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Infinite chromatic games ☆

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ABSTRACT

In the paper we introduce a new variant of the graph coloring game and a new graph parameter being the result of the new game. We study their properties and get some lower and upper bounds, exact values for complete multipartite graphs and optimal, often polynomial-time strategies for both players provided that the game is played on a graph with an odd number of vertices. At the end we show that both games, the new and the classic one, are related: our new parameter is an upper bound for the game chromatic number.

1. Introduction

All graphs considered in this paper are finite, undirected and simple, i.e. without loops or multiple edges. Following Diestel [3] we use the standard graph notation: a graph G has a vertex set $V(G)$ and an edge set $E(G)$, with their cardinalities $n(G)$ and $m(G)$, respectively. We denote by $\Delta(G)$, $\delta(G)$, $\chi(G)$, $\alpha(G)$ and $\alpha'(G)$ the maximum and minimum degree, the chromatic number, the independence number and the matching number of the graph G, respectively. The complement of a graph G is denoted by $\bar{G}$.

Map-coloring game can be traced to Scientific American 1981 article [5], but it has been analyzed extensively only since it was reinvented a decade later in [2] as the game played on graphs. As for today, there are many generalizations and variations of the graph coloring game, depending on what exactly is colored and what are the additional constraints on the graph structure, admissible coloring etc. See for example the survey [1], covering some variants, techniques and results for this type of problems.

The standard version of the graph coloring game is played between Alice and Bob on a graph G with a set C of k colors, with k fixed. We say that color $c \in C$ is *legal* for a vertex $v \in V(G)$ if no neighbor of v is colored with c. The game proceeds by Alice and Bob taking subsequent turns and coloring any uncolored vertex with a legal color until the entire graph is colored or there are no legal colors for all uncolored vertices. Alice wins in the former case and Bob in the latter.

The game chromatic number of a graph G, denoted by $\chi_g(G)$, is defined as the minimum k such that there exists a winning strategy for Alice, that is, it is certain that the entire graph will be colored regardless of the strategy of Bob. This parameter is well-defined, because Alice always wins if C contains at least as many colors as there are vertices of G.

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The graph coloring game was studied by many authors. For the class of planar graphs $\mathcal{P}$ it was proved in [7,9] that $8 \leq \max\{\chi_g(G): G \in \mathcal{P}\} \leq 17$. For the class of outerplanar graphs $\mathcal{OP}$, it was shown in [6] that $6 \leq \max\{\chi_g(G): G \in \mathcal{OP}\} \leq 7$. For k-trees $\mathcal{KT}$ it is the case that $\max\{\chi_g(G): G \in \mathcal{OP}\} = 3k + 2$ for $k \geq 2$ (see [10]). Similarly, it was shown in [8] that $\max\{\chi_g(G): G \in \mathcal{C}\} = 5$ for cacti graphs $\mathcal{C}$. Finally, for interval graphs it was proved in [4] that $\chi_g(G) \leq 3\omega(G) - 2$, where $\omega(G)$ is the clique number of G, and that there are examples of graphs with $\chi_g(G) \geq 2\omega(G) - 1$.

In the paper we introduce a related graph coloring game, under the name *infinite graph coloring game*, played on a graph G. In this game Alice and Bob do not use any fixed set of available colors, but instead have an infinite set of colors at their disposal. Moreover, Alice wants to color the graph using as few colors as possible and Bob, conversely, wants to maximize the number of colors used. The *infinite game chromatic number* $\chi_g^\infty(G)$ is then defined as the number of colors obtained when both players use their optimal strategies.

We analyze basic properties of the infinite game chromatic number. For example, we provide the exact values for certain classes of graphs, such as paths, cycles and complete k-partite graphs. More important, we provide exact values for $\chi_g^\infty(G)$ provided that $n(G)$ is odd, and upper and lower bound for $n(G)$ even. Most of these values are obtainable in a polynomial time. Then, we proceed to the proof that there is a relation between the graph coloring game and the infinite graph coloring game. We show i.a. that the infinite game chromatic number is always an upper bound for the game chromatic number and that there are graphs for which these numbers are identical.

2. Basic properties of $\chi_g^\infty(G)$

Throughout this paper we proceed in a manner known from many other proofs of properties of games on graphs: if we want to prove that $\chi_g^\infty(G) \geq k$, then it is sufficient to show that there exists a strategy for Bob, which—regardless of Alice's play—enforces at least k colors. Similarly, if we want to prove that $\chi_g^\infty(G) \leq k$, we present a possible strategy for Alice, such that all possible outcomes are guaranteed to be restricted to use at most k colors.

Theorem 1. *The inequality $\chi_g^\infty(G) \geq \max\{\chi(G), \lfloor \frac{1}{2}n(G) \rfloor + 1\}$ holds for all graphs G.*

Proof. It is obvious that $\chi_g^\infty(G) \geq \chi(G)$. To prove the second inequality, let Bob adopt the "always use a new color" strategy, i.e. every time Bob moves, he selects an uncolored vertex and colors it with a color not assigned previously to any vertex. The strategy guarantees that there will be at least $\lfloor \frac{1}{2}n(G) \rfloor + 1$ colors used as Alice has to use one new color at the beginning and Bob performs exactly $\lfloor \frac{1}{2}n(G) \rfloor$ moves. □

As we will see later, there are many graphs for which $\chi_g^\infty(G) = \lfloor \frac{1}{2}n(G) \rfloor + 1$ (especially all graphs with low maximum degree, see Section 3.2). Now we proceed to upper bounds for $\chi_g^\infty(G)$.

