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ORIGINAL PAPER

Independent Domination Subdivision in Graphs

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Abstract

A set S of vertices in a graph G is a dominating set if every vertex not in S is ad jacent to a vertex in S. If, in addition, S is an independent set, then S is an independent dominating set. The independent domination number $i(G)$ of G is the minimum cardinality of an independent dominating set in G. The independent domination subdivision number $\mathrm{sd_i}(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the independent domination number. We show that for every connected graph G on at least three vertices, the parameter $\mathrm{sd_i}(G)$ is well defined and differs significantly from the well-studied domination subdivision number $\mathrm{sd}_\gamma(G)$. For example, if G is a block graph, then $\mathrm{sd}_\gamma(G) \leq 3$, while $\mathrm{sd_i}(G)$ can be arbitrary large. Further we show that there exist connected graph G with arbitrarily large maximum degree $\Delta(G)$ such that $\mathrm{sd_i}(G) \geq 3\Delta(G) - 2$, in contrast to the known result that $\mathrm{sd}_\gamma(G) \leq 2\Delta(G) - 1$ always holds. Among other results, we present a simple characterization of trees T with $\mathrm{sd_i}(T) = 1$.

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1 Introduction

Domination subdivision in graphs is very well studied in the literature. In this paper, we study independent domination subdivisions in graphs. A set S of vertices in a graph G is a *dominating set* if every vertex not in S is adjacent to a vertex in S. If, in addition, S is an independent set, then S is an *independent dominating set*, abbreviated ID-set, of G. The *domination number*, denoted $\gamma(G)$, of G is the minimum cardinality of a dominating set of G, and the *independent domination number*, denoted $i(G)$, of G is the minimum cardinality of an ID-set in G. An independent set of vertices in a graph G is a dominating set of G if and only if it is a maximal independent set. Thus, $i(G)$ is equivalently the minimum cardinality of a maximal independent set of vertices in G. An ID-set of cardinality $i(G)$ is called an *i-set of G*. A survey on independent domination in graphs can be found in [6].

An edge uw in a graph G is *subdivided* by deleting the edge uw, and adding a new vertex v and two new edges uv and vw. The added vertex v of degree 2 is called a *subdivision vertex*. The *domination subdivision number* of a connected graph G of order at least 3, denoted $\mathrm{sd}_\gamma(G)$, is the minimum number of edges that must be subdivided (where no edge in G can be subdivided more than once) in order to create a graph whose domination number exceeds that of G. The domination subdivision number of a graph G was first formally defined in the 2000 paper by Haynes, Hedetniemi and Hedetniemi [7], although the concept is attributed by the authors in [7] to Arumugan, and studied further, for example, in [1, 4, 8, 12].

The *independent domination subdivision number* of a connected graph G of order at least 3, denoted $\mathrm{sd}_\mathrm{i}(G)$, is the minimum number of edges that must be subdivided (where no edge in G can be subdivided more than once) in order to create a graph whose independent domination number exceeds that of G. This parameter was introduced by Sharada and Soner at an international conference on discrete mathematics in India in 2008, and subsequently studied by them in their 2010 paper [11] in which they characterize the trees T satisfying $\mathrm{sd}_\mathrm{i}(T) = 3$.

Although the independent domination subdivision number differs significantly from the well-studied domination subdivision number, it has not received as much attention in the literature. In this paper, we provide a more in-depth study of the independent domination subdivision number of a graph. We proceed as follows. In Sect. 1.1 we provide the necessary graph theory notation and terminology. Our main results are presented in Sect. 2, and selected known results are given in Sect. 3. In Sect. 4, we present a general upper bound on the independent domination subdivision number. In Sects. 5 and 6 we study the independent domination subdivision number of special classes of graphs such as complete bipartite graphs and block graphs. In Sect. 7, we characterize trees T with $\mathrm{sd}_\mathrm{i}(T) = 1$.

1.1 Notation

For notation and graph theory terminology we generally follow [9]. Specifically, let G be a graph with vertex set $V(G)$ of order $n(G) = |V(G)|$ and edge set $E(G)$ of size $m(G) = |E(G)|$. If G is clear from the context, we simply write V and E rather

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than $V(G)$ and $E(G)$. A *neighbor* of a vertex v in G is a vertex adjacent to v. The *open neighborhood* of a vertex v in G, denoted $N_G(v)$, is the set of all neighbors of v in G, while the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. If the graph G is clear from the context, we write $N(v)$ and $N[v]$ rather than $N_G(v)$ and $N_G[v]$, respectively. We denote the *degree* of a vertex v in G by $d_G(v) = |N_G(v)|$. A vertex of degree 1 in G is a *leaf* of G, and a vertex with a leaf neighbor is a *support vertex* of G. A *strong support vertex* of G is a support vertex with at least two leaf neighbors. The minimum and maximum degrees among all vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

For a set $S \subset V(G)$ of vertices in a graph G and a vertex $v \in S$, we define the S-*external private neighborhood* of a vertex v, abbreviated $\mathrm{epn}_G(v, S)$, as the set of all vertices in $V(G)\backslash S$ that are adjacent to v but to no other vertex of S; that is, $\mathrm{epn}_G(v, S) = \{w \in V(G)\backslash S \mid N_G(w) \cap S = \{v\}\}$. We define an *S-external private neighbor* of v to be a vertex in $\mathrm{epn}_G(v, S)$.

The *distance* between two vertices u and v in a connected graph G, denoted $d_G(u, v)$ or simply $d(u, v)$ if the graph G is clear from context, is the minimum length of a (u, v)-path in G. The *diameter* of G is the maximum distance among all pairs of vertices of G. The *distance* from a vertex v to the set S in G, denoted by $d_G(v, S)$, is the minimum distance from v to a vertex of S; that is, $d_G(v, S) = \min\{d(u, v) \mid u \in S\}$.

We denote the path and cycle on n vertices by P_n and C_n, respectively. A *complete graph* on n vertices is denoted by K_n, while a *complete bipartite graph* with partite sets of size n and m is denoted by $K_{n,m}$. A *star* is the graph $K_{1,k}$, where $k \geq 1$. A *double star* is a tree with exactly two (adjacent) non-leaf vertices. Further if one of these vertices is adjacent to ℓ_1 leaves and the other to ℓ_2 leaves, then we denote the double star by $S(\ell_1, \ell_2)$. For example, the double star $S(1, 1)$ is the path P_4. A *block graph* is a graph in which every block is a complete graph. In particular, every tree is a block graph.

A *rooted tree* T distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. The set of children of v is denoted by $C(v)$. A *descendant* of v is a vertex $u \neq v$ such that the unique (r, u)-path contains v. In particular, every child of v is a descendant of v. A *grandchild* of v is the descendant of v at distance 2 from v. We let $D(v)$ denote the set of descendants of v, and we define $D[v] = D(v) \cup \{v\}$.

We use the standard notation $[k] = \{1, \ldots, k\}$.

2 Main Result

In this paper, we continue the study of the independent domination subdivision number of a graph. We first establish a general upper bound on the independent domination subdivision number of a connected graph of order at least 3 in terms of its maximum degree and its independent domination number.

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Theorem 2.1 *If G is a connected graph of order at least* 3, *then* $\mathrm{sd_i}(G) \leq \Delta(G) \cdot i(G)$.

As a consequence of Theorem 2.1, the independent domination subdivision number is well defined on the class of connected graphs of order at least 3. We state this formally as follows.

Corollary 2.2 *For every connected graph G of order at least* 3, *the parameter* $\mathrm{sd_i}(G)$ *is well defined.*

We next determine the independent domination subdivision number of a complete regular bipartite graph.

