IMPLEMENTATION OF THE BOUNDARY ELEMENT METHOD TO TWO-DIMENSIONAL HEAT TRANSFER WITH THERMAL BRIDGE EFFECTS

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(Received 26 February 2013; revised manuscript received 9 April 2013)

Abstract: The work presents an application of the boundary element method applied to a two-dimensional conductive heat transfer. The algorithm of the method is explained and its advantages are outlined. Green's function as a fundamental solution for Poisson's equation in two dimensions was used and the direct approach was applied. The presented results concern building construction elements as typical cases of thermal bridges. Some properties of the boundary element method which give new possibilities were considered. For instance, forcing selected values of temperature on inner edges of the considered domain or local increasing of the temperature field resolution. The simulations were performed with the author's own algorithm.

Keywords: heat transfer, boundary element method (BEM), temperature field, thermal bridge, Green's function

1. Introduction

In recent years there has been a trend to build energy-efficient or even passive houses where the energy usage should be minimized with the thermal comfort estimated at an optimal level. It is mainly achieved by appropriate thermal insulation and installation of heating systems based on renewable sources of energy. It is important to avoid or minimize the thermal bridge effect. Therefore, proper construction of building elements is essential and this can be obtained by detailed analysis of heat transfer in places where a thermal bridge can exist. Generally in building engineering it can be assumed that temperature does not change in one direction, simplifying the problem to a two-dimensional heat transfer [1]. The temperature distribution in building elements is also important to predict areas where vapor condensation may occur [2].

It would be quite difficult and time-consuming to carry out an in-depth analysis of the heat transfer balance of a building. Appropriate standards have been developed to simplify the process and to determine how to perform calculations. A thermal bridge is a place of significant attention where standard guidelines may fail. Disregarding the thermal bridge can considerably underestimate the heat loss through the considered element. Then, heat transfer can be found by numerical calculations and heat transfer coefficients for particular examples are presented in thermal bridge catalogues [3]. Commercial software which is used for such purpose is mainly based on traditional mesh numerical methods such as the finite element method (FEM), the finite difference method (FDM) or the finite volume method (FVM).

The present work is concerned with the boundary element method in heat transfer through building construction elements which are typical thermal bridges.

2. Theory

The steady-state two-dimensional heat transfer is expressed by Poisson's equation:

$$\lambda\left(\frac{\partial^2 T(x_p,y_p)}{\partial x_p^2}+\frac{\partial^2 T(x_p,y_p)}{\partial y_p^2}\right)+q_v(x_p,y_p)=0 \tag{1}$$

where λ is the thermal conductivity, (x_p,y_p) are the coordinates of point p, q_V is the heat source function and T is the temperature. A fundamental solution (Green's function) was used in the following form:

$$T^*\left(\vec{\xi},\vec{p}\right)=\frac{1}{2\pi\lambda}\ln\frac{1}{r} \tag{2}$$

where $\vec{p}=(x_p,y_p)$, $\vec{\xi}=(x_\xi,y_\xi)$ are the points belonging to the considered domain Ω, and $r=\sqrt{(x_p-x_\xi)^2+(y_p-y_\xi)^2}$ is the distance between the two points.

The direct approach was adopted to derive an integral formulation of a partial differential Equation (1). The weighted residual method was implemented [4]:

$$\iint\limits_{\Omega}\left[\lambda\vec{\nabla}^2 T(\vec{p})+q_V(\vec{p})\right]T^*\left(\vec{\xi}\right)d\Omega=0 \tag{3}$$

Subsequently, having in mind the definition of the directional derivative and using the Dirac delta function properties and Green's second identity, the boundary integral equation can be written as:

$$B(\vec{\xi})T(\vec{\xi})+\oint\limits_{\Gamma}q(\vec{p})T^*\left(\vec{\xi},\vec{p}\right)d\Gamma=\oint\limits_{\Gamma}T(\vec{p})q^*\left(\vec{\xi},\vec{p}\right)d\Gamma+\iint\limits_{\Omega}q_V(\vec{p})T^*\left(\vec{\xi},\vec{p}\right)d\Omega \tag{4}$$