Proposition 2. *For any graph G it holds that $\chi_g^\infty(G) \leq n(G)$. Furthermore, $\chi_g^\infty(G) = n(G)$ if and only if one of the following conditions hold:*

(i) $n(G)$ is odd and G is a complete graph,
(ii) $n(G)$ is even and $\Delta(\bar{G}) \leq 1$.

Proof. It is clear that $\chi_g^\infty(G) \leq n(G)$. To complete the proof, it suffices to consider the following three cases:

(1) $\Delta(\bar{G}) \geq 2$. Then there exists a vertex v of G and two its non-neighbors v_1, v_2. Alice colors v in her first move with color 1. There are two possible continuations of the game:

  (a) Bob reused the color 1 in his first move. As there are $n(G) - 2$ remaining moves and only one color used so far, the number of colors used in the game will not exceed $n(G) - 1$.
  (b) Bob introduced a new color in his first move. Then at least one of v_1, v_2 is uncolored and Alice may choose it and color with 1. As there are $n(G) - 3$ remaining moves and only two colors used so far, the number of colors used in the game will not exceed $n(G) - 1$.

  In both cases $\chi_g^\infty(G) < n(G)$.

(2) $\Delta(\bar{G}) = 1$. Then $V(G) = V_0 \cup V_1$ where V_i is the set of vertices of G that are of degree i in $\bar{G}$. There is also a one-to-one function $f: V_1 \to V_1$ such that $vf(v) \in E(\bar{G})$. There are two subcases to consider:

  (a) $n(G)$ is even. Then Bob may adopt the following strategy: if Alice colored v in her last move and $v \in V_0$ then color any uncolored vertex of V_0 with a new color, otherwise color $f(v)$ with a new color. In this case V_0 has an even number of vertices so the strategy can be applied and moreover Alice is forced to use a new color in every move. Hence $\chi_g^\infty(G) = n(G)$.
  (b) $n(G)$ is odd. Then Alice may adopt the following strategy: start by coloring a vertex from V_0 and then mimic Bob's moves: if he colored a vertex v in his last move and $v \in V_0$, color any uncolored vertex of V_0, otherwise color $f(v)$ with the color used by Bob in his last move. In this case V_0 has an odd number of vertices so the

strategy can be applied and moreover Bob will be forced to color at least once a vertex from V_1. Therefore, Alice will repeat his color i.e. she will reuse an already existing color at least once. Hence $\chi_g^{\infty}(G) < n(G)$.

(3) $\Delta(\bar{G}) = 0$. Then G is a complete graph and obviously $\chi_g^{\infty}(G) = n(G)$. □

Theorem 3. *The inequality* $\chi_g^{\infty}(G) \leq n(G) + 1 - \lceil \frac{1}{2}\alpha(G) \rceil$ *holds for every graph G.*

Proof. It is easy to see that our claim holds (and it is tight) for empty graphs so we assume that G is not empty. Alice's strategy consists of two parts: the first move and all other moves. In her first move, Alice:

(a) makes a partition of $V(G)$ into a maximum independent set V_0 and a nonempty set V_1—note that such partition exists since G is not empty,
(b) chooses any vertex of V_1 and colors it with 1.

Alice's next moves depend on last Bob's move in a following way:

(c) if Bob colored a vertex from V_1 and there is an uncolored vertex in V_1, Alice chooses it and colors with a new color,
(d) if Bob colored a vertex from V_1 and all vertices in V_1 are colored, Alice colors any uncolored vertex of V_0 with a new color,
(e) if Bob colored a vertex from V_0 with a new color c and there is an uncolored vertex in V_0, Alice chooses it and colors with c,
(f) if Bob colored a vertex from V_0 with previously used color and there is an uncolored vertex in V_0, Alice chooses it and colors with a new color,
(g) if Bob colored a vertex from V_0 and all vertices in V_0 are colored, Alice colors any uncolored vertex of V_1 with a new color.

It is easy to see that the above strategy works, as it is always possible to color vertices with new colors or the same color that was used previously on exactly one vertex that is not a neighbor of the one being colored (rule (e)). Moreover, the rule (d) will be applied at most once during the game, since there is exactly one moment in the game when the last uncolored vertex of V_1 becomes colored. Hence, there are only two cases to be considered:

(1) The rule (d) was not used. Then all vertices of V_0, possibly except of the one colored last, may be divided into pairs u, v where u was colored by Bob and v was colored by Alice in consecutive moves. In all such cases Alice used rules (e) or (f) so the number of new colors introduced in these two moves equals 1. This implies that the number of new colors used in V_0 is at most $\frac{1}{2}|V_0|$ if $|V_0|$ is even or $1 + \frac{1}{2}(|V_0| - 1)$ otherwise.