Theorem 2.3 *For* $k \geq 4$, $\mathrm{sd_i}(K_{k,k}) = 3k - 2$.

As a consequence of Theorem 2.3, we remark that $\mathrm{sd_i}(G)$ can be arbitrarily large, even when restricted to the class of connected bipartite graphs G.

Corollary 2.4 *There exist connected graphs G with arbitrarily large maximum degree* $\Delta(G)$ *such that* $\mathrm{sd_i}(G) \geq 3\Delta(G) - 2$.

The following result shows that the independent domination subdivision number can be arbitrary large, even for the classes of block graphs.

Theorem 2.5 *There exist block graphs G such that* $\mathrm{sd_i}(G) = k$ *for any arbitrary positive* (*fixed*) *integer k.*

Our final aim is to give a simple characterization of trees T with $\mathrm{sd_i}(T) = 1$. For this purpose, let $\mathcal{N}(T)$ be the set of vertices in the tree T that belong to no i-set of T; that is,

$$\mathcal{N}(T) = \{v \in V(T) \mid v \text{ is in no i-set of } T\}.$$

Theorem 2.6 *For a tree T of order at least* 3, $\mathrm{sd_i}(T) = 1$ *if and only if T has at least one of the following two properties.*

P_1: *A leaf of T belongs to the set $\mathcal{N}(T)$.*
P_2: *An edge of T has both its ends in $\mathcal{N}(T)$.*

3 Known Results

In 2000 Haynes, Hedetniemi, and Hedetniemi [7] established the following upper bound on the domination subdivision number of a graph.

Theorem 3.1 [7] *Let G be a connected graph of order $n \geq 3$. If u and v are two arbitrary adjacent vertices each having degree at least* 2, *then* $\mathrm{sd}_\gamma(G) \leq d_G(u) + d_G(v) - 1$.

As a consequence of Theorem 3.1, the domination subdivision number $\mathrm{sd}_\gamma(G)$ is defined for every connected graph G of order $n \geq 3$. As a further consequence of

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Theorem 3.1, we have the following upper bound on the domination subdivision number of a graph.

Corollary 3.2 [7] *If G is a connected graph of order $n \geq 3$, then* $\mathrm{sd}_\gamma(G) \leq \delta(G) + \Delta(G) - 1$.

In 2008 Favaron et al. [5] gave a construction showing that the domination subdivision number can be made arbitrarily large.

Theorem 3.3 [5] *For each pair of positive integers r and q such that $r + q \geq 4$, there exists a graph G with $\delta(G) = r$ and $\mathrm{sd}_\gamma(G) \geq r + q$.*

Dettlaff [3] proved in her PhD thesis that the domination subdivision number of a block graph G is at most 3.

Theorem 3.4 [3] *If G is a block graph, then $1 \leq \mathrm{sd}_\gamma(G) \leq 3$.*

As an immediate consequence of Theorem 3.4, we have that if T is a tree, then $1 \leq \mathrm{sd}_\gamma(T) \leq 3$. Recall that P_n denotes a path on n vertices. For $n \geq 3$, we note that $i(P_n) = \lceil n/3 \rceil$, implying the following result first observed by Sharada and Soner [11].

Observation 3.5 [11] *For $n \geq 3$,*

$$\mathrm{sd_i}(P_n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 3 \\ 3 & \text{if } n \equiv 1 \pmod 3 \\ 2 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

4 A General Upper Bound

We note that if $G = K_2$, then the independent domination number of G does not change when its only edge is subdivided. Further we note that it possible that the subdivision of an edge in a connected graph may decrease its independent domination number. For example, if $k \geq 2$ and G is a tree with exactly two non-leaf vertices both of which have k leaf neighbors, that is, if G is a double star $S(k, k)$, then $i(G) = k + 1$ but subdividing the edge joining the two non-leaf vertices produces a graph G^* with $i(G^*) = 2 < i(G)$.

However we show that the independent domination subdivision number is well defined on the class of connected graphs of order at least 3. For each such graph, we show there always exists some set of edges that can be subdivided (where each edge can be subdivided at most once) in order to increase its independent domination number.

Theorem 4.1 *If G is a connected graph of order at least 3, then there exists an i-set I of G such that the graph G^* obtained from G by subdividing every edge incident with a vertex of I satisfies $i(G^*) > i(G)$.*

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Proof Let G be a connected graph of order $n \geq 3$. Let I be an i-set of G with the minimum possible number of vertices of degree 1, and let $\bar{I}$ be the complement of the set I, and so $\bar{I} = V(G) \backslash I$. We proceed further with the following claim. □

Claim 1 *At least one vertex of I is not a leaf.*

Proof Suppose, to the contrary, that every vertex of I is a leaf. If two vertices of I have a common neighbor, say v, then the set $(I \backslash N_G(v)) \cup \{v\}$ is an ID-set of G of cardinality less than $|I|$, contradicting the minimality of the set I. Hence, no two vertices of I have a common neighbor. Thus, $|I| = |\bar{I}| = \frac{1}{2}n$ and each vertex outside I has a unique neighbor in I. Let x be an arbitrary vertex of I, and let y be its unique neighbor (in $\bar{I}$). Since G is a connected graph of order at least 3, we note that the vertex y has degree at least 2 in G. Thus the set $(I \backslash \{x\}) \cup \{y\}$ is a i-set of G that contains fewer leaves than the set I, contradicting our choice of the set I. □

By Claim 1, at least one vertex in the independent set I has degree 2 or more in the graph G. Let

$$I_1 = \{v \in I \mid d_G(v) = 1\}$$

and let $I_2 = I \backslash I_1$. By our earlier observations, $|I_2| \geq 1$. Possibly, $I_1 = \emptyset$. Let G^* be the graph obtained from G by subdividing every edge incident with a vertex of I. For each vertex $v \in I$, let

$$N_v^* = N_{G^*}(v)$$

be the set consisting of all neighbors of the vertex v in G^*. Every independent dominating set of G^* must contain at least one vertex in $N_{G^*}[v] = N_v^* \cup \{v\}$ in order to dominate the vertex v for every vertex $v \in I$. Thus, $i(G^*) \geq |I| = i(G)$. We show that $i(G^*) > i(G)$. Suppose, to the contrary, that $i(G^*) = i(G)$. Let I^* be an i-set of G^*, and so I^* is an ID-set of G^* and $|I^*| = i(G^*) = |I|$. In what follows we present a series of claims describing some structural properties of G which culminate in the implication of its non-existence.

Claim 2 $I_2 \subseteq I^*$.

Proof Suppose, to the contrary, that the vertex $v \in I_2$ does not belong to the set I^* for some vertex $v \in I$. By definition of the set I_2, we note that $d_G(v) \geq 2$. Let $v_1, v_2, \ldots, v_k$ be the neighbors of v in the graph G^*. We note that the set $N_v^* = \{v_1, , v_2, \ldots, v_k\}$ of neighbors of v in G^* is an independent set, and each vertex in this set is a subdivided vertex of degree 2.

By supposition, $v \notin I^*$. Renaming vertices if necessary, we may assume that $v_1 \in I^*$ in order to dominate the vertex v. We note that every vertex in I is at distance at least 4 from every other vertex of I in the graph G^*. In particular, for distinct vertices x and y in I, the sets N_x^* and N_y^* are disjoint and there is no edge joining a vertex in N_x^* and a vertex in N_y^*. In order to dominate the vertex v_2, either $v_2 \in I^*$ or the (unique) neighbor of v_2 that belongs to the set $\bar{I}$ belongs to the set I^*. Thus, the set I^* contains at least one vertex from each of the sets $N_{G^*}[w]$ for every

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vertex $w \in I \backslash \{v\}$, and contains at least two additional vertices to dominate the vertices in $N_{G^*}[v]$. Therefore, $|I^*| \geq |I| + 1$, contradicting our supposition that $|I^*| = |I|$. □

By Claim 2, $I_2 \subseteq I^*$.