where $B\left(\vec{\xi}\right)$ is a coefficient dependent on the boundary shape, q is the heat flux density normal to the edge Γ (q^* is derived from T^*). Assuming that there are no heat sources, the smooth edge ($B\left(\vec{\xi}\right)=\frac{1}{2}$) divided into N elements, integrals in Equation (4) can be substituted by the sums of integrals over all the boundary elements indexed from $j=0$ to $j=N$. Thus, for certain edge node ξ^i, Equation (4) takes the following form:

$$\frac{1}{2}T\left(\vec{\xi^i}\right)+\sum_{j=1}^{N}q_j\int\limits_{\Gamma_j}T^*\left(\vec{\xi^i},\vec{p}\right)d\Gamma_j=\sum_{j=1}^{N}T_j\int\limits_{\Gamma_j}q^*\left(\vec{\xi^i},\vec{p}\right)d\Gamma_j \tag{5}$$

There will be a system of N such boundary equations (one per each edge node, Figure 1a), where either temperature T or heat flux density q are unknown values. Implementing appropriate boundary conditions allows us to solve the system, and integrals $\int\limits_{\Gamma_j}T^*\left(\vec{\xi^i},\vec{p}\right)d\Gamma_j$ and $\int\limits_{\Gamma_j}q^*\left(\vec{\xi^i},\vec{p}\right)d\Gamma_j$ can be determined using Gaussian quadratures. Finally, when all the boundary values are known, the temperature at any chosen point (Figure 1b) in the interior of the considered domain can be calculated with the following formula:

$$T\left(\vec{\xi_i}\right)=\sum_{j=1}^{N}T_j\int\limits_{\Gamma_j}q^*\left(\vec{\xi^i},\vec{p}\right)d\Gamma_j-\sum_{j=1}^{N}q_j\int\limits_{\Gamma_j}T^*\left(\vec{\xi^i},\vec{p}\right)d\Gamma_j \tag{6}$$

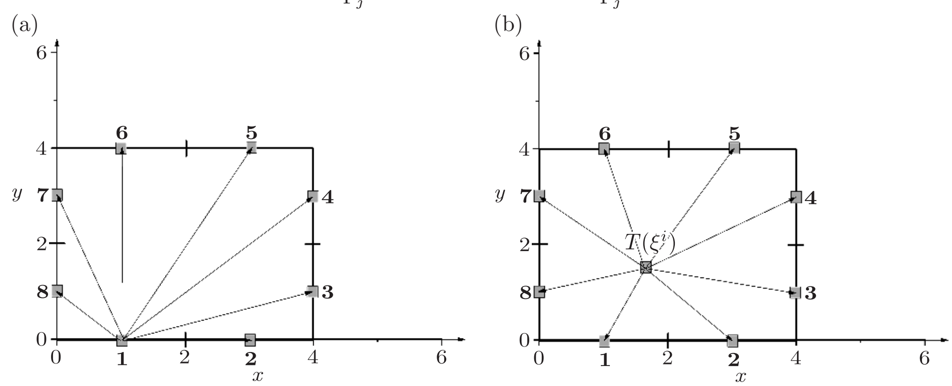

Figure 1. Graphical scheme of boundary integral equation (a) and integral equation for domain interior (b)

It is also possible to consider a domain consisting of several homogenous sub-domains. In such a case the procedure for every sub-domain is the same as explained above.

Assuming the continuity condition of temperature $T_a=T_b$ and heat flux density $q_a=-q_b$ for the joint edges, the systems of equations for every sub-domain can be joined. The system of equations can be written in a matrix form, and then, the integrals of the same node must be stored in the same column.

3. Block scheme of the algorithm

Calculations were carried out in a program written in the C++ language according to the following algorithm:

1) *Define the functions of the integrals;*
2) *Define the function which assigns coordinates of points ξ on the boundary when calculating the boundary integrals;*

3) *Define the function which assigns coordinates of points p on the boundary when calculating integrals in the interior of the domain;*
4) *Assign the input data (thermal conductivity and boundary conditions);*
5) *Calculate the boundary integrals;*
6) *Form the main matrix of the system;*
7) *Calculate the values of the right hand side matrix;*
8) *Solve the system of equations using Gaussian elimination;*
9) *Assign the results to the appropriate edges and form tables with all the boundary values;*
10) *Calculate the temperature values in the domain interior.*

4. Results

4.1. Corner

Four rectangular subdomains were used to create a shape of a corner of a wall as shown in Figure 2. A perfect contact between the subdomains was assumed. Such a structure allows simulating various scenarios. The two basic scenarios are: a wall without insulation and a wall with insulation. The boundary conditions for the following examples were taken from [5], so that the results could be compared.