(2) The rule (d) was used once to color vertex v. Then the above reasoning holds for $V_0 \setminus \{v\}$ and the number of new colors used in V_0 is at most $1 + \frac{1}{2}(|V_0| - 1)$ if $|V_0|$ is odd or $1 + \frac{1}{2}|V_0|$ otherwise.

In both cases the number of new colors used to color V_0 is at most $1 + \lfloor \frac{1}{2}|V_0| \rfloor$. Therefore the number of colors used in the game is at most $|V_1| + 1 + \lfloor \frac{1}{2}|V_0| \rfloor = n(G) + 1 - |V_0| + \lfloor \frac{1}{2}|V_0| \rfloor = n(G) + 1 - \lceil \frac{1}{2}|V_0| \rceil = n(G) + 1 - \lceil \frac{1}{2}\alpha(G) \rceil$. □

Theorem 4. *The inequality* $\chi_g^{\infty}(G) \leq \lfloor \frac{1}{2}n(G) \rfloor + \chi(G)$ *holds for every graph G.*

Proof. Alice's strategy consists of two parts: the first move and all other moves. In her first move, Alice:

(a) makes a partition of $V(G)$ into independent sets $V_1, V_2, \ldots, V_{\chi(G)}$, which exists by the definition of the chromatic number,
(b) chooses any vertex $v_1 \in V_1$, colors it with $c_1 = 1$ and sets $B_0 = \emptyset$, $A_1^1 = \{c_1\}$ and $A_1^i = \emptyset$ for $2 \leq i \leq \chi(G)$.

In her jth move, where $j \geq 2$, played after Bob's $(j-1)$th move (consisting of coloring of a vertex u_{j-1} with color b_{j-1}), Alice:

(c) sets $B_{j-1} = B_{j-2} \cup \{b_{j-1}\}$ and chooses i_j for which there is an uncolored vertex $v_j \in V_{i_j}$,
(d) chooses a new color c_j if $A_{j-1}^{i_j} \subseteq B_{j-1}$ or any element c_j of $A_{j-1}^{i_j} \setminus B_{j-1}$ otherwise,
(e) colors v_j with c_j, sets $A_j^{i_j} = A_{j-1}^{i_j} \cup \{c_j\}$ and $A_j^k = A_{j-1}^k$ for $k \neq i_j$.

We will use induction on j to show that the above strategy works, B_{j-1} is the set of colors used by Bob in his first $j-1$ moves, A_j^i is the set of colors used by Alice in her first j moves on vertices from V_i, $|A_j^i \setminus B_{j-1}| \leq 1$ for $1 \leq i \leq \chi(G)$ and $(A_j^k \cap A_j^l) \setminus B_{j-1} = \emptyset$ for $k \neq l$.

The above statement is clearly true for $j = 1$, so assume that $j > 1$ and that it holds for $j - 1$. Since B_{j-2} is the set of colors used by Bob up to move $j-2$ and B_{j-1} arises from B_{j-2} by including the color used by Bob in his $(j-1)$th move, then B_{j-1} is the set of colors used by Bob in his first $j-1$ moves. Similar reasoning shows that A_j^i must be the set of colors used by Alice in her first j moves on vertices from V_i, provided that Alice can color v_j with c_j. To show that this move is playable, let us consider the following two cases:

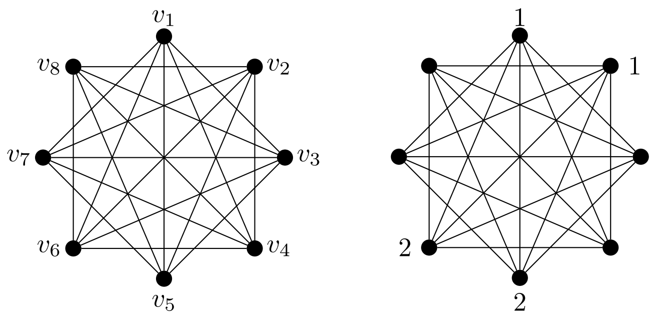

Fig. 1. The graph $\bar{C_8}$ (left) and a partial coloring obtained during the game (right).

(1) $A^{i_j}_{j-1} \subseteq B_{j-1}$. Then c_j is a new color and as such can be used to color any vertex of G.
(2) $c_j \in A^{i_j}_{j-1} \setminus B_{j-1}$. Then c_j is a color not used by Bob up to move $j-1$. If Alice cannot color v_j with c_j, it means that Alice colored some neighbor of v_j, say v_k in her kth move, with color $c_j = c_k$. Since V_{i_k} is an independent set, we know that $i_k \neq i_j$ and $c_j \in (A^{i_j}_{j-1} \cap A^{i_k}_{j-1}) \setminus B_{j-1} \subseteq (A^{i_j}_{j-1} \cap A^{i_k}_{j-1}) \setminus B_{j-2} = \emptyset$—a contradiction. Hence Alice can color v_j with c_j.

The same two cases must be considered to show that $|A^i_j \setminus B_{j-1}| \leq 1$ for $1 \leq i \leq \chi(G)$ and $(A^k_j \cap A^l_j) \setminus B_{j-1} = \emptyset$ for $k \neq l$.