Claim 3 *The set I_1 dominates the set $\bar{I}$ in the graph G.*

Proof Suppose, to the contrary, that there is a vertex $w \in \bar{I}$ that is adjacent to no vertex of I_1 in the graph G. Thus, no neighbor of a vertex in the set I_1 is adjacent to the vertex w in the graph G^*. Therefore, the vertex w is not dominated by any vertex in I^* that belongs to one of the sets N_x^* for every $x \in I_1$. By Claim 2, we have $I_2 \subseteq I^*$, implying that $I^* \cap N_v^* = \emptyset$ for every vertex $v \in I_2$. Thus, no vertex of I^* belongs to one of the sets N_x^* for every $x \in I_2$. Thus, in order to dominate the vertex w in G^*, the set I^* must contain at least one vertex that belongs to the set $\bar{I}$, implying that $|I^*| \geq |I| + 1$, a contradiction. □

By Claim 3, the set I_1 dominates the set $\bar{I}$ in the graph G. Thus, every vertex in $\bar{I}$ has at least one neighbor in G that belong to the set I_1.

Claim 4 *Every vertex in $\bar{I}$ has at least one neighbor in G that belong to the set I_2.*

Proof Suppose, to the contrary, that there is a vertex $w \in \bar{I}$ that has no neighbor in G that belongs to the set I_2. Let I_w be the set of neighbors of w that belong to the set I, that is, $I_w = I \cap N_G(w)$. By supposition, every neighbor of w in I belongs to the set I_1 noting that I is a dominating set of G. Thus, $I_w \subseteq I_1$ and $|I_w| \geq 1$. Since G is a connected graph of order at least 3, we note that the vertex w has degree at least 2 in G. The set $I_w^* = (I \backslash I_w) \cup \{w\}$ is an ID-set of G, implying by our earlier observations that $i(G) \leq |I_w^*| = |I| - |I_w| + 1 \leq |I| = i(G)$. Hence we must have equality throughout this inequality chain. In particular, this implies that $|I_w| = 1$ and the set I_w^* is an i-set of G. However, the set I_w^* contains fewer vertices of degree 1 that does the set I, contradicting our choice of the set I. □

By Claim 3, every vertex in $\bar{I}$ has at least one neighbor in G that belong to the set I_1. By Claim 4, every vertex in $\bar{I}$ has at least one neighbor in G that belong to the set I_2. Let w be an arbitrary vertex in $\bar{I}$ and let I_w be the set of neighbors of w in G that belong to the set I, that is,

$$I_w = I \cap N_G(w).$$

Thus, $|I_w| = |I_w \cap I_1| + |I_w \cap I_2| \geq 1 + 1 = 2$. The set $I_w^* = (I \backslash I_w) \cup \{w\}$ is an ID-set of G of cardinality $|I_w^*| = |I| - |I_w| + 1 \leq |I| - 2 + 1 < |I|$, contradicting the minimality of the set I. This completes the proof of Theorem 4.1. □

Theorem 2.1 now follows as an immediate consequence of Theorem 4.1.

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5 Complete Bipartite Graphs

In this section, we show that the independent domination subdivision number can be arbitrary large, even for special classes of graphs such as connected bipartite graph. We also show that the independent domination subdivision number differs significantly from the domination subdivision number. The following result determines the independent domination subdivision number of a complete bipartite graph.

Theorem 5.1 *For $k \geq 2$,*

$$\mathrm{sd_i}(K_{k,k}) = \begin{cases} 3k-3 & \text{if } k \in \{2,3\} \\ 3k-2 & \text{if } k \geq 4. \end{cases}$$

Proof For $k \geq 2$, let $G \cong K_{k,k}$ and let X and Y be the partite sets of G. If $k = 2$, then $K_{k,k} = C_4$ and $\mathrm{sd_i}(C_4) = 3 = 3(k-1)$. Hence in what follows, we may assume that $k \geq 3$. We note that $i(G) = k$ and that both X and Y are i-sets of G. We establish first a lower bound on $\mathrm{sd_i}(G)$. □

Claim 5 *If $k = 3$, then $\mathrm{sd_i}(G) \geq 3k-3$, while if $k \geq 4$, then $\mathrm{sd_i}(G) \geq 3k-2$.*

Proof Let E' be a smallest possible set of edges of G that need to be subdivided in order to increase the independent domination number of G. Thus, $E' \subseteq E(G)$ and $\mathrm{sd_i}(G) = |E'|$. We show that $|E'| \geq 3(k-1)$. Let G' be a graph obtained from G by subdividing the edges in the set E'. By our choice of the set E', we note that $i(G') > i(G) = k$.

If the set X is a dominating set of G', then since X is an independent set it is an ID-set of G', implying that $i(G') \leq |X| = i(G)$, a contradiction. Hence, the set X is not a dominating set of G'. Therefore there must exist a vertex $y \in Y$ not dominated by the set X in G'. This implies that every edge incident with y in G is subdivided when constructing G'. Analogously, reversing the roles of X and Y, we have that the set Y is not a dominating set of G' and there must exist a vertex $x \in X$ such that every edge incident with x in G is subdivided when constructing G'. Let E_x and E_y be the edges of G incident with x and y, respectively. Thus, $E_x \cup E_y \subseteq E'$ and every edge of $E_x \cup E_y$ is subdivided when constructing G'. We note that $|E_x \cup E_y| = d_G(x) + d_G(y) - 1 = 2k-1$. Let $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$, where $x = x_1$ and $y = y_1$. Let $Y_1 = Y \backslash \{y_1\}$ and let $X_1 = X \backslash \{x_1\}$.

Claim 5.1 *If no vertex in X_1 dominates the set Y_1 or if no vertex in Y_1 dominates the set X_1, then $\mathrm{sd_i}(G) \geq 3k-2$.*

Proof Suppose that no vertex in X_1 dominates the set Y_1. This implies that at least one edge joining each vertex of X_1 with some vertex of Y_1 is subdivided when constructing the graph G'. Since the set X_1 is an independent set, at least $|X_1|$ such edges are subdivided. Further each such edge incident with a vertex of X_1 and a

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vertex of Y_1 does not belong to the set $E_x \cup E_y$. Hence, $|E'| \geq |E_x \cup E_y| + |X_1| = (2k-1) + (|X|-1) = (2k-1) + (k-1) = 3k-2$. Analogously interchanging the roles of X and Y, if no vertex in Y_1 dominates the set X_1, then $|E'| \geq 3k-2$. The desired result follows noting that $\mathrm{sd_i}(G) = |E'|$. □

By Claim 5.1, we may assume that there exists a vertex $x' \in X_1$ that dominates the set Y_1, for otherwise $\mathrm{sd_i}(G) \geq 3k-2$ and the desired result follows. Analogously interchanging the roles of X and Y, we may assume that there exists a vertex $y' \in Y_1$ that dominates the set X_1. Renaming vertices if necessary, we may assume that $x' = x_2$ and $y' = y_2$. Thus, the only edge incident with x_2 in G that is subdivided when constructing G' is the edge x_2y_1, and the only edge incident with y_2 in G that is subdivided when constructing G' is the edge x_1y_2. We note that the vertex x_2 dominates the set Y_1 in G', and the vertex y_2 dominates the set X_1 in G'.

