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INFLUENCE OF VORTICES ON A PROGRESSIVE QUASI-PLANE ACOUSTIC WAVE

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The projecting of the quasi-plane flow into specific modes yields in a set of coupled equations accounting for all possible interactions of the basic types of motion. A particular case of interaction considers vortices affecting the character of sound propagation. The new dynamic equations describing the propagation of a progressive acoustic beam interacting with a vortex background are derived and discussed. Since two acoustic branches become separated, these equations include the first order derivative with respect to time. It is the main result of the present paper. Illustrations on the scattered acoustic pressure referring to the different types of vortex flow are presented.

Key words: nonlinear acoustics, vortex flow, acoustic scattering.

1. Introduction

The general hydrodynamic flow is known to consist of different types of motion: the acoustic, vortex and entropy ones. In a flow over the uniform background, only acoustic waves associate with the change of pressure of the fluid. The dispersion relation and links of perturbations specific for every mode are determined by a linearized system of basic equations of conservation of mass, momentum and energy ([1] and papers referred there). There is a wide variety of interactions between different types of motion that have been observed experimentally [2–5]. They are typically nonlinear since the modes may interact due to nonlinear coupling of the correspondent terms of the dynamic equations. The governing equations of conservation in the differential form are nonlinear due to nonlinearity of the convective term in the momentum equation $\mathbf{V}(\nabla\mathbf{V})$, the nonlinearity of thermodynamic equations of state, and the damping nonlinearity in the energy balance equation. Some of these effects are related to the transfer of energy (acoustic heating) or momentum (acoustic streaming) from acoustic into non-acoustic motions due to attenuation.

The interaction of the acoustic and vortex flows belongs to the most complex problems of the nonlinear hydrodynamics ([6] and papers referred there). The first point of the problem is acoustic streaming, a phenomenon presupposing that the acoustic wave is dominant. Vortex flow following the acoustic wave grows with time due to nonlinear dissipation of the acoustic momentum. On the contrary, scattering of the acoustic wave at the vortices (being a special kind of obstacles) occurs even in a non-dissipative flow. Many applications of hydrodynamics and aerodynamics deal with generation of sound by a turbulent flow appearing at the background of aircrafts or submarines and ships. Since a vortex flow is highly noisy, the nonlinear mathematics is still a very difficult lock to pick.

Scattering of sound by vortices is a common subject in the nonlinear acoustics. The governing equation comes from the works of LIGHTHILL [7]. It is written for the acoustic pressure without distinguishing the branches of acoustic motion and therefore includes a second order derivative with respect to time and a spatial Laplace operator. In other words, the governing equation is a wave equation with a nonlinear source in the right-hand side. The advance in the nonlinear theory is a possibility to split the governing equation in separate ones for every acoustic branch. That is the main result of the present paper. Projecting which has been worked out by the author yields in the first order dynamic equations with respect to time for everyone of the two acoustic modes [8–10]. This is important in problems related to quasi-plane motion, where beam dynamics of a chosen direction of propagation is a subject of investigation. A remarkable achievement of the last decades is the equation of Khokhlov–Zabolotskaya–Kuznetsov (KZK) describing the nonlinear propagation of a progressive beam [11, 12]. It accounts for the diffraction and damping of the sound beam. The KZK equation is an immediate result of acting by a corresponding projector on the basic system of conservation laws [9]. In a similar manner, the equations governing the scattering of acoustic beams are the result of projecting. A subject of the present paper is to derive these equations and to give examples of their solutions.

2. Basic modes and correspondent projectors of the quasi-plane flow

The mass, momentum and energy conservation equations for a thermoviscous flow are:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}(\rho \mathbf{v}) &= 0, \\
\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\boldsymbol{\nabla})\mathbf{v}\right] &= -\boldsymbol{\nabla} p + \eta \Delta \mathbf{v} + \left(\varsigma + \frac{\eta}{3}\right) \boldsymbol{\nabla}(\boldsymbol{\nabla}\mathbf{v}), \\
\rho\left[\frac{\partial e}{\partial t} + (\mathbf{v}\boldsymbol{\nabla})e\right] + p\boldsymbol{\nabla}\mathbf{v} - \chi\Delta T &= \varsigma\left(\boldsymbol{\nabla}\mathbf{v}\right)^2 + \frac{\eta}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)^2.
\end{aligned} \tag{1}$$