(1) $A^{i_j}_{j-1} \subseteq B_{j-1}$. Then $A^{i_j}_j \setminus B_{j-1} = \{c_j\}$, $A^i_j = A^i_{j-1}$ for $i \neq i_j$, $B_{j-2} \subseteq B_{j-1}$ and it is easy to see that our claim holds.
(2) $c_j \in A^{i_j}_{j-1} \setminus B_{j-1}$. Then $A^i_j = A^i_{j-1}$ for all i, $B_{j-2} \subseteq B_{j-1}$ and it is easy to verify that our claim holds.

To complete the proof, it suffices to see how many colors were used in the game played with the above strategy. There are two cases to consider:

(1) Alice's jth move was the last move of the game. Then the number of colors used by both sides equals $|B_{j-1} \cup \bigcup_{i=1}^{\chi(G)} A^i_j| \leq |B_{j-1}| + \sum_{i=1}^{\chi(G)} |A^i_j \setminus B_{j-1}| \leq \lfloor \frac{1}{2} n(G) \rfloor + \chi(G)$.
(2) Bob's jth move was the last move of the game. Let B be the set of colors used by Bob. Then the number of colors used by both sides equals $|B \cup \bigcup_{i=1}^{\chi(G)} A^i_j| \leq |B| + \sum_{i=1}^{\chi(G)} |A^i_j \setminus B| \leq |B| + \sum_{i=1}^{\chi(G)} |A^i_j \setminus B_{j-1}| \leq \lfloor \frac{1}{2} n(G) \rfloor + \chi(G)$. □

At first glance the "always use a new color" strategy seems optimal for Bob. It appears that it is not true.

Proposition 5. *There exist graphs such that the strategy "use always a new color" is not optimal for Bob, that is, there exists a graph G such that no matter how Bob plays using this strategy, the players use strictly less than $\chi_g^\infty(G)$ colors.*

Proof. Let us consider a game played on the graph $\bar{C_8}$ shown in the left part of Fig. 1. Without loss of generality, we may assume that Alice started the game by choosing vertex v_1 and assigning to it the color 1.

If Bob uses a new color in each of his moves, Alice can reuse colors at least 3 times. To see this, let us consider the next moves in the game. Due to symmetries in the graph, there are only two cases to consider:

1. Bob colors v_2 or v_3 with 2. Then Alice colors the only uncolored vertex in the set $\{v_2, v_3\}$ with 2. The game proceeds as follows:
   (a) If Bob colors v_4 or v_5 with 3, Alice colors the only uncolored vertex in the set $\{v_4, v_5\}$ with 3, too. Whatever vertex v_i is chosen by Bob as next, it has at least one uncolored non-neighbor and Alice may choose it and repeat the color Bob just assigned to v_i.
   (b) If Bob colors v_7 or v_8 with 3, Alice colors the only uncolored vertex in the set $\{v_7, v_8\}$ with 3, too. Whatever vertex v_i is chosen by Bob as next, it has at least one uncolored non-neighbor and Alice may choose it and repeat the color Bob just assigned to v_i.
   (c) If Bob colors v_6 with 3, Alice colors v_5 with 3. If in the next move Bob colors v_8 with 4, Alice colors v_7 with 4. Otherwise, Alice colors v_8 with 1.
2. Bob colors v_4 or v_5 with 2. Then Alice colors the only uncolored vertex in the set $\{v_4, v_5\}$ with 2. The game proceeds as follows:
   (a) If Bob colors v_2 or v_3 with 3, Alice colors the only uncolored vertex in the set $\{v_2, v_3\}$ with 3, too. Whatever vertex v_i is chosen by Bob as next, it has at least one uncolored non-neighbor and Alice may choose it and repeat the color Bob just assigned to v_i.

(b) If Bob colors v_6, v_7 or v_8 with 3, Alice chooses its uncolored non-neighbor and colors it with 3. If in the next move Bob colors v_2 with 4, Alice colors v_3 with 4. Otherwise Alice colors v_2 with 1.

In all above cases, the obtained colorings use 5 colors.

However, when Bob is allowed to reuse colors, he answers Alice's first move with assigning color 2 to v_5. Then, Alice has two possibilities: either repeat a color or use a new one.

1. In the first case Bob just mirrors the move Alice did, creating a situation presented in the right part of Fig. 1. It is easy to see that Bob can force Alice to use only new colors in the remaining moves, simply by choosing to color the only uncolored neighbor of the vertex Alice just colored.
2. In the second case Bob follows the "always use a new color" strategy.

In all the above cases, the obtained colorings use at least 6 colors, which completes the proof. □

It remains an open problem to decide whether "reuse an existing color if possible" is always an optimal strategy for Alice.

3. Special classes of graphs

3.1. Complete multipartite graphs

Theorem 6. *Let exactly* $2l$ *numbers among* $r_1, r_2, \dots, r_k$ *be odd for some* $l \geq 0$. *Then* $\chi_g^\infty(K_{r_1,r_2\dots,r_k}) = \lfloor \frac{1}{2}n(G) \rfloor + k - l$.