Let $X_2 = X \backslash \{x_1, x_2\}$ and let $Y_2 = Y \backslash \{y_1, y_2\}$. Let $y_{1,i}$ be the subdivided vertex of G' corresponding to the edge x_iy_1 of G for $i \in [k]$. If the set X_2 dominates the set Y_2 in G', then the set $X_2 \cup \{x_1, y_{1,2}\}$, for example, is an ID-set of G' noting that the set X_2 dominates the vertex y_2, implying that $i(G') \leq |X_2| + |\{x_1, y_{1,2}\}| = |X_2| + 2 = |X| = i(G)$, a contradiction. Hence, the set X_2 does not dominates the set Y_2 in G'. Therefore there must exist a vertex $y'' \in Y_2$ not dominated by the set X_2 in G'. This implies that every edge joining y'' with a vertex of X_2 in G is subdivided when constructing G'. Analogously interchanging the roles of X and Y, the set Y_2 does not dominates the set X_2 in G', implying that there must exist a vertex $x'' \in X_2$ not dominated by the set Y_2 in G'. This implies that every edge joining x'' with a vertex of Y_2 in G is subdivided when constructing G'. Renaming vertices if necessary, we may assume that $x'' = x_3$ and $y'' = y_3$.

Let E_{x_3} be the set of edges of G joining x_3 with vertices of Y_2, and let E_{y_3} be the set of edges of G joining y_3 with vertices of X_2. Thus, $E_{x_3} \cup E_{y_3} \subset E'$ and every edge of $E_{x_3} \cup E_{y_3}$ is subdivided when constructing G'. We note that $|E_{x_3} \cup E_{y_3}| = (k-2) + (k-2) - 1 = 2k-5$. By our earlier observations, $E_x \cup E_y \cup E_{x_3} \cup E_{y_3} \subseteq E'$, and so $\mathrm{sd_i}(G) = |E'| \geq |E_x \cup E_y \cup E_{x_3} \cup E_{y_3}| = |E_x \cup E_y| + |E_{x_3} \cup E_{y_3}| = (2k-1) + (2k-5) = 4k-6$. Thus if $k = 3$, then $\mathrm{sd_i}(G) \geq 4k-6 = 3k-3$, while if $k \geq 4$, then $\mathrm{sd_i}(G) \geq 4k-6 \geq 3k-2$. This completes the proof of Claim 5. □

Claim 6 *If* $k = 3$, *then* $\mathrm{sd_i}(G) \leq 3k-3$.

Proof Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$. Let $x = x_1$ and $y = y_1$, and let E_x and E_y be the set of edges of G incident with x and y, respectively. Let $E^* = E_x \cup E_y \cup \{x_2y_2\}$. Let G^* be the graph obtained from G by subdividing the edges of G that belong to the set E^*. Let $y_{1,i}$ be the subdivided vertex of G^* resulting from subdividing the edge x_iy_1 for $i \in [3]$, and let $x_{1,i}$ be the subdivided vertex of G^* resulting from subdividing the edge x_1y_i for $i \in [3]$. We note that $x_{1,1} = y_{1,1}$. Let w be the subdivided vertex of G^* resulting from subdividing the edge x_2y_2. We show that $i(G^*) > i(G) = 3$. Let I^* be an i-set of G^*.

In order to dominate the vertex $x_{1,1}$, we have that $I^* \cap \{x_1, y_1, x_{1,1}\} \neq \emptyset$. If $x_{1,1} \in I^*$, then in order to dominate the four subdivided vertices incident with x or y

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different from $x_{1,1}$, we need at least four additional vertices in I^*, and so in this case $|I^*| \geq 5$. If $\{x, y\} \subset I^*$, then we need at least two additional vertices in I^* to dominate the four vertices in $\{x_2, x_3, y_2, y_3\}$, and so $|I^*| \geq 4$. Hence, we may assume that exactly one of x and y belong to the set I^*, for otherwise $|I^*| \geq 4$, as desired. By symmetry, we may assume that $x \in I^*$. In order to dominate the vertex y, at least one of $y_{1,2}$ and $y_{1,3}$ belongs to I^*. If $y_{1,2} \in I^*$, then we need at least two additional vertices in I^* to dominate w and y_3, while if $y_{1,3} \in I^*$, then we need at least two additional vertices in I^* to dominate $y_{1,2}$ and y_2. Thus in both cases, $|I^*| \geq 4$. Hence, $i(G^*) = |I^*| \geq 4 > i(G)$, implying that $\mathrm{sd_i}(G) \leq |E^*| = 6 = 3k - 3$. □

Claim 7 *If* $k \geq 4$, *then* $\mathrm{sd_i}(G) \leq 3k - 2$.

Proof Let $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$. Let $x = x_1$ and $y = y_1$, and let E_x and E_y be the set of edges of G incident with x and y, respectively. Let $e_i = x_i y_i$ for $i \in [k]$, and let $E' = \{e_2, \ldots, e_k\}$. Let $E^* = E_x \cup E_y \cup E'$. We note that $|E^*| = |E_x \cup E_y| + |E'| = (2k - 1) + (k - 1) = 3k - 2$. Let G^* be the graph obtained from G by subdividing the edges of G that belong to the set E^*. Let $y_{1,i}$ be the subdivided vertex of G^* resulting from subdividing the edge $x_i y_1$ for $i \in [k]$, and let $x_{1,i}$ be the subdivided vertex of G^* resulting from subdividing the edge $x_1 y_i$ for $i \in [k]$. We note that $x_{1,1} = y_{1,1}$. The set $X \cup \{y_1\}$ is an ID-set of G^* (as is the set $Y \cup \{x_1\}$), and so $i(G^*) \leq |X| + 1 = k + 1 = i(G) + 1$. We show that $i(G^*) = k + 1 = i(G) + 1$.

Let N_x and N_y be the set of subdivided vertices adjacent to x and y, respectively, in G^*, and so $N_x = \{x_{1,i} \mid i \in [k]\}$ and $N_y = \{y_{1,i} \mid i \in [k]\}$. Let w_i be the subdivided vertex in G' corresponding to the edge $x_i y_i$ of G for $i \in [k]$. In particular, we note that $N_x \cap N_y = \{w_1\}$, and so $w_1 = x_{1,1} = y_{1,1}$. Let $Y_1 = Y \backslash \{y_1\}$ and let $X_1 = X \backslash \{x_1\}$. Let I^* be an i-set of G^*. In order to dominate the vertex w_1, we have that $I^* \cap \{x_1, y_1, w_1\} \neq \emptyset$.

Suppose firstly that $I^* \cap \{x, y\} = \emptyset$. In this case, $w_1 \in I^*$. Since $x \notin I^*$ and $y \notin I^*$, in order to dominate the $2(k - 1)$ subdivided vertices in G^* that belong to the set $(N_x \cup N_y) \backslash \{w_1\}$, we need at least $2(k - 1)$ additional vertices in I^*, implying that $|I^*| \geq 1 + 2(k - 1) = 2k - 1 \geq k + 1$.