Here, $\mathbf{v}$ denotes the velocity of the fluid, ρ, p are the density and pressure, e, T are internal energy per unit mass and temperature; ς, η, χ are bulk and shear viscosities and thermal conductivity (all supposed to be constants), x_i are space coordinates. Two thermodynamic functions $e(p,\rho)$ and $T(p,\rho)$ complete the system (1). Without loss of generality, let us consider an ideal gas treated by the thermodynamic functions as follows:

$$e(p,\rho)=\frac{p}{\rho(\gamma-1)},\qquad T(p,\rho)=\frac{p}{C_v\rho(\gamma-1)}\,, \tag{2}$$

where C_v and $\gamma=C_p/C_v$ mean the specific heat per unit mass at constant volume and the specific heats ratio, correspondingly.

In the quasi-plane geometry characterized by a small diffraction parameter μ which expresses the relation of the longitudinal (along the y-axis) and transverse (in the (x,z) plane) scales of perturbation, the equivalent system in the dimensionless variables (background quantities are marked by zero, perturbations are primed):

$$\begin{aligned}\mathbf{v}_*,\ \mathbf{x}_*,\ \rho_*,\ p_*,\ t_*:\quad & \mathbf{v}=c\mathbf{v}_*,\quad p'=c^2\rho_0p_*,\quad \rho'=\rho_0\rho_*,\\ & \mathbf{x}=(\lambda x_*/\sqrt{\mu},\ \lambda y_*,\ \lambda z_*/\sqrt{\mu}),\quad t=\lambda t_*/c,\end{aligned} \tag{3}$$

($c=\sqrt{\gamma p_0/\rho_0}$ is a small-signal sound velocity, λ means a characteristic scale of longitudinal perturbations), may be rewritten in the following form (asterisks for dimensionless variables will be omitted everywhere later):

$$\frac{\partial}{\partial t}\psi+L\psi=\varphi+\varphi_{tv}, \tag{4}$$

where ψ is a column of perturbations

$$\psi=\begin{pmatrix} v_x & v_y & v_z & p & \rho\end{pmatrix}^{\mathrm{T}}, \tag{5}$$

and L is the linear matrix operator:

$$L=\begin{pmatrix}
-\delta_1^1\mu\dfrac{\partial^2}{\partial x^2}-\delta_1^2\Delta & -\delta_1^1\sqrt{\mu}\dfrac{\partial^2}{\partial x\partial y} & -\delta_1^1\mu\dfrac{\partial^2}{\partial x\partial z} & \sqrt{\mu}\partial/\partial x & 0\\
-\delta_1^1\sqrt{\mu}\dfrac{\partial^2}{\partial x\partial y} & -\delta_1^1\dfrac{\partial^2}{\partial y^2}-\delta_1^2\Delta & -\delta_1^1\sqrt{\mu}\dfrac{\partial^2}{\partial y\partial z} & \partial/\partial y & 0\\
-\delta_1^1\mu\dfrac{\partial^2}{\partial x\partial z} & -\delta_1^1\sqrt{\mu}\dfrac{\partial^2}{\partial z\partial y} & -\delta_1^1\mu\dfrac{\partial^2}{\partial z^2}-\delta_1^2\Delta & \sqrt{\mu}\partial/\partial z & 0\\
\sqrt{\mu}\partial/\partial x & \partial/\partial y & \sqrt{\mu}\partial/\partial z & -\delta_2^1\Delta & -\delta_2^2\Delta\\
\sqrt{\mu}\partial/\partial x & \partial/\partial y & \sqrt{\mu}\partial/\partial z & 0 & 0
\end{pmatrix} \tag{6}$$

with the dimensionless coefficients

$$\delta_1^1=\frac{(\zeta+\eta/3)}{\rho_0c\lambda}\,,\qquad \delta_1^2=\frac{\eta}{\rho_0c\lambda}\,,\qquad \delta_2^1=\frac{\chi}{\rho_0c\lambda C_v}\,,\qquad \delta_2^2=-\frac{\chi}{\rho_0c\lambda C_p}\,.$$