Proof. Let $V_1, V_2, \dots, V_k$ be a partition of vertices of $K_{r_1,r_2\dots,r_k}$ into independent sets of sizes $r_1, r_2, \dots, r_k$. Note that, if a color c is used for a vertex in V_i, then it cannot be used in any V_j, where $j \neq i$. Therefore, as long as there are uncolored vertices in V_i, players may always reuse c for other vertices in V_i.

The strategy for Alice is to minimize the number of parts that she colors first. On the other hand, it seems reasonable for Bob to try to force Alice to begin coloring vertices from as many sets V_i as possible.

($\leq$) A strategy for Alice, which ensures that no more than $\lfloor \frac{1}{2}n(G) \rfloor + k - l$ colors are used:

(1) if possible, pick a vertex from a set V_i which already has some vertices colored—and use one of colors previously used in V_i,
(2) if possible, pick any uncolored vertex in a set V_i with an odd number of vertices and use a new color,
(3) otherwise, pick any uncolored vertex in any set V_i and use a new color.

The strategy guarantees that Alice colors the first vertex only in at most l sets V_i of odd cardinality. Note that Alice starts coloring a set only when all sets are either colored completely or not colored at all—and only in the case when the sum of cardinalities of all colored sets is even. Alice starts coloring a set with an odd number of vertices. Therefore, she does not start coloring any new set until some yet uncolored set with an odd number of vertices will have the first vertex colored by Bob. And since there are $k - 2l$ sets of even cardinality, we know that there are at most $\lfloor \frac{1}{2}n(G) \rfloor + k - l$ colors in any resulting coloring.

($\geq$) A strategy for Bob is strikingly similar:

(1) if possible, pick an uncolored vertex from the same set V_i which has already some vertices colored,
(2) if possible, pick any uncolored vertex in a set V_i with an odd number of vertices,
(3) otherwise, pick any uncolored vertex.

Always use a new color.

Playing this way Bob can force Alice to start at most half of sets V_i with an odd parity and all sets V_i with an even parity. To see the former fact, suppose Bob starts coloring a set with an odd number of vertices. This means that there were no partially colored sets in the graph. Since Bob does not start coloring a new set unless he is forced to, the next set with an odd number of vertices will have the first vertex definitely colored by Alice. To see the latter one, let Alice start coloring any vertex in a set V_i with even number of vertices. After Bob's answer the total number of vertices in sets V_i which are partially and completely colored is an even number. Bob's strategy ensures that the next yet uncolored set will be colored first by Alice.

Therefore Alice will use at least $(k - 2l) + l = k - l$ new colors. And since in all moves Bob use a new color, we have at least $\lfloor \frac{1}{2}n(G) \rfloor + k - l$ colors in total. □

Note that from this follows immediately that $K_{r_1,r_2,\dots,r_k}$ with all r_i $(1 \leq i \leq k)$ even is a tight example for upper bound from Theorem 4.

Theorem 7. *Let exactly* $2l + 1$ *numbers among* $r_1, r_2, \dots, r_k$ *be odd for some* $l \geq 0$. *Then* $\chi_g^\infty(K_{r_1,r_2,\dots,r_k}) = \lfloor \frac{1}{2}n(G) \rfloor + l + 1$.

Proof. We use exactly the same strategies as in the previous proof.

($\leq$) Alice is certain to use only $l+1$ colors, as she may force Bob to start all sets of even cardinality and $\lfloor \frac{2l+1}{2} \rfloor = l$ sets of odd cardinality. Therefore it is true that there are at most $\lfloor \frac{1}{2} n(G) \rfloor + l + 1$ colors in any resulting coloring.

($\geq$) Playing this way Bob guarantees that Alice starts coloring at least $l+1$ sets of odd cardinality, which is sufficient to use at least $\lfloor \frac{1}{2} n(G) \rfloor + l + 1$ colors in total. □

Observe that there is a striking asymmetry between the results from Theorems 6 and 7: in the former case $\chi_g^\infty(G)$ becomes smaller with the growth of l, when n and k are held constant, but in the latter case it behaves exactly the other way round, with the maximum at $l = k$.

3.2. Graphs with bounded maximum degree

Theorem 8. *If a graph G satisfies condition $\Delta(G) \leq \frac{1}{3}(n(G)-1)$ then $\chi_g^\infty(G) = \lfloor \frac{1}{2} n(G) \rfloor + 1$.*

Proof. The lower bound follows from Theorem 1, therefore we only need to prove the upper bound. Suppose that Alice uses the following strategy:

(1) in the first move, pick any vertex v_1 and assign to it color 1,
(2) in her kth move with $2 \leq k \leq \Delta(G)+1$, pick any uncolored vertex v_k which is not adjacent to vertex u_{k-1} picked in the previous move by Bob and assign to it the same color (if Bob used a new color) or assign to it any new color (otherwise).
(3) in any further moves, pick any uncolored vertex and assign to it any color which was already used, but which does not appear in the neighborhood of this vertex.