Suppose secondly that $|I^* \cap \{x, y\}| = 1$, and so $w_1 \notin I^*$. By symmetry, we may assume that $x \in I^*$, and so $y \notin I^*$. In order to dominate $k - 1$ vertices in $N_y \backslash \{w_1\}$ we note that $|I^* \cap \{x_i, y_{1,i}\}| = 1$ for each $i \in [k] \backslash \{1\}$. Further in order to dominate the vertex y, we note that $I^* \cap (N_y \backslash \{w_1\}) \neq \emptyset$. Renaming vertices if necessary, we may assume that the vertex $y_{1,2} \in I^*$, and so $x_2 \notin I^*$. But then at least one additional vertex is needed in I^* in order to dominate the subdivided vertex w_2 in G^*, implying that $|I^*| \geq 1 + (|N_y| - 1) + 1 = |N_y| + 1 = k + 1$.

Suppose thirdly that $\{x, y\} \subset I^*$. In this case, we need at least $k - 1$ additional vertices in I^* to dominate the $k - 1$ vertices $w_2, w_3, \ldots, w_k$. Therefore, $|I^*| \geq 2 + (k - 1) = k + 1$. In all cases, we have shown that $i(G^*) = |I^*| \geq k + 1 > i(G)$, implying that $\mathrm{sd_i}(G) \leq |E^*| = 3k - 2$. □

The desired result of Theorem 5.1 now follows immediately from Claims 5, 6, and 7 . □

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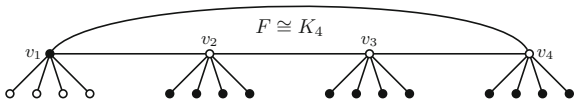

Fig. 1 The graph G_4

Theorem 2.3 is a special case of Theorem 5.1 when $k \geq 4$. By Theorem 2.3, we remark that the result of Theorem 3.1 for the domination subdivision number does not hold for the independent domination subdivision number. Thus, the independent domination subdivision number differs significantly from the domination subdivision number.

6 Block Graphs

We consider next the class of block graphs, where we recall that a *block graph* is a graph in which every block is a complete graph. In this section we prove Theorem 2.5. Recall its statement.

Theorem 2.5. *There exist block graphs G such that* $\mathrm{sd_i}(G) = k$ *for any arbitrary (fixed) positive integer k.*

Proof For $k = 1$, we take $G = P_n$ where $n \equiv 0 \pmod 3$. For $k = 2$, we take $G = P_n$ where $n \equiv 2 \pmod 3$, while for $k = 3$, we take $G = P_n$ where $n \equiv 1 \pmod 3$. By Observation 3.5, we note that $\mathrm{sd_i}(G) = k$ in each of these cases. Hence, we may assume that $k \geq 4$, for otherwise the result is immediate. For $k \geq 4$, let G_k be obtained from a complete graph $F \cong K_k$ on k vertices with vertex set $V(F) = \{v_1, v_2, \ldots, v_k\}$ by attaching k pendant edges to each vertex of the complete graph F. For example, when $k = 4$ the graph G_k is illustrated in Fig. 1, where the darkened vertices form an i-set of G_4. We note that $i(G_k) = 1 + (k-1)k = k^2 - k + 1$ since every i-set of G_k contains one vertex of F and all $(k-1)k$ leaves of G_k not adjacent to the selected vertex of F.

We show firstly that $\mathrm{sd_i}(G) \geq k$. Let $E' \subset E(G_k)$ be an arbitrary subset of $k-1$ edges. Let G' be the graph obtained from G by subdividing the $k-1$ edges in the set E'.

Suppose firstly that E' contains at least one edge of the complete graph F. Renaming vertices if necessary, we may assume that $v_1v_2 \in E'$. Let v be the subdivided vertex in G' resulting from subdividing the edge v_1v_2. Let W be the set of $k-2$ subdivided vertices of G' different from v. Let W_1 be the set of subdivided vertices of G' resulting from subdividing pendant edges of G incident with v_1 or v_2 (possibly, $W_1 = \emptyset$). Let W_2 be the set of vertices in $W \backslash W_1$ that have a leaf neighbor in G', and let W_3 be the set of vertices in $W \backslash (W_1 \cup W_2)$ that are adjacent to neither v_1 nor v_2 in G'. We note that $|W_1| + |W_2| + |W_3| \leq |W|$. Let L_1 be the set of leaves of G' incident with a vertex in W_1, and let L_2 be the set of leaves in G' adjacent to a vertex in $V(F) \backslash \{v_1, v_2\}$. We note that $|L_1| = |W_1|$, $|L_2| = (k-2)k - |W_2|$. We now consider the set

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$$D = \{v_1, v_2\} \cup L_1 \cup L_2 \cup W_2 \cup W_3.$$

The set D is an ID-set of G', and so

$$\begin{aligned} i(G') \le |D| &= 2 + |L_1| + |L_2| + |W_2| + |W_3| \\ &= 2 + |W_1| + ((k-2)k - |W_2|) + |W_2| + |W_3| \\ &= 2 + |W_1| + (k-2)k + |W_3| \\ &\le k^2 - 2k + 2 + |W| \\ &= k^2 - 2k + 2 + (k-2) \\ &= k^2 - k \\ &< k^2 - k + 1 \\ &= i(G_k). \end{aligned}$$

This, if E' contains at least one edge of the complete graph F, then $i(G') < i(G_k)$, a contradiction. Hence, E' contains no edge from the complete graph F. Thus, every edge of E' is a pendant edge of G. Since $|E'| < k$, at least one vertex of F is such that none of its incident pendant edges belong to the set E'. Renaming vertices if necessary, we may assume that v_1 is such a vertex of F. In this case, we let L consist of all leaves of G that are not adjacent to v_1. The set $L \cup \{v_1\}$ is an ID-set of G', implying that $i(G') \le |L| + 1 = (k-1)k + 1 = k^2 - k + 1 = i(G_k)$. Thus, if $E' \subset E(G_k)$ is an arbitrary subset of $k-1$ edges, then we have shown that removing the set of edges E' from G_k does not increase the independent domination number. This proves that $\mathrm{sd_i}(G) \ge k$, as desired.

To prove that $\mathrm{sd_i}(G) \le k$, let u_i be an arbitrary leaf neighbor of the vertex v_i in G for all $i \in [k]$. Let $e_i = u_i v_i$ for $i \in [k]$, and let $E^* = \{e_1, e_2, \ldots, e_k\}$. Let G^* be the graph obtained from G by subdividing the k edges that belong to the set E^*. Let I^* be an arbitrary i-set of G^*. By the minimality of I^*, the set I^* contains exactly one vertex from the complete graph. Renaming vertices if necessary, we may assume that $v_1 \in I^*$, implying that $u_1 \notin I^*$. In order to dominate the leaves of G^* that are not adjacent to v_1 in G^* and are different from u_1, the set I^* contains at least $(k-1)k$ additional vertices. Thus, $i(G^*) = |I^*| \ge 2 + (k-1)k = k^2 - k + 2 > i(G_k)$. Hence, $\mathrm{sd_i}(G_k) \le |E^*| = k$. As observed earlier, $\mathrm{sd_i}(G) \ge k$. Consequently, $\mathrm{sd_i}(G) = k$. $\square$

Recall that by Theorem 3.4, if G is a block graph, then $1 \le \mathrm{sd}_\gamma(G) \le 3$. However by Theorem 2.5, we can construct a block graph G such that $\mathrm{sd_i}(G)$ is arbitrary large. This shows once again that the independent domination subdivision number differs significantly from the domination subdivision number.

7 Trees

In this section, we present a proof of Theorem 2.6. We remark that our characterization given in Theorem 2.6 is along similar lines to that presented by Benecke and Mynhardt [2, Theorem 2] for the domination subdivision number. (In [2, Theorem 2], however, we note that the set $\mathcal{N}(T)$ is defined with respect to the

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domination number; that is, $\mathcal{N}(T) = \{v \in V(T) \mid v \text{ is in no } \gamma\text{-set of } T\}$.) We also remark that our proof of the necessity part of the proof of Theorem 2.6 is along similar lines to the proof presented in [2, Theorem 2].