The dimensionless operators ∇, Δ are: $\nabla = \begin{pmatrix} \sqrt{\mu}\partial/\partial x & \partial/\partial y & \sqrt{\mu}\partial/\partial z \end{pmatrix}$, $\Delta = \mu\partial^2/\partial x^2 + \partial^2/\partial y^2 + \mu\partial^2/\partial z^2$, φ is a quadratic nonlinear column:

$$\varphi = \begin{pmatrix} -(\mathbf{v}\nabla)v_x + \sqrt{\mu}\rho\partial p/\partial x \\ -(\mathbf{v}\nabla)v_y + \rho\partial p/\partial y \\ -(\mathbf{v}\nabla)v_z + \sqrt{\mu}\rho\partial p/\partial z \\ -\gamma p(\nabla\mathbf{v}) - (\mathbf{v}\nabla)p \\ -\rho(\nabla\mathbf{v}) - (\mathbf{v}\nabla)\rho \end{pmatrix}, \tag{7}$$

and φ_{tv} is a quadratic nonlinear column $O(\beta)$ appearing in the viscous flow [10].

For a linear flow defined by the linearized version of the system (4)

$$\frac{\partial}{\partial t}\psi + L\psi = 0, \tag{8}$$

a solution may be found as a sum of planar waves: $v_x = \widetilde{v}_x(\mathbf{k})\exp(i\omega t - i\mathbf{k}\mathbf{x})$, ... where

$$\mathbf{k} = (k_x,\, k_y,\, k_z)$$

is the wave vector. In the Fourier space, $-ik_x$ means $\partial/\partial x$, $i\omega$ means $\partial/\partial t$, and (8) yields in the five roots of the dispersion relation representing three basic types of motion in a compressible fluid: two acoustic beams, two vortices and an entropy mode. Only acoustic modes are progressive, the two other ones are related to motions of imaginary frequency close to zero.

The modes of a linear flow in $\mathbf{k}$-space are determined by the relations of amplitudes of planar waves $\tilde{v}_x(k_x,\, k_y,\, k_z),\ \ldots$. The corresponding eigenvectors fix these relations [10]:

$$\begin{aligned} \widetilde{\psi}_1 &= \begin{pmatrix} \sqrt{\mu}k_x/k_y \\ 1 - \mu(k_x^2 + k_z^2)/(2k_y^2) + i\beta k_y/2 \\ \sqrt{\mu}k_z/k_y \\ 1 + i(\delta_2^1 + \delta_2^2)k_y \\ 1 \end{pmatrix}\widetilde{\rho}_1, \\ \widetilde{\psi}_2 &= \begin{pmatrix} -\sqrt{\mu}k_x/k_y \\ -1 + \mu(k_x^2 + k_z^2)/(2k_y^2) + i\beta k_y/2 \\ -\sqrt{\mu}k_z/k_y \\ 1 - i(\delta_2^1 + \delta_2^2)k_y \\ 1 \end{pmatrix}\widetilde{\rho}_2, \end{aligned} \tag{9}$$

$$\widetilde{\psi}_3 = \begin{pmatrix} 0 \\ -i\delta_2^2 k_y \\ 0 \\ 0 \\ 1 \end{pmatrix} \widetilde{\rho}_3, \quad \widetilde{\psi}_4 = \begin{pmatrix} ik_y \\ -i\sqrt{\mu}k_x \\ 0 \\ 0 \\ 0 \end{pmatrix} \widetilde{\varphi}_4, \quad \widetilde{\psi}_5 = \begin{pmatrix} 0 \\ -i\sqrt{\mu}k_z \\ ik_y \\ 0 \\ 0 \end{pmatrix} \widetilde{\varphi}_5, \tag{9 [cont.]}$$

where $\beta = \delta_1^1 + \delta_1^2 + \delta_2^1 + \delta_2^2 = \dfrac{\zeta + 4\eta/3}{\rho_0 c\lambda} + \dfrac{\chi}{\rho_0 c\lambda}\left(\dfrac{1}{C_v} - \dfrac{1}{C_p}\right)$ is an overall attenuation, and the factor $(-ik_y)^{-1}$ represents the operator $\int \mathrm{d}y$ in $\mathbf{k}$-space. The first two eigenvectors are acoustic modes, the rightwards and leftwards progressive ones, the third one is an entropy mode, and two last ones are vortex modes.