We will show that the above strategy works and during phases (1) and (2) both players use exactly $\Delta(G)+1$ colors. To this aim consider Alice's kth move, where $2 \leq k \leq \Delta(G)+1$. Vertex u_{k-1} has at most $\Delta(G)$ neighbors, $2k-2$ other vertices are colored and $\Delta(G) + 2k - 2 \leq 3\Delta(G) < n(G)$, so there is an uncolored vertex that is not adjacent to u_{k-1}. It can be easily colored using the rule described in phase (2) and it is easy to see that Alice introduces a new color if and only if Bob did not introduce a new color in his last move. Therefore after kth Alice's move the number of colors in the game equals k and when phase (2) ends there are $\Delta(G)+1$ colors in the game.

Phase (3) also works as every vertex has at most $\Delta(G)$ neighbors and there are already $\Delta(G)+1$ colors used. Hence the number of colors used in the whole game equals $\Delta(G)+1$ colors introduced during phases (1) and (2) and at most $\lceil \frac{1}{2}(n(G) - 2\Delta(G) - 1) \rceil$ colors introduced in phase (3) by Bob, which gives at most $\lceil \frac{1}{2}(n(G)-1) \rceil + 1 = \lfloor \frac{1}{2} n(G) \rfloor + 1$ colors. □

Corollary 9. *If G is a subcubic graph with at least 10 vertices then $\chi_g^\infty(G) = \lfloor \frac{1}{2} n(G) \rfloor + 1$.*

Proof. It follows immediately from Theorem 8 and definition of subcubic graphs as graphs with maximum degree not exceeding 3. □

Exhaustive computer search that we performed on all subcubic graphs with at most 9 vertices shows that there are only four subcubic graphs with $\chi_g^\infty(G) > \lfloor \frac{1}{2} n(G) \rfloor + 1$, namely K_3, C_4, K_4 without one edge and K_4. Combining this with Proposition 2 we get the following formula:

Corollary 10. *If G is a subcubic graph then*

$$\chi_g^\infty(G) = \begin{cases} 3 & \text{if } G \text{ is } K_3, \\ 4 & \text{if } G \text{ is } C_4,\ K_4 \text{ without an edge or } K_4, \\ \lfloor \frac{1}{2} n(G) \rfloor + 1 & \text{otherwise.} \end{cases} \qquad \square$$

4. General results

Theorem 11. *For any graph G with an odd number of vertices it is true that $\chi_g^\infty(G) \leq n(G) - \alpha'(\bar{G})$.*

Proof. Let $M \subseteq E(\bar{G})$ be a maximum matching in $\bar{G}$. Then all vertices outside of M form a clique in G—let us denote this set by K. Alice adopts the following strategy:

(1) start in K by picking any vertex and assign to it color 1,
(2) if Bob colored a vertex in K in his last move, pick any uncolored vertex in K and assign to it a new color,
(3) if Bob colored a vertex $u \notin K$ in his last move, then pick v such that $uv \in M$ and use exactly the same color as Bob used for u, when it was a new color or use a new color, otherwise.

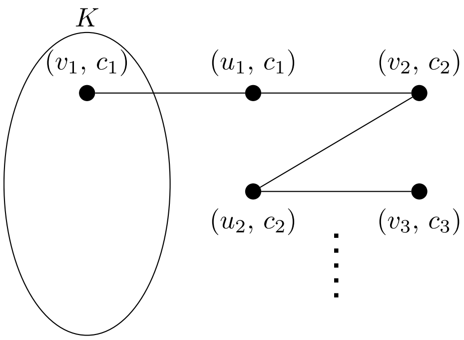

Fig. 2. An example path used in a construction of function f.

This strategy works since K has an odd number of vertices and $V(G) \setminus K$ has an even number of vertices. The number of colors will never exceed $|K|$ for K and $|M|$ for $V(G) \setminus K$, because for each edge of M we use exactly one new color. Therefore in total it will not exceed $|K| + |M| = n(G) - 2\alpha'(\bar{G}) + \alpha'(\bar{G}) = n(G) - \alpha'(\bar{G})$. □

Theorem 12. *For all graphs G it holds that* $\chi_g^{\infty}(G) \geq n(G) - \alpha'(\bar{G})$.

Proof. Let us use M and K exactly as defined in the proof above. Bob divides the game into two phases: (1) there are uncolored vertices in K and (2) all vertices in K are colored. Bob's strategy for phase (1) is as follows:

(a) if Alice picked a vertex from K in her last move, pick any uncolored vertex of K and color it with a new color,
(b) if Alice picked a vertex $u \notin K$ in her last move, then pick v such that $uv \in M$ and color it with a new color.

During phase (2) Bob picks any uncolored vertex and colors it with a new color.

It is easy to see that the above strategy works. Let us consider the moment when the last vertex of K is colored. By the strategy, when it happens all edges in M have either none or both vertices colored—and note that every edge has different colors at its endpoints. Denote by $M' \subseteq M$ the set of edges with colored both ends. Let us also denote by C_A^K, C_B^K, C_A^M, C_B^M colors used by Alice and Bob in K and M', respectively.