We are now in a position to present a proof of Theorem 2.6. Recall its statement.

Theorem 2.6. *For a tree T of order at least* 3, $\mathrm{sd_i}(T) = 1$ *if and only if T has at least one of the following two properties.*

P_1: A leaf of T belongs to the set $\mathcal{N}(T)$.
P_2: An edge of T has both its ends in $\mathcal{N}(T)$.

Proof We first prove the sufficiency. We proceed with the following two claims.

Claim 8 *If the tree T has property P_1, then* $\mathrm{sd_i}(T) = 1$.

Proof Suppose that the tree T has property P_1. Let u be a leaf of T such that $u \in \mathcal{N}(T)$, and let v be its neighbor. Consider the tree T' obtained from T by subdividing the edge uv and let w be the resulting subdivided vertex adjacent to u and v in T'. Let I' be an i-set of T'. We note that either $u \in I'$ or $w \in I'$. Suppose firstly that $w \in I'$. In this case, the set $I = I' \backslash \{w\}) \cup \{u\}$ is an ID-set of T. Since $u \in \mathcal{N}(T)$, the set I is not an i-set of T, implying that $i(T') = |I'| \geq i(T) + 1$. Suppose secondly that $u \in I'$. If $v \notin I'$, then we can replace u in I' with the vertex w, and obtain $i(T') \geq i(T) + 1$, as in the previous case. If $v \in I'$, then the set $I' \backslash \{u\}$ is an ID-set of T, implying that $i(T) \leq |I'| - 1 = i(T') - 1$. Thus in all cases we have $i(T') > i(T)$. Therefore, subdividing only one edge of T, namely the edge uv, increases the independent domination number. Thus, $\mathrm{sd_i}(T) = 1$. □

Claim 9 *If the tree T has property P_2, then* $\mathrm{sd_i}(T) = 1$.

Proof Suppose that the tree T has property P_2. Let $e = uv$ be an edge of T such that both $u \in \mathcal{N}(T)$ and $v \in \mathcal{N}(T)$. We note that in this case neither u nor v is a leaf in T, and so $d_T(u) \geq 2$ and $d_T(v) \geq 2$. Let T_u and T_v be the components of the tree $T - uv$ that contain the vertex u and v, respectively. Let I be an i-set of T, and let I_u and I_v be the restriction of the set I to the trees T_u and T_v, and so $I_u = I \cap V(T_u)$ and $I_v = I \cap V(T_v)$. By our earlier observations, $u \notin I_u$ and $v \notin I_v$. If I_u is not an i-set of T_u, then $|I_u| > i(T_u)$. In this case, if I_u^* is an i-set of T_u, then the set $I_u^* \cup I_v$ is an ID-set of T, and so $i(T) \leq |I_u^*| + |I_v| < |I_u| + |I_v| = |I| = i(T)$, a contradiction. Hence, I_u is an i-set of T_u, and so $i(T_u) = |I_u|$. Analogously, $i(T_v) = |I_v|$. If there is an i-set of T_u that contains the vertex u, then such a set can be extended to an i-set of T by adding to it the set I_v, contradicting the supposition that $u \in \mathcal{N}(T)$. Hence, $u \in \mathcal{N}(T_u)$. Analogously, $v \in \mathcal{N}(T_v)$.

We now consider the tree T' obtained from T by subdividing the edge uv and let w be the resulting subdivided vertex adjacent to u and v in T'. Let I' be an i-set of T'. In order to dominate the vertex w, we note that $I' \cap \{u, v, w\} \neq \emptyset$. Let I'_u and I'_v be the restriction of the set I' to the subtrees T_u and T_v of T', and so $I'_u = I' \cap V(T_u)$ and $I'_v = I' \cap V(T_v)$.

Claim 9.1 *If $w \notin I'$, then* $\mathrm{sd_i}(T) = 1$.

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Proof Suppose that $w \notin I'$. We show that in this case, we must have $|I' \cap \{u, v\}| = 1$. Suppose, to the contrary, that $\{u, v\} \subseteq I'$. We note that I'_u and I'_v are ID-sets of T_u and T_v, respectively. By supposition, $u \in I'_u$ and $v \in I'_v$. By our earlier observations, $u \in \mathcal{N}(T_u)$, implying that the set I'_u is not an i-set of T_u, and so $|I'_u| \geq i(T_u) + 1$. Analogously since $v \in \mathcal{N}(T_v)$, we have $|I'_v| \geq i(T_v) + 1$. Thus, $i(T') = |I'| = |I'_u| + |I'_v| \geq i(T_u) + i(T_v) + 2 = i(T) + 2$. However the set $I_u \cup I_v \cup \{w\}$ is an ID-set of T', implying that $i(T') \leq |I_u| + |I_v| + 1 = i(T_u) + i(T_v) + 1 = i(T) + 1$. Thus we have shown that $i(T) + 2 \leq i(T') \leq i(T) + 1$, a contradiction.

Hence, $|I' \cap \{u, v\}| = 1$, and so exactly one of u and v belong to the set I'. Renaming u and v if necessary, we may assume that $u \in I'$ (and therefore that $v \notin I'$). In this case, the set I' is an ID-set of T. However since $u \in \mathcal{N}(T)$, the set I' is not an i-set of T, implying that $i(T') = |I'| > i(T)$. Therefore, subdividing only one edge of T, namely the edge $e = uv$, increases the independent domination number. Thus, $\mathrm{sd_i}(T) = 1$. □

By Claim 9.1, we may assume that $w \in I'$, for otherwise the desired result, namely $\mathrm{sd_i}(T) = 1$, follows. Thus, $u \notin I'$ and $v \notin I'$ since the set I' is an independent set. If the set I'_u is not an ID-set of T_u, then this implies that I'_u dominates all vertices of T_u except for the vertex u. In this case, the set $I = (I' \backslash \{w\}) \cup \{u\}$ is an ID-set of T. However since $u \in \mathcal{N}(T)$, the set I is not an i-set of T, implying that $i(T') = |I'| = |I| > i(T)$. Hence we may assume that the set I'_u is an ID-set of T_u, and analogously that the set I'_v is an ID-set of T_v. Thus, $i(T') = |I'| = |I'_u| + |I'_v| + 1 \geq i(T_u) + i(T_v) + 1 = i(T) + 1$. Hence if $w \in I'$, then $i(T') > i(T)$. This completes the proof of Claim 9. □

The proof of the sufficiency follows from Claims 8 and 9 . To prove the necessity, suppose that $\mathrm{sd_i}(T) = 1$. Thus by subdividing only one edge of T, say the edge uv, we increase the independent domination number. Let T' be the tree obtained from T by subdividing the edge uv and let w be the resulting subdivided vertex adjacent to u and v in T'. Let I' be an i-set of T'. By supposition, $i(T') > i(T)$.

Suppose that either u or v, say u, is a leaf of T. If the vertex u belongs to some i-set I of T, then the set $(I \backslash \{u\}) \cup \{w\}$ is an ID-set of T', implying that $i(T') \leq |I| = i(T)$, a contradiction. Hence, $u \in \mathcal{N}(T)$. Therefore we may assume that neither u nor v is a leaf of T, for otherwise the tree T has property P_1 as desired. Thus, $d_T(u) \geq 2$ and $d_T(v) \geq 2$. If both $u \in \mathcal{N}(T)$ and $v \in \mathcal{N}(T)$, then T has property P_2 as desired. Hence we may further assume that at least one of u and v, say u, belongs to some i-set of T.