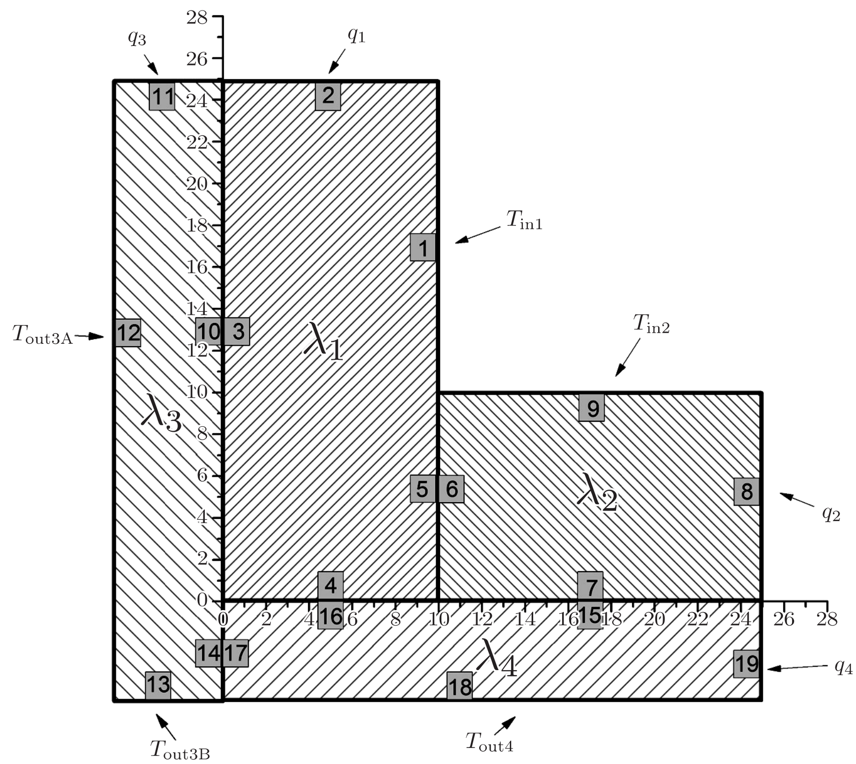

Figure 2. Domain simulating corner geometry

15 nodes were assigned for a wall thickness: 5 for the outer part and 10 for the inner part. The outer and inner edges consisted of 30 and 15 nodes, respectively. First, a corner of a simple 15cm brick wall ($\lambda = 0.77\frac{\mathrm{W}}{\mathrm{m{\cdot}K}}$) was simulated. In order to force on the inner edges (no. 10, 16 and 15 in Figure 2) the same boundary conditions (Dirichlet type) as on the outer ones, a very high value of λ_3 and λ_4 have been assigned (domain 3 and 4 would show no heat resistance).

In the second case a 5cm layer of styrofoam insulation ($\lambda = 0.04\frac{\mathrm{W}}{\mathrm{m{\cdot}K}}$) was added. The temperature fields inside the domain and values of the heat flux density in both cases are presented in Figure 3 and Figure 4, respectively.

In some figures isotherms seem to cross the outer boundary of the domain, what would be unphysical. This is due to an approximation in graphical post-processing software [6] and that the temperature on the boundary is not displayed.

The heat rate for 1m of the wall height was estimated as $\dot{Q} = 115.132\,\mathrm{W}$ for the first case and 7.6 times smaller value for insulated wall $\dot{Q} = 15.02\,\mathrm{W}$ (If straight 30cm section of a wall would be considered, those values would be respectively 97.02W and 9.13W).