All calculations of the modes and projectors were undertaken with an accuracy up to terms of the order μ^1, β^1. Any linear flow $\widetilde{\psi}$ being a solution of the linearized matrix equation (8), is a sum of independent modes. Every specific mode may be decomposed from the overall vector of perturbations by linear matrix projectors [8–10]:

$$\widetilde{P}_1\widetilde{\psi} = \widetilde{\psi}_1, \qquad ..., \qquad \widetilde{P}_5\widetilde{\psi} = \widetilde{\psi}_5. \tag{10}$$

They form a full set of orthogonal operators and satisfy their common properties.

Acoustic projectors calculated with the accuracy of order μ, β have the form:

$$\widetilde{P}_{1,2} = \left(\begin{matrix} \mu\dfrac{k_x^2}{2k_y^2} & \sqrt{\mu}\dfrac{k_x}{2k_y} \\ \sqrt{\mu}\dfrac{k_x}{2k_y} & \dfrac{1}{2}\left(1 \pm \dfrac{i\beta}{2}k_y \mp i(\delta_2^1 + \delta_2^2)k_y - \mu\dfrac{k_x^2 + k_z^2}{2k_y^2}\right) \\ \mu\dfrac{k_xk_z}{2k_y^2} & \sqrt{\mu}\dfrac{k_z}{2k_y} \\ \pm\sqrt{\mu}\dfrac{k_x}{2k_y} & \pm\dfrac{1}{2}\left(1 - \mu\dfrac{k_x^2 + k_z^2}{2k_y^2}\right) \\ \pm\sqrt{\mu}\dfrac{k_x}{2k_y} & \dfrac{1}{2}\left(\pm 1 - i(\delta_2^1 + \delta_2^2)k_y \mp \mu\dfrac{k_x^2 + k_z^2}{2k_y^2}\right) \end{matrix}\right.$$

$$\left.\begin{matrix} \mu\dfrac{k_xk_z}{2k_y^2} & \pm\sqrt{\mu}\dfrac{k_x}{2k_y} & 0 \\ \sqrt{\mu}\dfrac{k_z}{2k_y} & \dfrac{1}{2}\left(\pm 1 - i\delta_2^2 k_y - \mu\dfrac{k_x^2 + k_z^2}{2k_y^2}\right) & \dfrac{i\delta_2^2 k_y}{2} \\ \mu\dfrac{k_z^2}{2k_y^2} & \pm\sqrt{\mu}\dfrac{k_z}{2k_y} & 0 \\ \pm\sqrt{\mu}\dfrac{k_z}{2k_y} & \dfrac{1}{2}(1 \mp \dfrac{i\beta}{2}k_y \pm i\delta_2^1 k_y) & \pm\dfrac{i\delta_2^2 k_y}{2} \\ \pm\sqrt{\mu}\dfrac{k_z}{2k_y} & \dfrac{1}{2}(1 \mp \dfrac{i\beta}{2}k_y \mp i\delta_2^2 k_y) & \pm\dfrac{i\delta_2^2 k_y}{2} \end{matrix}\right). \tag{11}$$

3. Dynamic equation for the rightwards beam affected by vortices

Nonlinear dynamic equations follow from the basic nonlinear system (4) while acting on it by the corresponding projector. The projecting does not need a temporal averaging, so a wide variety of problems concerning the propagation of the aperiodic acoustic wave may be solved (acoustic heating and streaming [10]). The other remarkable possibility of projecting is to yield in dynamic equations for every branch of acoustic and vortex modes separately.

The definition of the modes in the uniform and isotropic media does not depend on the presence of boundaries and the boundary or initial conditions. Specific for the boundary valued or initial problems modes are the superposition of the modes (9) satisfying the boundary regime or initial conditions. A flow in closed volumes supposes the existence of two acoustic modes to support the boundary conditions on the rigid boundaries. Standing waves of a finite spectrum in closed tubes are an example of superposition. The knowledge of the initial amplitude of every mode allows to solve the problem by a set of successful approximations basing on a dominant mode at the beginning of evolution (or, alternatively, at the boundary).