Let I be an i-set of T that contains the vertex u. Let $N_T(v) \backslash \{u\} = \{v_1, \ldots, v_k\}$ where $k = d_T(v) - 1 \geq 1$. Let T_i be the component of $T - vv_i$ that contains the vertex v_i for $i \in [k]$. Let I_i be the restriction of I to the tree T_i for $i \in [k]$; that is, $I_i = I \cap V(T_i)$. Further, let E_v be the set of k edges incident with v different from the edge uv, and let T_v be the component of $T - E_v$ that contains the vertex v. Finally, let I_v be the restriction of the set I to the tree T_v, and so $I_v = I \cap V(T_v)$. We note that

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$$I = I_v \cup \left(\bigcup_{i \in [k]} I_i \right) \quad \text{and} \quad i(T) = |I| = |I_v| + \sum_{i=1}^{k} |I_i|.$$

We proceed further with the following series of claims.

Claim 10 *The vertex u is the only neighbor of v that belongs to the set I.*

Proof Suppose, to the contrary, that the vertex $v_i \in I$ for at least one $i \in [k]$. This implies that the set I is also an ID-set in the tree T', and so $i(T') \leq i(T)$, a contradiction. □

By Claim 10, the vertex u is the only neighbor of v that belongs to the set I; that is, $v \in \mathrm{epn}_T(u, I)$.

Claim 11 *The following hold for all $i \in [k]$.*

(a) *The set I_i is an i-set of T_i.*
(b) *The vertex v_i does not belong to any i-set of T_i for all $i \in [k]$*

Proof Since $v \notin I$, we note that the set I_i is an ID-set of T_i. If I_i is not an i-set of T_i for some $i \in [k]$, then replacing the set I_i in I with an i-set of T_i produces an independent set of T of cardinality less than $|I|$, a contradiction. Hence, I_i is an i-set of T_i for all $i \in [k]$. This proves part (a). To prove part (b), suppose, to the contrary, that there is an i-set of T_i that contains the vertex v_i for some $i \in [k]$. Such an i-set can be extended to an i-set of T by adding to it the set I_v and the i-sets I_j of T_j for $j \in [k] \backslash \{i\}$. However as observed in the proof of Claim 10, such an i-set of T is also an ID-set of T', and so $i(T') \leq i(T)$, a contradiction. Hence, vertex v_i does not belong to any i-set of T_i for all $i \in [k]$; that is, $v_i \in N(T_i)$ for all $i \in [k]$. □

Claim 12 *$v \in \mathcal{N}(T_v)$ and $v \in \mathcal{N}(T)$.*

Proof Suppose that there is an i-set, say D_v, of T_v that contains the vertex v. In particular, we note that $i(T_v) = |I_v| = |D_v|$. In this case, the set $(I \backslash I_v) \cup (D_v \backslash \{v\}) \cup \{w\}$ is an ID-set of T', implying that $i(T') \leq |I| - |I_v| + |D_v| - 1 + 1 = |I| = i(T)$, a contradiction. Hence, the vertex v does not belong to any i-set of T_v; that is, $v \in \mathcal{N}(T_v)$.

Suppose next that there is an i-set, say S, of T that contains the vertex v. Let S_v be the restriction of S to the tree T_v, and so $S_v = S \cap V(T_v)$. Since $v \in S_v$, we note that the set S_v is an ID-set of T_v. However since $v \in \mathcal{N}(T_v)$, the set S_v is not an i-set of T_v, implying that $|S_v| \geq i(T_v) + 1$. Let S_i be the restriction of S to the tree T_i for $i \in [k]$; that is, $S_i = S \cap V(T_i)$. If $|S_i| \geq i(T_i)$ for all $i \in [k]$, then

$$i(T) = |S| = |S_v| + \sum_{i=1}^{k} |S_i| \geq i(T_v) + 1 + \sum_{i=1}^{k} i(T_i) = i(T) + 1,$$

a contradiction. Hence, $|S_i| < i(T_i)$ for at least one $i \in [k]$. For such an $i \in [k]$, the set S_i is not an ID-set of T_i. However, S_i is an ID-set of $T_i - v_i$, implying that $S_i \cup \{v_i\}$ is an ID-set of T_i of cardinality $|S_i| + 1 \leq i(T_i)$. Consequently, $|S_i| = i(T_i) - 1$ and

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the set $S_i \cup \{v_i\}$ is an i-set of T_i that contains the vertex v_i, contradicting Claim 11(b). Hence, there is no i-set of T that contains the vertex v; that is, $v \in \mathcal{N}(T)$. □

By Claims 11 and 12 , for every edge vv_i of T where $i \in [k]$, we have $v \in \mathcal{N}(T)$ and $v_i \in \mathcal{N}(T)$; that is, the edge vv_i of T has both its ends in $\mathcal{N}(T)$ for $i \in [k]$. Hence, the tree T has property P_2. This completes the proof of the necessity. □

As a consequence of Theorem 2.6 we present a simple proof that the independent domination subdivision number of a tree is at most 3. For this purpose, if v is a support vertex in a tree T, we denote by $L_v(T)$ the set of leaf neighbors of v in T. If the tree T is clear from context, we simply write L_v rather that $L_v(T)$.

Theorem 7.1 *If T is a tree of order $n \geq 3$, then $1 \leq \mathrm{sd_i}(T) \leq 3$.*

Proof Let T be a tree of order $n \geq 3$. If $\mathrm{sd_i}(T) = 1$, then the desired result follows. Hence we may assume that $\mathrm{sd_i}(T) \geq 2$. Thus by Theorem 2.6, the tree T has neither property P_1 nor P_2. In particular, for each leaf v of T there exists an i-set, D_v, of T such that $v \in D_v$, and for each edge uv of T there exists an i-set, D_{uv}, of T such that $|\{u, v\} \cap D_{uv}| = 1$. Since T is a tree of order $n \geq 3$ satisfying $\mathrm{sd_i}(T) > 1$, we note that T not a star, and so $\mathrm{diam}(T) \geq 3$.

Suppose that $\mathrm{diam}(T) = 3$, and so T is a double star $S(\ell_1, \ell_2)$ for some integers $\ell_2 \geq \ell_1 \geq 1$. If $\ell_2 > \ell_1$, then T has property P_1, a contradiction. Hence, $\ell_2 = \ell_1$. If $\ell = 1$, then $T = P_4$ and $\mathrm{sd_i}(T) = 3$. If $\ell \geq 2$, then subdividing two non-adjacent edges (that have no vertex in common) results in a tree T' with $i(T') = \ell_1 + 2 > \ell_1 + 1 = i(T)$, implying that $\mathrm{sd_i}(T) = 2$. Hence we may assume that $\mathrm{diam}(T) \geq 4$, for otherwise $2 \leq \mathrm{sd_i}(T) \leq 3$ and the desired result holds.

Among all longest paths in T, let $P : v_1 v_2 \ldots v_{\mathrm{diam}(T)+1}$ be chosen so that the degree of v_2 is as large as possible. We now root the tree T at the vertex $v_{\mathrm{diam}(T)+1}$. We note that every child of v_2 in the rooted tree T is a leaf; that is, $L_{v_2} = C(v_2)$. Let A and B be the set of children and grandchildren, respectively, of the vertex v_3 in the rooted tree T. We note that each vertex of B is a leaf of T and is at distance 2 from v_3 in T. Further, every descendant of v_3 belongs to the set $A \cup B$. We proceed further with the following series of claims.