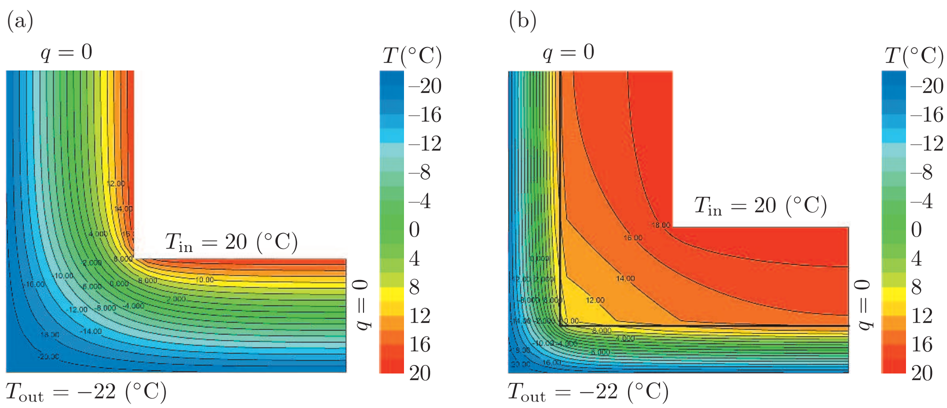

Figure 3. The temperature field of a corner of a brick wall ($\lambda = 0.77\frac{\mathrm{W}}{\mathrm{m{\cdot}K}}$) without insulation (a) and with 5cm styrofoam insulation ($\lambda = 0.04\frac{\mathrm{W}}{\mathrm{m{\cdot}K}}$) (b)

Thanks to the applied geometry of the domain a few more scenarios were considered and their results are shown in Figure 5.

4.2. Corner in higher resolution

One of the main advantages of the BEM is that the temperature at each point of the domain is estimated independently, based on the values of all the boundary conditions. It means that in the case where knowledge of the detailed temperature field in one specific area is significant, there is no need to create a dense mesh in a whole domain but only there where it is needed. However, when approaching the boundary, the resolution of the boundary elements should be increased to avoid large error.

As an example of such procedure, the resolution of a domain (Figure 6) 1cm deep and long on the inner side of the uninsulated corner of a brick wall,

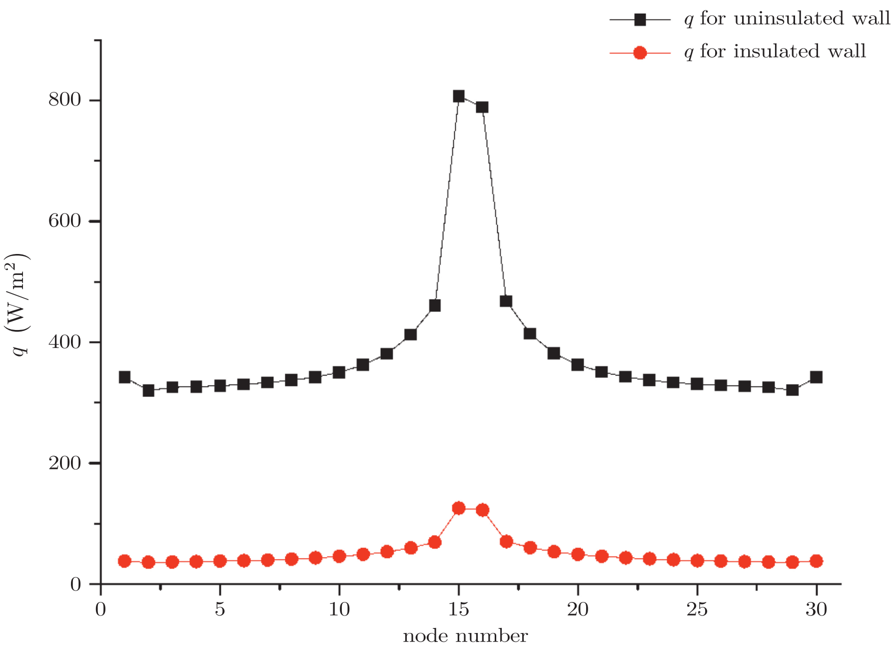

Figure 4. Heat flux density obtained for boundary nodes on the inner edge of the corner with and without insulation. Nodes are numbered 1 to 30 beginning from the right side of edge no. 9 to top of edge no. 1 in Figure 2

