Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Independence in uniform linear triangle-free hypergraphs



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ARTICLE INFO

Article history: Received 2 October 2014 Accepted 5 January 2016 Available online 1 February 2016

Keywords: Independence Hypergraph Linear Uniform Double linear Triangle-free

ABSTRACT

The independence number $\alpha(H)$ of a hypergraph H is the maximum cardinality of a set of vertices of H that does not contain an edge of H. Generalizing Shearer's classical lower bound on the independence number of triangle-free graphs Shearer (1991), and considerably improving recent results of Li and Zang (2006) and Chishti et al. (2014), we show that

$$\alpha(H) \ge \sum_{u \in V(H)} f_r(d_H(u))$$

for an *r*-uniform linear triangle-free hypergraph *H* with $r \ge 2$, where

$$f_r(0) = 1$$
, and
 $f_r(d) = \frac{1 + ((r-1)d^2 - d)f_r(d-1)}{1 + (r-1)d^2}$ for $d \ge 1$.

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1. Introduction

We consider finite hypergraphs H, which are ordered pairs (V(H), E(H)) of two sets, where V(H) is the finite set of vertices of H and E(H) is the set of edges of H, which are subsets of V(H). The order n(H) of H is the cardinality of V(H). The degree $d_H(u)$ of a vertex u of H is the number of edges of H that contain u. The average degree d(H) of H is the arithmetic mean of the degrees of its vertices. Two distinct vertices of H are adjacent or neighbors if some edge of H contains both. The neighborhood $N_H(u)$ of a vertex u of H is the set of vertices of H that are adjacent to u. For a set X of vertices of H, the hypergraph H - Xarises from H by removing from V(H) all vertices in X and removing from E(H) all edges that intersect X. If every two distinct edges of H share at most one vertex, then H is linear. If H is linear and for every two distinct non-adjacent vertices u and v of H, every edge of H that contains u contains at most one neighbor of v, then H is double linear. If there are not three distinct vertices u_1, u_2 , and u_3 of H and three distinct edges e_1, e_2 , and e_3 of H such that $\{u_1, u_2, u_3\} \setminus \{u_i\} \subseteq e_i$ for $i \in \{1, 2, 3\}$, then H is triangle-free. A set I of vertices of H is a (weak) independent set of H if no edge of H is contained in I. The (weak) independence number $\alpha(H)$ of H is the maximum cardinality of an independent set of H. If all edges of H have cardinality r, then H is r-uniform. If H is 2-uniform, then H is referred to as a graph.

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http://dx.doi.org/10.1016/j.disc.2016.01.006 0012-365X/© 2016 Elsevier B.V. All rights reserved.





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The independence number of (hyper)graphs is a well studied computationally hard parameter. Caro [4] and Wei [14] proved a classical lower bound on the independence number of graphs, which was extended to hypergraphs by Caro and Tuza [5]. Specifically, for an *r*-uniform hypergraph *H*, Caro and Tuza [5] proved

$$\alpha(H) \geq \sum_{u \in V(H)} f_{CT(r)}(d_H(u)),$$

where $f_{CT(r)}(d) = {\binom{d+\frac{1}{d}}{d}}^{-1}$. Thiele [13] generalized Caro and Tuza's bound to general hypergraphs; see [3] for a very simple probabilistic proof of Thiele's bound. Originally motivated by Ramsey theory, Ajtai et al. [2] showed that $\alpha(G) = \Omega\left(\frac{\ln d(G)}{d(G)}n(G)\right)$ for every triangle-free graph *G*. Confirming a conjecture from [2] concerning the implicit constant, Shearer [11] improved this bound to $\alpha(H) \ge f_{S_1}(d(G))n(G)$, where $f_{S_1}(d) = \frac{d \ln d - d + 1}{(d-1)^2}$. In [11] the function f_{S_1} arises as a solution of the differential equation

$$(d+1)f(d) = 1 + (d-d^2)f'(d)$$
 and $f(0) = 1$.

In [12] Shearer showed that

$$\alpha(G) \geq \sum_{u \in V(G)} f_{S_2}(d_G(u))$$

for every triangle-free graph G, where f_{S_2} solves the difference equation

$$(d+1)f(d) = 1 + (d-d^2)(f(d) - f(d-1))$$
 and $f(0) = 1$

Since $f_{S_1}(d) \le f_{S_2}(d)$ for every non-negative integer d, and f_{S_1} is convex, Shearer's bound from [12] is stronger than his bound from [11].