Acting by the the fourth row of the projector P_1 of (11) on both parts of the system (4) and taking into account the inputs of the acoustic modes and vortices in the right-hand nonlinear part of (4) in accordance to links (9), give a following dynamic equation:

$$\frac{\partial p_1}{\partial t} + \frac{\partial p_1}{\partial y} + \frac{\mu}{2}\int \Delta_\perp p_1 \,\mathrm{d}y - \frac{\beta}{2}\frac{\partial^2 p_1}{\partial y^2} + \frac{\gamma+1}{2} p_1 \frac{\partial}{\partial y} p_1 = Q_1, \tag{12}$$

with a source representing mixed acoustic-vortex and quadratic vortex terms Q_1:

$$Q_1 = -\frac{1}{2}\boldsymbol{\nabla}\int \big((\mathbf{U}\boldsymbol{\nabla})\mathbf{U} + (\mathbf{V}\boldsymbol{\nabla})\mathbf{U} + (\mathbf{U}\boldsymbol{\nabla})\mathbf{V}\big)\mathrm{d}y - \frac{1}{2}(\mathbf{U}\boldsymbol{\nabla})(p_1+p_2). \tag{13}$$

In the formula (13), $\mathbf{V} = \mathbf{v}_1 + \mathbf{v}_2$ is a part of velocity corresponding to both the acoustic modes, $\mathbf{U}$ denotes vortex velocity. The quadratic acoustic term (13) is written for the rightwards progressive beam, the effects of scattering of sound by sound are not taken into account. Similarly, acting by the fourth row of P_2 of (11) on (4) yields in dynamic equation for the pressure of the leftwards propagating beam:

$$\frac{\partial p_2}{\partial t} - \frac{\partial p_2}{\partial y} - \frac{\mu}{2}\int \Delta_\perp p_2 \,\mathrm{d}y - \frac{\beta}{2}\frac{\partial^2 p_2}{\partial y^2} - \frac{\gamma+1}{2} p_2 \frac{\partial}{\partial y} p_2 = Q_2, \tag{14}$$

with a source Q_2:

$$Q_2 = \frac{1}{2}\boldsymbol{\nabla}\int \left((\mathbf{U}\boldsymbol{\nabla})\mathbf{U} + (\mathbf{V}\boldsymbol{\nabla})\mathbf{U} + (\mathbf{U}\boldsymbol{\nabla})\mathbf{V}\right)\mathrm{d}y - \frac{1}{2}(\mathbf{U}\boldsymbol{\nabla})(p_1+p_2). \tag{15}$$

The right-hand sides of the dynamic equations (12) and (14) are evaluated with the accuracy $O(\mu, \beta)$. In fact there are the famous Khokhlov–Zabolotskaya–Kuznetsov equations with a source. All the cross terms in the sources (13), (15) show how the well-known formula for the scattering of the acoustic wave follows from combining of Eqs. (12) and (14). Acting on their sum by the operator $\partial/\partial t$, and on their difference by the operator $\boldsymbol{\nabla}$, and combining the equations, one comes finally in the leading order to an equation for the acoustic pressure $P = p_1 + p_2$:

$$\begin{aligned}\frac{\partial^2 P}{\partial t^2} - \Delta P &= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial y}\right) Q_1 + \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right) Q_2 \\ &= -\frac{\partial}{\partial t}(\mathbf{U}\boldsymbol{\nabla})P + \boldsymbol{\nabla}\big((\mathbf{U}\boldsymbol{\nabla})\mathbf{U} + (\mathbf{V}\boldsymbol{\nabla})\mathbf{U} + (\mathbf{U}\boldsymbol{\nabla})\mathbf{V}\big). \qquad (16)\end{aligned}$$

Furthemore, one should take into account that the vortex flow is solenoidal $\boldsymbol{\nabla}\mathbf{U} = 0$, and the acoustic velocity is divergent $\boldsymbol{\nabla} \times \mathbf{V} = 0$, that follows also from the links (9), and the equalities from the vector analysis

$$\begin{aligned}\boldsymbol{\nabla}(\mathbf{V}\mathbf{U}) &= (\mathbf{V}\boldsymbol{\nabla})\mathbf{U} + (\mathbf{U}\boldsymbol{\nabla})\mathbf{V} + \mathbf{V} \times (\boldsymbol{\nabla} \times \mathbf{U}) + \mathbf{U} \times (\boldsymbol{\nabla} \times \mathbf{V}), \\ \boldsymbol{\nabla}(P\mathbf{U}) &= (\mathbf{U}\boldsymbol{\nabla})P + P(\boldsymbol{\nabla}\mathbf{U}).\end{aligned} \qquad (17)$$