We will prove that $|C_A^K \cap C_B^M| \leq |C_A^M \setminus (C_B^M \cup C_B^K \cup C_A^K)|$. To this end, we will define an injective function $f : C_A^K \cap C_B^M \to C_A^M \setminus (C_B^M \cup C_B^K \cup C_A^K)$. Let us pick $c_1 \in C_A^K \cap C_B^M$. There exists exactly one $v_1 \in K$, which has been assigned c_1 by Alice (because K is a clique in G). Since $c_1 \in C_B^M$, there exists an edge $u_1v_2 \in M$ such that u_1 was colored with c_1 by Bob. Due to the Bob's strategy, it means that v_2 was colored by Alice—let us denote this color by c_2.

Clearly, $c_2 \in C_A^M$. Moreover, $c_2 \notin C_A^K \cup C_B^K$, because otherwise it would mean that v_2 has a non-neighbor $w \in K$ in G—but then there would be an augmenting path v_1, u_1, v_2, w in $\bar{G}$, so M would not be a maximum matching in $\bar{G}$. Now if $c_2 \notin C_B^M$, then clearly $c_2 \in C_A^M \setminus (C_B^M \cup C_B^K \cup C_A^K)$, so we just assign $c_2 = f(c_1)$.

Otherwise, there is a vertex $u_2 \in V(G) \setminus K$, to which Bob assigned color c_2. Due to the strategy there has to be a vertex v_3 such that Alice colored it with some c_3. We have now $c_3 \in C_A^M \setminus (C_B^K \cup C_A^K)$ so similarly as before: either $c_2 \notin C_B^M$ (and then we assign $c_3 = f(c_1)$), or $c_2 \in C_B^M$—and we repeat the step, as shown in Fig. 2. By the strategy of Bob, all c_i are different, so obviously the whole procedure has to end after at most $|M'|$ steps and the value of $f(c_1)$ is well defined.

This function is injective, because if there was $c_1 \neq c_1'$ such that $f(c_1) = f(c_1')$, we would have both paths joining, as shown in Fig. 3. However, it would mean that the first vertex (counting from v_1 and v_1') belonging to both paths is either some v_j or u_j. The first case is impossible—since paths are defined in such a way that $u_{j-1}v_j \in M$, if v_j is common for both paths, then so is u_{j-1}. But if u_j is common for both paths, then a path from v_1 through u_j to v_1' is an augmenting path for M in G—which contradicts the maximality of M.

Therefore, our function is injective and $|C_A^K \cap C_B^M| \leq |C_A^M \setminus (C_B^M \cup C_B^K \cup C_A^K)|$—and the number of colors used at the moment of coloring the last vertex of K can be bounded in the following way:

$$\begin{aligned} |C_B^M \cup C_A^M \cup C_B^K \cup C_A^K| &= |C_B^M| + |C_B^K| \\ &\quad + |C_A^K \setminus C_B^M| + |C_A^M \setminus (C_B^M \cup C_B^K \cup C_A^K)| \\ &\geq |M'| + |C_B^K| + |C_A^K \setminus C_B^M| + |C_A^K \cap C_B^M| \\ &= |M'| + |C_B^K| + |C_A^K| = |M'| + |K|. \end{aligned}$$

Here we used the fact that by Bob's strategy C_B^K is disjoint from C_B^M, and by the fact that in clique the players cannot repeat colors it is disjoint from C_A^K—and therefore $|C_A^K| + |C_B^K| = |K|$.

Note also that in later game there are only vertices belonging to a matching $M \setminus M'$ left, so Bob will use exactly $|M| - |M'|$ new colors. Thus, the final number of colors will be at least $|M| + |K| = n(G) - \alpha'(\bar{G})$. □

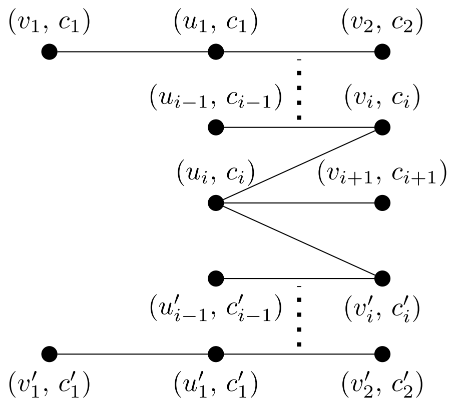

Fig. 3. An example structure occurring whenever $c_1 \neq c'_1$ and $f(c_1) = f(c'_1)$.

Corollary 13. *For graphs with an odd number of vertices it holds that* $\chi_g^\infty(G) = n(G) - \alpha'(\bar{G})$. □

Note that when $n(G)$ is even, the equality $\chi_g^\infty(G) = n(G) - \alpha'(\bar{G})$ may not hold. For example, for the complement G of the graph $K_{2,2,\ldots,2}$ (k twos) we have $\alpha'(G) = k$. But at the same time we know that $\chi_g^\infty(K_{2,2,\ldots,2}) = 2k$, which directly follows from Proposition 2.

On the other hand, we see that Corollary 13 provides a stronger bound than Theorem 8 for graphs with odd number of vertices:

Proposition 14. *If* $n(G)$ *is odd,* $n(G) \geq 3$ *and* $\Delta(G) \leq \lfloor \frac{1}{2} n(G) \rfloor - 1$ *then* $\chi_g^\infty(G) = \lfloor \frac{1}{2} n(G) \rfloor + 1$.