Li and Zang [10] adapted Shearer's approach to hypergraphs and obtained the following.

Theorem 1 (*Li* and Zang [10]). Let r and m be positive integers with $r \ge 2$.

If H is an r-uniform double linear hypergraph such that the maximum degree of every subhypergraph of H induced by the neighborhood of a vertex of H is less than m, then

$$\alpha(H) \geq \sum_{u \in V(H)} f_{LZ(r,m)}(d_H(u)),$$

where

$$f_{LZ(r,m)}(x) = \frac{m}{B} \int_0^1 \frac{(1-t)^{\frac{m}{m}}}{t^b (m-(x-m)t)} dt,$$

$$a = \frac{1}{(r-1)^2}, b = \frac{r-2}{r-1}, and B = \int_0^1 (1-t)^{(\frac{m}{m}-1)} t^{-b} dt.$$

Note that for $r \ge 2$, an *r*-uniform linear hypergraph *H* is triangle-free if and only if it is double linear and the maximum degree of every subhypergraph of *H* induced by the neighborhood of a vertex of *H* is less than 1. Therefore, since $f_{S_1} = f_{LZ(2,1)}$ and f_{S_1} is convex, Theorem 1 implies Shearer's bound from [11]. Nevertheless, since $f_{S_1}(d) < f_{S_2}(d)$ for every integer *d* with $d \ge 2$, Shearer's bound from [12] does not quite follow from Theorem 1.

In [6] Chishti et al. presented another version of Shearer's bound from [11] for hypergraphs.

Theorem 2 (Chishti et al. [6]). Let r be an integer with $r \ge 2$. If H is an r-uniform linear triangle-free hypergraph, then

$$\alpha(H) \ge f_{CZPI(r)}(d(H))n(H)$$

where

$$f_{CZPI(r)}(x) = \frac{1}{r-1} \int_0^1 \frac{1-t}{t^b (1-((r-1)x-1)t)} dt$$

and $b = \frac{r-2}{r-1}$.

Since $f_{S_1} = f_{CZPI(2)}$, for r = 2, the last result coincides with Shearer's bound from [11].

A drawback of the bounds in Theorems 1 and 2 is that they are very often weaker than Caro and Tuza's bound [5], which holds for a more general class of hypergraphs. See Fig. 1 for an illustration.



Fig. 1. The values of $f_{LZ(r, 1)}(d)$ (line), $f_{CZPI(r)}(d)$ (dashed line), $f_{CT(r)}(d)$ (empty circles), and $f_r(d)$ (solid circles) for $0 \le d \le 40$ and r = 3 (left) and r = 4 (right).

In the present paper we extend Shearer's approach from [12] and establish a lower bound on the independence number of a uniform linear triangle-free hypergraph that considerably improves Theorems 1 and 2 and is systematically better than Caro and Tuza's bound.

For further related results we refer to Ajtai et al. [1], Duke et al. [7], Dutta et al. [8] and Kostochka et al. [9]. Note that our main result provides explicit values when applied to a specific hypergraph but that we do not completely understand its asymptotics. In contrast to that, results as in [1,7,8] are essentially asymptotic statements but are of limited value when applied to a specific hypergraph.

2. Results

For an integer *r* with $r \ge 2$, let $f_r : \mathbb{N}_0 \to \mathbb{R}_0$ be such that

$$f_r(0) = 1$$
 and
 $f_r(d) = \frac{1 + ((r-1)d^2 - d)f_r(d-1)}{1 + (r-1)d^2}$

for every positive integer *d*.