Claim 13 *If $d_T(v_2) \geq 3$, then the vertex v_3 is a support vertex.*

Proof Let $d_T(v_2) \geq 3$ and suppose, to the contrary, that the vertex v_3 is not a support vertex of T. Thus each child of v_3 has at least one leaf neighbor. Since the child v_2 of v_3 has at least two leaf neighbors, we note therefore that $|B| \geq |A| + 1$. As observed earlier, every leaf belongs to some i-set of T. Let I be an i-set of T that contains the leaf v_1. We note that every child of v_2 is a leaf, and so, $L_{v_2} = C(v_2)$. Since $v_2 \notin I$, we note that $L_{v_2} \subset I$. If $v_3 \notin I$, then $(I \backslash L_{v_2}) \cup \{v_2\}$ is an ID-set of T of smaller cardinality than $|I| = i(T)$, a contradiction. Hence, $v_3 \in I$. This in turn implies that $B \subset I$. If v_4 has at least two neighbors in I, then let $I^* = (I \backslash (B \cup \{v_3\})) \cup A$. If v_3 is the only neighbor of v_4 in I, then let $I^* = (I \backslash (B \cup \{v_3\}) \cup (A \cup \{v_4\})$. In both cases, I^* is an ID-set of T, and so $i(T) = |I^*| \leq |I| - |B| + |A| < |I| = i(T)$, a contradiction. □

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Claim 14 *If* $d_T(v_2) \geq 3$, *then* $\mathrm{sd_i}(T) = 2$.

Proof Let $d_T(v_2) \geq 3$. By Claim 14, the vertex v_3 is a support vertex. Let y be an arbitrary leaf neighbor of v_3, and let T' be the tree obtained from T by subdividing the edges v_1v_2 and v_3y with the new subdivided vertices u_1 and u_2, respectively. Let I' be an i-set of T'. Suppose firstly that $v_3 \in I'$. In this case, $y \in I'$, $v_2 \notin I'$ and $|\{u_1, v_1\} \cap I'| = 1$. If $u_1 \in I'$, then we can replace u_1 with v_1 in I'. Hence we can choose I' so that $v_1 \in I'$. We now let $I^* = I' \backslash \{y\}$. Suppose secondly that $v_3 \notin I'$. In this case, $|\{u_2, y\} \cap I'| = 1$ and $|D[v_2] \cap I'| = 2$, where note that $D[v_2] = L_{v_2}(T) \cup \{v_2\}$. Hence we can choose I' so that $y \in I'$ and $\{v_1, v_2\} \subset I'$. We now let $I^* = I' \backslash \{v_1\}$. In both cases, $|I^*| = |I'| - 1$ and the set I^* is an ID-set of T, and so $i(T) \leq |I^*| < |I'| = i(T')$, implying that $\mathrm{sd_i}(T) = 2$. □

Claim 15 *If* $d_T(v_2) = 2$ *and* v_3 *is a support vertex*, *then* $\mathrm{sd_i}(T) \leq 3$.

Proof Let $d_T(v_2) = 2$ and suppose that v_3 is a support vertex. Let y be an arbitrary leaf neighbor of v_3, and let T' be the tree obtained from T by subdividing the edges v_1v_2, v_2v_3 and v_3y with the new subdivided vertices u_1, u_2 and u_3, respectively. Let I' be an i-set of T'. Suppose firstly $v_3 \in I'$. In this case, $\{u_1, y\} \subset I'$ and we let $I^* = (I' \backslash \{u_1, y\}) \cup \{v_1\}$. Suppose secondly that $v_3 \notin I'$. In this case, $|\{u_1, v_1\} \cap I'| = 1$, $|\{u_2, v_2\} \cap I'| = 1$ and $|\{u_3, y\} \cap I'| = 1$. Hence we can choose the set I' so that $\{u_1, u_2, u_3\} \subseteq I'$, and we let $I^* = (I' \backslash \{u_1, u_2, u_3\}) \cup \{v_2, y\}$. In both cases, $|I^*| = |I'| - 1$ and the set I^* is an ID-set of T, and so $i(T) \leq |I^*| < |I'| = i(T')$, implying that $\mathrm{sd_i}(T) \leq 3$. □

Claim 16 *If* $d_T(v_2) = 2$ *and* v_3 *is not a support vertex*, *then* $\mathrm{sd_i}(T) \leq 3$.

Proof Let $d_T(v_2) = 2$ and suppose that v_3 is not a support vertex. By our choice of the vertex v_2, we note that each child of v_3 is therefore a support vertex of degree 2. Let T' be the tree obtained from T by subdividing the edges v_1v_2, v_2v_3 and v_3v_4 with the new subdivided vertices u_1, u_2 and u_3, respectively. Let I' be an i-set of T'. Suppose firstly $v_4 \in I'$. In this case, $u_3 \notin I'$, $|\{u_1, v_1\} \cap I'| = 1$ and $|\{u_2, v_2, v_3\} \cap I'| = 1$. Hence we can choose the set I' so that $\{u_1, u_2\} \subseteq I'$, and we let $I^* = (I' \backslash \{u_1, u_2\}) \cup \{v_2\}$. Suppose secondly $v_4 \notin I'$ and $v_3 \in I'$. In this case, $\{u_1, v_3\} \subset I'$ and we let $I^* = (I' \backslash \{u_1, v_3\}) \cup \{v_2\}$. Suppose thirdly $u_3 \in I'$. In this case, $|\{u_1, v_1\} \cap I'| = 1$ and $|\{u_2, v_2\} \cap I'| = 1$. Hence we can choose the set I' so that $\{u_1, u_2\} \subseteq I'$. Further we can choose I' so that if $d_T(v_3) \geq 3$, then the set I' contains all grandchildren of v_3 different from v_1. We now let $I^* = (I' \backslash \{u_1, u_2, u_3\}) \cup \{v_1, v_3\}$. In all three cases, $|I^*| = |I'| - 1$ and the set I^* is an ID-set of T, and so $i(T) \leq |I^*| < |I'| = i(T')$, implying that $\mathrm{sd_i}(T) \leq 3$. □

The proof of Theorem 7.1 follows from Claims 14, 15 and 16 . □

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8 Open Problems

We close with some open problems that we have yet to settle. By Theorem 2.1, if G is a connected graph of order at least 3, then $\mathrm{sd_i}(G) \leq \Delta(G) \cdot i(G)$. A natural question is to find a general upper bound on the independent domination subdivision number of a connected graph G in terms of its maximum degree $\Delta(G)$.

Problem 1 *Does there exists a constant C such that every connected graph G satisfies $\mathrm{sd_i}(G) \leq C \cdot \Delta(G)$? If so, determine the smallest such constant C.*

By Corollary 2.4 we know that the constant C in Problem 1, if it exists, satisfies $C \geq 3$. By Theorem 7.1, if T is a tree of order $n \geq 3$, then $1 \leq \mathrm{sd_i}(T) \leq 3$, and in Theorem 2.6 the trees T satisfying $\mathrm{sd_i}(T) = 1$ are characterized. It remains an open problem to characterize the trees T satisfying $\mathrm{sd_i}(T) = 2$ or $\mathrm{sd_i}(T) = 3$. We state this problem formally as follows.

By Theorem 2.5, the independent domination subdivision number of a block graph can be arbitrary large. However as remarked earlier, the independent domination subdivision number of a tree is at most 3.

Problem 2 *Determine classes of graphs different from the class of trees for which the independent domination subdivision number is bounded above by a constant.*

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