical results for HH, CH, and CC edges with respect to both the bi-Helmholtz and Helmholtz operators. All the computations for HH edge are carried out by using Navier’s technique, while the Hermite-Ritz method is used for other boundary conditions. These results clearly reveals that the natural frequencies of all modes and all boundary conditions decrease with an increase in the power-law exponent \( k \), that means when the beam is purely ceramic i.e., at \( k = 0 \) possesses the highest natural frequencies and when the beam is purely metal i.e., at \( k = \infty \) retains the lowest natural frequencies. This reduction is due to the fact that as we go on increasing the power-law exponent \( k \), the beam becomes more flexible, retaining less natural frequencies. This reduction is more remarkable with higher modes and at \( k < 2 \).

**Table 5** Natural frequencies (MHz) for Helmholtz operator and Bi-Helmholtz operator with respect to power-law index.

(a) Hinged-Hinged (HH) boundary condition

<table>
<thead>
<tr>
<th>k</th>
<th>( \omega_1 )</th>
<th>( \omega_2 )</th>
<th>( \omega_3 )</th>
<th>( \omega_4 )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Ho-Ho</td>
<td>Ho-Ho</td>
<td>Ho-Ho</td>
<td>Ho-Ho</td>
</tr>
<tr>
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<td>10.7925</td>
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</tr>
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<td>20.5373</td>
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</tr>
<tr>
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<td>8.7060</td>
<td>8.7042</td>
<td>20.1140</td>
<td>19.9962</td>
</tr>
</tbody>
</table>

(b) Clamped-Hinged (CH) boundary condition

32
(c) Clamped-Clamped (CC) boundary condition

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$\omega_3$</th>
<th>$\omega_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ho B-Ho Ho</td>
<td>Ho B-Ho Ho</td>
<td>Ho B-Ho Ho</td>
<td>Ho B-Ho Ho</td>
</tr>
<tr>
<td>0.2</td>
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<td>28.4981 28.0498</td>
<td>47.2095 45.4679</td>
<td>69.8833 62.6547</td>
</tr>
<tr>
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</tr>
<tr>
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<td>25.5648 25.1599</td>
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</tr>
<tr>
<td>2</td>
<td>10.8668 10.8035</td>
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</tr>
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<td>23.9305 23.5472</td>
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<td>57.8970 52.0277</td>
</tr>
<tr>
<td>5</td>
<td>10.4516 10.3871</td>
<td>23.4172 23.0390</td>
<td>38.4169 37.1130</td>
<td>56.5535 50.8202</td>
</tr>
</tbody>
</table>

\begin{figure}
\centering
\includegraphics[width=\textwidth]{natural_frequency_vs_power-law_exponent}
\end{figure}
Fig. 13 Variation of first four natural frequencies ($\omega$) with power-law index ($k$) for HH edge

Fig. 14 Variation of first four natural frequencies ($\omega$) with power-law index ($k$) for CH edge

Fig. 15 Variation of first four natural frequencies ($\omega$) with power-law index ($k$) for CC edge
5.6 Effect of elastic foundation

This subsection is devoted to analyzing the effect of elastic foundation, i.e., non-dimensional Winkler ($K_w$), and Pasternak ($K_g$) elastic parameters on natural frequencies of the FG nanobeam. In this regard, a comprehensive study has been undertaken by varying the elastic parameters, and the results are noted in tabular form, which can be seen in Table 6. The tabular results are incorporated for HH, CH, and CC boundary conditions with power-law exponent ($k$) = 1, porosity volume fraction ($\theta$) = 0.1, and nonlocal parameter ($\alpha$) = 0.1. Different combinations for elastic foundations are considered, and results are noted for the first four natural frequencies by considering both bi-Helmholtz and Helmholtz operators. From these results, it’s quite clear that the natural frequencies increase with the increase in elastic constants except the second mode of CC edge, where some irregularities occur with few combinations for elastic foundations, and these growths are more remarkable with higher modes. The increase in natural frequencies can be explained by the fact that the higher values of elastic parameters make the beam stiffer resulting higher value of natural frequencies.

Table 6 Natural frequencies (MHz) for Helmholtz operator and Bi-Helmholtz operator with respect to elastic foundation

(a) Hinged-Hinged (HH) boundary condition

<table>
<thead>
<tr>
<th>($K_w, K_g$)</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$\omega_3$</th>
<th>$\omega_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ho</td>
<td>B-Ho</td>
<td>Ho</td>
<td>B-Ho</td>
</tr>
<tr>
<td>(0, 0)</td>
<td>4.1382</td>
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<td>14.4871</td>
</tr>
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<td>5.0232</td>
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<td>14.7655</td>
</tr>
<tr>
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</tr>
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<td>15.5708</td>
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<td>9.9270</td>
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<tr>
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<td>11.4529</td>
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<td>18.1587</td>
<td>17.9965</td>
</tr>
<tr>
<td>(1000, 0)</td>
<td>13.4180</td>
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<tr>
<td>(1, 50)</td>
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<td>23.0508</td>
</tr>
<tr>
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<td>13.3372</td>
<td>29.3030</td>
<td>29.2027</td>
</tr>
<tr>
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<td>28.6542</td>
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</table>
(b) Clamped-Hinged (CH) boundary condition

<table>
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<th>$(K_w, K_g)$</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$\omega_3$</th>
<th>$\omega_4$</th>
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(c) Clamped-Clamped (CC) boundary condition

<table>
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<th>$\omega_4$</th>
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<tr>
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<td>42.2842</td>
<td>82.5095</td>
<td>82.4740</td>
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6. Concluding remarks

In this investigation, a computationally efficient method, namely the Hermite-Ritz method has been employed to compute the frequency response of the proposed model. Bi-Helmholtz type of nonlocal operator has been incorporated to seize the effect small scale effect. HH, CH, and CC boundary conditions are considered in this investigation, and closed-form solution is also obtained for HH boundary condition by utilizing Navier’s technique. Validation and convergence of the proposed model/and method have been conducted successfully. Conclusions obtained from the parametric study are summarized as follow;

- The natural frequencies are decreasing with the increase in the nonlocal parameters except for the first and second modes of CH and CC boundary conditions concerning bi-Helmholtz operator. Also, this decrease is very significant in the case of higher modes.
Natural frequencies of the FG nanobeam increase with the rise in porosity volume fraction index and the increase in natural frequencies is more substantial in higher modes.

The results obtained by both the bi-Helmholtz and Helmholtz operators are almost equal in lower modes. But, in higher modes, the Helmholtz operator possesses more natural frequencies than bi-Helmholtz operator.

The natural frequencies reduce with the increase in the power-law exponent ($k$), which means at $k = 0$ the beam possesses the highest natural frequencies and at $k = \infty$ the beam retains the lowest natural frequencies.

The natural frequencies increase with the increase in elastic parameters except for the second mode of CC edge, where some irregularities occur with few combinations for elastic foundations, and these growths are more remarkable with higher modes.

**Acknowledgment**

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**References**