Lemma 3. If r and d are integers with $r \ge 2$ and $d \ge 0$, then $f_r(d) - f_r(d+1) \ge f_r(d+1) - f_r(d+2)$. **Proof.** Substituting within the inequality $f_r(d) - 2f_r(d+1) + f_r(d+2) \ge 0$ first $f_r(d+2)$ with

$$\frac{1 + \left((r-1)(d+2)^2 - (d+2) \right) f_r(d+1)}{1 + \left((r-1)(d+2)^2 - (d+2) \right) f_r(d+1)}$$

 $1 + (r - 1)(d + 2)^2$

and then $f_r(d + 1)$ with

$$\frac{1 + \left((r-1)(d+1)^2 - (d+1) \right) f_r(d)}{1 + (r-1)(d+1)^2},$$

and solving it for $f_r(d)$, it is straightforward but tedious to verify that it is equivalent to $f_r(d) \ge L(r, d)$ where

$$L(r, d) = \frac{(2r - 1)d + 3r}{r(d^2 + 5d + 5)}.$$

Therefore, in order to complete the proof, it suffices to show $f_r(d) \ge L(r, d)$. For d = 0, we have $f_r(0) = 1 > \frac{3}{5} = L(r, 0)$. Now, let $f(d) \ge L(r, d)$ for some non-negative integer d. Since $(r - 1)(d + 1)^2 - (d + 1) \ge 0$, we obtain by a straightforward yet tedious calculation

$$f(d+1) - L(r, d+1) = \frac{1 + ((r-1)(d+1)^2 - (d+1))f(d)}{1 + (r-1)(d+1)^2} - L(r, d+1)$$

$$\geq \frac{(1 + ((r-1)(d+1)^2 - (d+1))L(r, d)}{1 + (r-1)(d+1)^2} - L(r, d+1)$$
$$= \frac{2(1 + (r-1)(d+2)^2)}{r(d^2 + 7d + 11)(d^2 + 5d + 5)},$$

which is positive for $r \ge 2$. Therefore, $f(d + 1) \ge L(r, d + 1)$, which completes the proof by an inductive argument.

The following is our main result.

Theorem 4. *Let* r *be an integer with* $r \ge 2$ *.*

If H is an r-uniform linear triangle-free hypergraph, then

$$\alpha(H) \geq \sum_{u \in V(H)} f_r(d_H(u)).$$

Before we proceed to the proof, we compare our bound to the bounds of Caro and Tuza [5], Li and Zang [10], and Chishti et al. [6]. Fig. 1 illustrates some specific values. An inspection of Li and Zang's proof in [10] reveals that they actually prove a lower bound on the so-called *strong independence number*, which is defined as the maximum cardinality of a set of vertices that does not contain two adjacent vertices. Therefore, especially for large values of *r*, Theorem 1 is much weaker than Theorem 2. In fact, it is quite natural that it is worse by a factor of about r - 1.

As we show now, our bound is systematically better than Caro and Tuza's bound [5].

Lemma 5. If *r* and *d* are integers with $r \ge 3$ and $d \ge 2$, then $f_r(d) > f_{CT(r)}(d)$.

Proof. Note that $f_r(0) = f_{CT(r)}(0) = 1$, $f_r(1) = f_{CT(r)}(1) = \frac{r-1}{r}$, and $f_{CT(r)}(d) = \frac{d}{d + \frac{1}{r-1}} f_{CT(r)}(d-1)$ for $d \in \mathbb{N}$, which immediately implies that $f_{CT(r)}(d) < \frac{r-1}{r}$ for $d \ge 2$. Now, if $f_r(d-1) \ge f_{CT(r)}(d-1)$ for some $d \ge 2$, then

$$\begin{split} f_r(d) - f_{CT(r)}(d) &= \frac{1 + \left((r-1)d^2 - d \right) f_r(d-1)}{1 + (r-1)d^2} - f_{CT(r)}(d) \\ &\geq \frac{1 + \left((r-1)d^2 - d \right) f_{CT(r)}(d-1)}{1 + (r-1)d^2} - f_{CT(r)}(d) \\ &= \frac{1 + \left((r-1)d^2 - d \right) \frac{1 + (r-1)d}{(r-1)d} f_{CT(r)}(d)}{1 + (r-1)d^2} - f_{CT(r)}(d) \\ &= \frac{1 - \frac{r}{r-1} f_{CT(r)}(d)}{1 + (r-1)d^2} \\ &> 0, \end{split}$$

that is, $f_r(d) > f_{CT(r)}(d)$, which completes the proof by an inductive argument. \Box

For r = 2, Lemma 5 would state that Shearer's bound [12] is better than Caro [4] and Wei's bound [14], which is known. We proceed to the proof of Theorem 4.