was increased. Two cases were considered – firstly when boundary elements, 1cm long, were imposed on the boundary, and secondly when the length was 0.1cm. In both cases there were 300 points spaced by 0.1cm. For that reason, huge errors occurred (Figure 7a) close to the edges in the first case (also the inner one between the sub-domains). They were higher between boundary nodes. Such errors were not observed in the second case (Figure 7b). For comparison, a full map from Figure 3a was built of 400 nodes spaced 1cm between each other, that is, where the temperature was represented by only 3 nodes, now there were 300 nodes – what means that the resolution was increased 100 times. The result was obtained by increasing the number of boundary elements only 10 times.

4.3. Partly insulated wall

Dirichlet boundary conditions were assigned in the previous examples, however, they are not very realistic in building engineering problems. Therefore, in the present case, the Neuman boundary condition was applied. A 1m long segment of a 25cm thick brick wall ($\lambda = 0.77\frac{\mathrm{W}}{\mathrm{m{\cdot}K}}$) with an insulation layer, 50cm long and 10cm thick ($\lambda = 0.04\frac{\mathrm{W}}{\mathrm{m{\cdot}K}}$) was considered. $T_{\mathrm{out}} = -22°\mathrm{C}$ and heat transfer coefficient $\alpha_{\mathrm{out}} = 23\frac{W}{\mathrm{m}^2\cdot K}$, and $T_{\mathrm{in}} = 20°\mathrm{C}$ and $\alpha_{\mathrm{in}} = 8.1\frac{W}{\mathrm{m}^2\cdot K}$ were assumed on the outer and inner sides, respectively.

The numerically obtained heat flux value (for 1m of wall height) was $\dot{Q} = 57.718\,\mathrm{W}$. Simple analytical calculations give a smaller incorrect value: $q_1 = \frac{42}{0.12+0.325+0.04} = 86.6\frac{\mathrm{W}}{\mathrm{m}^2}$ and $q_2 = \frac{42}{0.12+0.325+2.5+0.04} = 14.07\frac{\mathrm{W}}{\mathrm{m}^2}$ therefore