Finally, the dynamic equation (16) goes to the next one:

$$\frac{\partial^2 P}{\partial t^2} - \Delta P = -\frac{\partial}{\partial t}\boldsymbol{\nabla}(P\mathbf{U}) + \boldsymbol{\nabla}\big((\mathbf{U}\boldsymbol{\nabla})\mathbf{U} + \boldsymbol{\nabla}(\mathbf{V}\mathbf{U}) - \mathbf{V} \times (\boldsymbol{\nabla} \times \mathbf{U})\big), \qquad (18)$$

which agrees to the dynamic equation by CHU, KOVASZNAY [1].

The deriving of the separate dynamic equations for the rightwards and leftwards propagating beams (12), (14) develops the up-to-date theory. In the frames of the quasi-plane geometry of the flow, it becomes possible not only to separate the wave equation for the overall acoustic pressure into two specific dynamic equations, but also to simplify them considerably in the leading order accounting for the specific links given by the eigenvectors (9).

4. Influence of vortex motion on the dynamics of a progressive acoustic beam

As we can conclude from the previous section, the sources on the right-hand side of Eqs. (12) and (14) caused by the interaction of acoustic and vortex modes appear even effects of viscosity and thermal conductivity are not taken into account. This exhibits the specific nonlinear interaction. The dynamic equations for the rightwards and leftwards progressive beams in the leading order are:

$$
\begin{aligned}
&\frac{\partial p_1}{\partial t} + \frac{\partial p_1}{\partial y} + \frac{\mu}{2}\int \Delta_\perp p_1 \,\mathrm{d}y - \frac{\beta}{2}\frac{\partial^2 p_1}{\partial y^2} + \frac{\gamma+1}{2} p_1 \frac{\partial}{\partial y} p_1 \\
&\qquad\qquad = -\int \left(\sqrt{\mu}\frac{\partial U_y}{\partial x}\frac{\partial U_x}{\partial y} + \left(\frac{\partial U_y}{\partial y}\right)^2 \right)\mathrm{d}y \\
&\qquad - \left(\sqrt{\mu}U_x\frac{\partial}{\partial x} + U_y\frac{\partial}{\partial y}\right)p_1 - \frac{1}{2}p_1\frac{\partial U_y}{\partial y} - \frac{1}{2}\sqrt{\mu}\frac{\partial}{\partial x}\int\left(p_1\frac{\partial U_x}{\partial y}\right)\mathrm{d}y, \\
&\frac{\partial p_2}{\partial t} - \frac{\partial p_2}{\partial y} - \frac{\mu}{2}\int \Delta_\perp p_2 \,\mathrm{d}y - \frac{\beta}{2}\frac{\partial^2 p_2}{\partial y^2} - \frac{\gamma+1}{2} p_2 \frac{\partial}{\partial y} p_2 \\
&\qquad\qquad = \int \left(\sqrt{\mu}\frac{\partial U_y}{\partial x}\frac{\partial U_x}{\partial y} + \left(\frac{\partial U_y}{\partial y}\right)^2 \right)\mathrm{d}y \\
&\qquad - \left(\sqrt{\mu}U_x\frac{\partial}{\partial x} + U_y\frac{\partial}{\partial y}\right)p_2 - \frac{1}{2}p_2\frac{\partial U_y}{\partial y} - \frac{1}{2}\sqrt{\mu}\frac{\partial}{\partial x}\int\left(p_2\frac{\partial U_x}{\partial y}\right)\mathrm{d}y.
\end{aligned}
\tag{19}
$$

In calculations of (19), the links of perturbations inside every mode accordingly to (9) are accounted: v_x, U_y are of order $O(\sqrt{\mu})$, v_y, U_x are of order $O(1)$, $v_{y,1} = p_1 + O(\mu,\beta)$, $v_{y,2} = -p_2 + O(\mu,\beta)$. The right-hand parts are evaluated with the accuracy $O(\sqrt{\mu})$, and only the vortex mode in the (x,y) plane is kept for simplicity which corresponds to the eigenvector $\tilde{\psi}_4$ from the set of (9). The Eqs. (12)–(15), and, consequently, (19) are the basic result of the present paper: they are governing equations for determining the quantities of the scattered acoustic pressure for every progressive acoustic mode separately.