Proof of Theorem 4. We prove the statement by induction on n(H). If H has no edge, then $\alpha(H) = n(H)$, which implies the desired result for $n(H) \le r - 1$. Now let $n(H) \ge r$. If H has a vertex x with $d_H(x) = 0$, then $f_r(d_H(x)) = 1$ and, by induction,

$$\alpha(H) \geq 1 + \alpha(H-x) \geq f_r(d_H(x)) + \sum_{u \in V(H) \setminus \{x\}} f_r(d_{H-x}(u)) = \sum_{u \in V(H)} f_r(d_H(u)).$$

Hence we may assume that *H* has no vertex of degree 0.

Since *H* is *r*-uniform and linear, for every two edges e_1 and e_2 with $e_1 \cap e_2 = \{u\}$ for some vertex *u* of *H*, the sets $e_1 \setminus \{u\}$ and $e_2 \setminus \{u\}$ are disjoint and of order r - 1. Therefore, for every vertex *u* of *H*, there is a set $\mathcal{R}(u)$ of r - 1 sets of neighbors of *u* such that every neighbor of *u* belongs to exactly one of the sets in $\mathcal{R}(u)$, and $|e \cap R| = 1$ for every edge *e* of *H* with $u \in e$ and every $R \in \mathcal{R}(u)$.

If *x* is a vertex of *H* and $R \in \mathcal{R}(x)$ is such that

$$1+\sum_{u\in V(H)\setminus(\{x\}\cup R)}f_r(d_{H-(\{x\}\cup R)}(u))\geq \sum_{u\in V(H)}f_r(d_H(u)),$$

then the statement follows by induction, because $\alpha(H) \ge 1 + \alpha(H - (\{x\} \cup R))$. Therefore, in order to complete the proof, it suffices to show that the following term is non-negative:

`

$$P = \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \left(1 + \sum_{u \in V(H) \setminus \{x\} \cup R} f_r(d_{H-(\{x\} \cup R)}(u)) - \sum_{u \in V(H)} f_r(d_H(u)) \right).$$

Since *H* is linear and triangle-free, we have $d_{H-(\{x\}\cup R)}(z) = d_H(z) - |N_H(z) \cap R|$ for every vertex *z* in $V(H) \setminus (\{x\}\cup R)$. Trivially, $d_{H-(\{x\}\cup R)}(z) = d_H(z)$ for $z \notin N_H(R)$, and hence *P* equals $P_1 + P_2$, where

$$P_1 = \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \left(1 - f_r(d_H(x)) - \sum_{y \in R} f_r(d_H(y)) \right) \text{ and}$$
$$P_2 = \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \sum_{z \in N_H(R) \setminus \{x\}} \left(f_r(d_H(z) - |N_H(z) \cap R|) - f_r(d_H(z)) \right).$$

Since for every vertex *u* of *H*, there are exactly $(r - 1)d_H(u)$ many vertices *v* of *H* such that *u* belongs to exactly one of the sets in $\Re(v)$, we have

$$P_1 = \sum_{x \in V(H)} \Big((r-1) - (r-1)(d_H(x) + 1)f_r(d_H(x)) \Big).$$

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Since $f_r(d-1) - f_r(d)$ is decreasing by Lemma 3, we have $f_r(d-n) - f_r(d) \ge n(f_r(d-1) - f_r(d))$ for all positive integers d and n with n < d. Therefore,

$$\begin{split} P_{2} &\geq \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \sum_{z \in N_{H}(R) \setminus \{x\}} |N_{H}(z) \cap R| \Big(f_{r}(d_{H}(z) - 1) - f_{r}(d_{H}(z)) \Big) \\ &= \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \sum_{z \in N_{H}(R) \setminus \{x\}} \sum_{y \in R} |N_{H}(z) \cap \{y\}| \Big(f_{r}(d_{H}(z) - 1) - f_{r}(d_{H}(z)) \Big) \\ &= \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \sum_{y \in R} \sum_{z \in N_{H}(R) \setminus \{x\}} |N_{H}(z) \cap \{y\}| \Big(f_{r}(d_{H}(z) - 1) - f_{r}(d_{H}(z)) \Big) \\ &= \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \sum_{y \in R} \sum_{z \in N_{H}(y) \setminus \{x\}} \Big(f_{r}(d_{H}(z) - 1) - f_{r}(d_{H}(z)) \Big). \end{split}$$