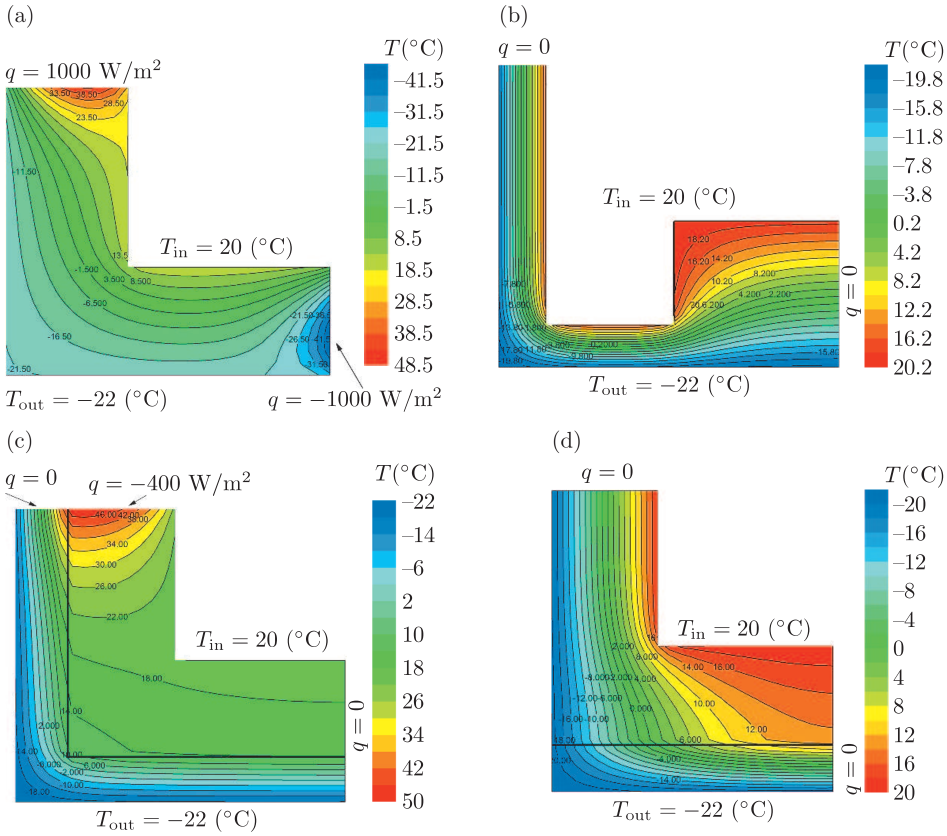

Figure 5. Temperature field of a corner of a: (a) brick wall ($\lambda = 0.77\frac{\mathrm{W}}{\mathrm{m \cdot K}}$) without insulation where one edge is being cooled and another heated with $q = 1000\frac{W}{\mathrm{m}^2}$; (b) uniform wall where domain no. 1 was "removed"; (c) brick wall with styrofoam insulation and the upper inner edge being heated with $q = 400\frac{W}{\mathrm{m}^2}$; (d) brick wall where only one side was insulated

$\dot{Q}_c = 0.5\mathrm{m} \cdot 86.6\frac{\mathrm{W}}{\mathrm{m}^2} \cdot 1\mathrm{m} + 0.5\mathrm{m} \cdot 14.07\frac{\mathrm{W}}{\mathrm{m}^2} \cdot 1\mathrm{m} = 50.335\,\mathrm{W}$. The difference between the values is $\Delta\dot{Q} = 7.383\,\mathrm{W}$. In Figure 8 a deviation from parallel isotherms is clearly noticed, what proves that one-dimensional heat transfer assumption is not applicable. In such cases a two dimensional thermal bridge should be considered and numerical calculations are of much help.

5. Conclusions

The presented examples of heat transfer in building construction elements well reflect the considered problem. The results obtained with the boundary element method correspond to the results obtained with other methods and are not characterized by a significant error. A clear advantage of the method is that, for a flat area, it is only an edge that is discretized and what comes after, there are fewer variables in the system of equations. The final solution in the form of a temperature field in the internal domain is determined only by the boundary values of temperature and heat flux density. Thus, the point

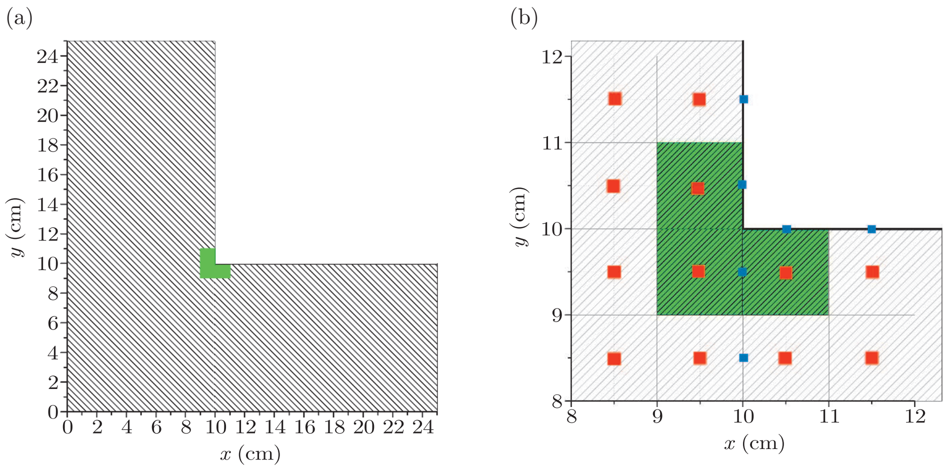

Figure 6. (a) Selected fragment of the corner, (b) selected fragment zoomed, where red squares indicate interior nodes and blue ones boundary nodes used to estimate the field temperature of the whole corner. The shaded green squares indicate a domain of 300 inner nodes (100 for each square) describing the field temperature of a chosen detail what corresponds to three red nodes