Proof. Since $\Delta(G) \leq \lfloor \frac{1}{2} n(G) \rfloor - 1$, we know that $\delta(\bar{G}) = n(G) - 1 - \Delta(G) \geq \lceil \frac{1}{2} n(G) \rceil$. This, by Dirac's theorem, implies that $\bar{G}$ is Hamiltonian and as such contains a matching of size $\lfloor \frac{1}{2} n(G) \rfloor$. The matching is clearly maximal, so Corollary 13 now gives $\chi_g^\infty(G) = n(G) - \lfloor \frac{1}{2} n(G) \rfloor = \lfloor \frac{1}{2} n(G) \rfloor + 1$. □

We note in passing that from the perspective of the players it may be important to distinguish between the strategies which can be obtained when they do not have unlimited computing power, but it is restricted to the execution of only polynomial-time algorithms. Recall that Theorems 1, 3 and 4 would require the knowledge of $\chi(G)$ or $\alpha(G)$, both parameters known to be NP-hard to compute in general. On the other hand, the strategies used in Theorems 11 and 12 are in this sense better for players, since $\alpha'(G)$ and the respective strategies for both Alice and Bob can be found in polynomial time. Similarly, all the strategies for particular classes of graphs presented in Section 3 can be run by players in polynomial time.

5. Relation between game chromatic number and infinite game chromatic number

Theorem 15. *For any graph G it holds that* $\chi_g(G) \leq \chi_g^\infty(G)$.

Proof. We will show that if $k \geq \chi_g^\infty(G)$ then there is a winning strategy for Alice in the standard chromatic game played on G with a set C of k colors. Note that without loss of generality we may assume that:

(1) $C = \{1, 2, \ldots, k\}$,
(2) when Bob and Alice use yet unused color, they pick always the smallest one possible.

Now it is sufficient to show that the optimal strategy for Alice in the infinite chromatic game can be used also in the chromatic game and it leads to her win.

Note that if at any point Alice or Bob cannot make move according to the rules of the chromatic game, it means that all colors in C are already in use. But now note that in the infinite chromatic game they would simply pick a new color, which contradicts the definition of k. Therefore it never happens and the whole graph gets colored which means that Alice wins. □

There are graphs for which the above inequality turns out to be equality. For instance, if $r_1 = r_2 = \cdots = r_{2l+1} = 1$ and $r_{2l+2} = r_{l+2} = \cdots = r_k = 2$ then $\chi_g^\infty(K_{r_1,r_2,\ldots,r_k}) = \chi_g(K_{r_1,r_2,\ldots,r_k}) = k$.

There are also graphs for which the inequality is strict. For example, it is known that $\chi_g^\infty(G) = 2n$ and $\chi_g(G) = 2n - 1$, where G is the complete graph K_{2n} with one edge e removed. The strategy to achieve the last equality is simple: in her moves Alice tries not to color any of the endpoint of e—unless Bob in his last move colored one, so Alice may color the other endpoint using the same color. Ultimately Alice may be forced to be first to color an endpoint of e in her final move, but then Bob will be forced to repeat this color in his final move.

The difference $\chi_g^\infty(G) - \chi_g(G)$ may be arbitrarily large. To prove that we need the following result.

Lemma 16. *For $k \geq 2$ and any $r_1, r_2, \ldots, r_k$ we have $\chi_g(K_{r_1,r_2,\ldots,r_k}) \leq 2k - 1$.*

Proof. Let $V_1, V_2, \ldots, V_k$ be a partition of $V(K_{r_1,r_2,\ldots,r_k})$ into independent sets such that $|V_i| = r_i$. The strategy for Alice is simple: in her first move she starts coloring each set V_i with first color available. Clearly, after at most $2k - 1$ moves in total (k by Alice, $k-1$ by Bob) it is achieved — and now she may proceed by repeating colors already used in the respective sets. Note that if the color set contains $2k - 1$ colors, then Alice can always achieve her goal and Bob is likewise limited to use this set of colors. □

Let $k \geq 2$, $r_1 = r_2 = \cdots = r_k = 4$ and $G = K_{r_1,r_2,\ldots,r_k}$. Using Lemma 16 and Theorem 6 we get $\chi_g^\infty(G) - \chi_g(G) \geq 3k - (2k - 1) = k - 1$.

6. Conclusions

The paper considers a chromatic game with a minor modification to the well-known one: there is no fixed set on allowed colors—but both players still aim to optimize the number of used colors in different ways. It turns out that there exists a fundamental asymmetry between the cases for graphs on even and odd number of vertices, as the optimal number of colors for the first one is always equal to $n(G) - \alpha'(\bar{G})$, but the second is only upper bounded by that value and may be significantly larger. Moreover, it was shown that this game may be used to bound from above the game chromatic number. Interestingly, even for the dense graphs it turns out that the difference between both chromatic numbers can be large, i.e. proportional to the number of vertices in the graph.

Finally, the problems of a complete account of values of $\chi_g^\infty(G)$ for even $n(G)$ and more precise analysis of a relation between $\chi_g^\infty(G)$ and $\chi_g(G)$ remains open for further investigation.

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