Let *T* be the set of all 4-tuples (x, R, y, z) with $x \in V(H)$, $R \in \mathcal{R}(x)$, $y \in R$, and $z \in N_H(y) \setminus \{x\}$. Note that $y \in N_H(z)$ for every (x, R, y, z) in *T*. Since *H* is linear, for a given vertex *z* of *H* and a given neighbor *y* of *z*, there are $(r - 1)d_H(y) - 1$ many vertices *x* of *H* with $y \in R$ for some *R* in $\mathcal{R}(x)$ and $z \in N_H(y) \setminus \{x\}$. Furthermore, by the properties of $\mathcal{R}(x)$, given *x* and *y*, the set *R* in $\mathcal{R}(x)$ with $y \in R$ is unique. Therefore,

$$P_{2} \geq \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \sum_{y \in R} \sum_{z \in N_{H}(y) \setminus \{x\}} \left(f_{r}(d_{H}(z) - 1) - f_{r}(d_{H}(z)) \right)$$
$$= \sum_{z \in V(H)} \sum_{y \in N_{H}(z)} \left((r - 1)d_{H}(y) - 1 \right) \left(f_{r}(d_{H}(z) - 1) - f_{r}(d_{H}(z)) \right).$$

Let \mathcal{E} be the edge set of the graph that arises from H by replacing every edge of H by a clique, that is, \mathcal{E} is the set of all sets containing exactly two adjacent vertices of H.

We obtain

$$P_{2} \geq \sum_{z \in V(H)} \sum_{y \in N_{H}(z)} ((r-1)d_{H}(y) - 1) (f_{r}(d_{H}(z) - 1) - f_{r}(d_{H}(z)))$$

=
$$\sum_{\{y,z\} \in \mathcal{E}} (h_{1}(y)h_{2}(z) + h_{1}(z)h_{2}(y)), \text{ where}$$

$$h_{1}(x) = (r-1)d_{H}(x) - 1 \text{ and}$$

$$h_{2}(x) = f_{r}(d_{H}(x) - 1) - f_{r}(d_{H}(x)).$$

If $d_H(y) \ge d_H(z)$, then $h_1(y) \ge h_1(z)$ and, by Lemma 3, $h_2(z) \ge h_2(y)$, which implies

$$(h_1(y) - h_1(z))(h_2(z) - h_2(y)) \ge 0.$$

Therefore, $h_1(y)h_2(z) + h_1(z)h_2(y) \ge h_1(y)h_2(y) + h_1(z)h_2(z)$.

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Since, for every vertex y of H, there are exactly $(r - 1)d_H(y)$ many vertices z of H with $\{y, z\} \in \mathcal{E}$, we obtain

$$P_{2} \geq \sum_{\{y,z\}\in\mathscr{E}} \left(h_{1}(y)h_{2}(z) + h_{1}(z)h_{2}(y) \right)$$

$$\geq \sum_{\{y,z\}\in\mathscr{E}} \left(h_{1}(y)h_{2}(y) + h_{1}(z)h_{2}(z) \right)$$

$$= \sum_{x\in V(H)} (r-1)d_{H}(x)h_{1}(x)h_{2}(x)$$

$$= \sum_{x\in V(H)} (r-1)d_{H}(x) \left((r-1)d_{H}(x) - 1 \right) \left(f_{r}(d_{H}(x) - 1) - f_{r}(d_{H}(x)) \right).$$

Combining these estimates, we see that

$$P = P_1 + P_2$$

$$\geq \sum_{x \in V(H)} \left((r-1) - (r-1)(d_H(x) + 1)f_r(d_H(x)) + (r-1)d_H(x) \left((r-1)d_H(x) - 1 \right) \left(f_r(d_H(x) - 1) - f_r(d_H(x)) \right) \right),$$

which is 0 by the definition of f_r . This completes the proof. \Box

It seems a challenging task to extend the presented results to non-uniform and/or non-linear triangle-free hypergraphs.

Acknowledgment

The first author has been partially supported by National Science Centre under contract DEC-2011/02/A/ST6/00201.

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