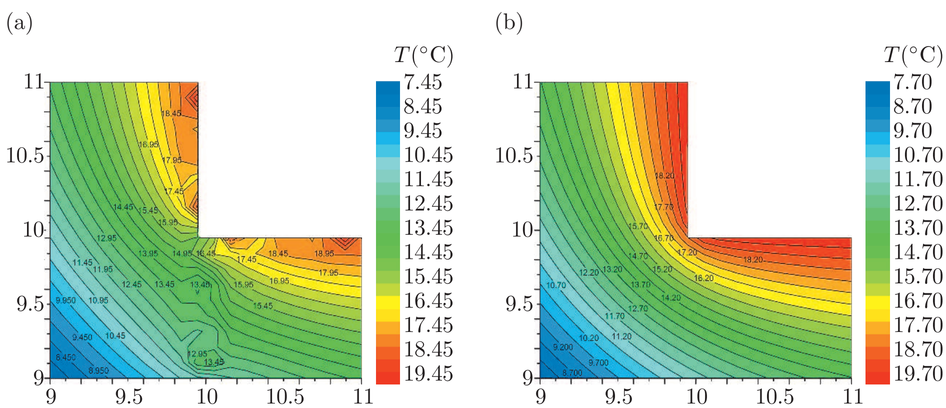

Figure 7. Results of field temperature for a small, 1 cm wide, domain of a corner of 10 cm brick wall without insulation. Calculations were carried out without (a) and with (b) increased number of boundary elements

coordinates and their quantity can be chosen freely. It means that if knowledge of the temperature field in a particular region in the inner domain is needed with a higher resolution, this can be easily done without any interference with the solving equation. However, when approaching the edge closer than the size of a constant boundary element, the error dramatically increases. Nevertheless, this can be expiated by an increasing number of boundary elements and solving the equation once more without considering all the inner domains which are not of interest.

On the other hand, it should be noted that the mathematical theory of the boundary element method is much more complicated than other competing mesh

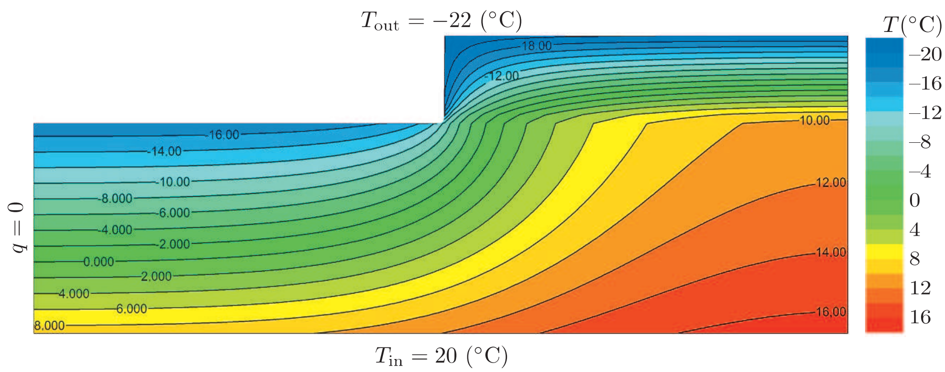

Figure 8. Temperature field for partly insulated wall

methods, hence, more effort and time will be required to familiarize with it. In the case of the present work, assigning coordinates to the corresponding points in order to determine geometries of new shapes was a laborious task. Therefore, some kind of a preprocessor would be much of help.

Moreover, it would be worth checking out the functionality of the method in the case of domains with more complex geometries than presented here. A constant boundary element was used, but other types (linear and parabolic) are also available and give more possibilities [4]. However, in the case of a constant boundary element there is no ambiguity in corners or places of contact with different boundary conditions, as values are assigned to points allocated in the centre of the constant boundary element and not on the edge.

In the case of domains with simple geometries the use of traditional mesh methods would be undoubtedly simpler. Nevertheless, the boundary element method is worth of interest and it may bring excellent results where mesh methods encounter difficulties.

Acknowledgements

The paper presents some of the results obtained in my Master Thesis supervised by Dr Sebastian Bielski to whom I would like to give sincere thanks for his support. I also wish to thank Dr hab. inż. Jacek Pozorski for his encouragement to write this article.

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