4.1. Scattering of the beam at the vortices

Let us concentrate on the first one from Eqs. (19). The further solution of the equation depends on the initial conditions. At least, the vortex mode should be enough developed with an amplitude of the fluid particles velocity of order of that in the pressure wave. It may itself be caused by losses in the momentum of flow. For example, it may develop at the background of an obstacle [6]. We leave here out of account the problem of generation of sound by vortices, and therefore do not consider the quadratic vortex term in the source Q_1. An effective generation may exist in a special case of a non-stationary vortex flow [6, 12]. In particular, it is a subject of aerodynamic researches of the atmosphere motion caused by aircrafts. Acoustic streaming is hardly expected to give rise to a pressure beam not only because of its comparatively small amplitude. The correspondent wavenumbers and frequencies should be close for successive interaction. The linear definition of the vortex flow gives a frequency proportional to viscosity $\omega = i\delta_1^2\left(k_y + \mu\frac{k_x^2+k_z^2}{2k_y}\right)^2$ which is an imagine value responsible for the damping of the vortex flow.

Scattering of the acoustic beam at the vortices occurs due to the cross terms on the right-hand side of the first equation in (19). As a standard procedure, the method of successive approximations will be used: a pressure of the acoustic rightwards beam is sought as a sum of non-scattered quantity $p_1^{(0)}$ and a scattered one $p_1^{(1)}$. The first equation of (19) splits into two ones:

$$\frac{\partial p_1^{(0)}}{\partial t}+\frac{\partial p_1^{(0)}}{\partial y}+\frac{\mu}{2}\int\limits_{\infty}^{y}\Delta_\perp p_1^{(0)}\,\mathrm{d}y-\frac{\beta}{2}\frac{\partial^2 p_1^{(0)}}{\partial y^2}+\frac{\gamma+1}{2}p_1^{(0)}\frac{\partial}{\partial y}p_1^{(0)}=0,$$

$$\frac{\partial p_1^{(1)}}{\partial t}+\frac{\partial p_1^{(1)}}{\partial y}=-\left(\sqrt{\mu}U_x\frac{\partial}{\partial x}+U_y\frac{\partial}{\partial y}\right)p_1^{(0)} \tag{20}$$

$$-\frac{1}{2}\sqrt{\mu}\frac{\partial}{\partial x}\int\limits_{\infty}^{y}\left(p_1^{(0)}\frac{\partial U_x}{\partial y}\right)\mathrm{d}y-\frac{1}{2}p_1^{(0)}\frac{\partial U_y}{\partial y}=Q(\mathbf{U},p_1^{(0)})=Q(x,y,t).$$

The last equation may be integrated along characteristic to find $p_1^{(1)}(x,\xi=y-t, z,\eta=y)$:

$$\frac{\partial p_1^{(1)}}{\partial \eta}=Q(x,\eta,z,\eta-\xi),\qquad p_1^{(1)}=\int\limits_{\infty}^{y}Q(x,\eta,z,\eta-\xi)\,\mathrm{d}\eta+F(x,\xi,z), \tag{21}$$

where $F(x,\xi,z)$ is a function independent of η and it should satisfy the boundary and initial conditions. Without loss of generality, we set it zero for all arguments. The domain of longitudinal integration is $[y,\infty]$ for sound vanishing at infinity.

It is well-known that the first equation of the set (20) (KZK equation) is very complex for an analytic solution [12]. In order to illustrate the results, any simple approximate solution is suitable. Acoustic pressure corresponding to the non-diffracting beam propagating in the positive direction of axis y is used as example:

$$p_1^{(0)}=P_0\exp(-\beta y)\,\sin(y-t)\,\exp(-x^2). \tag{22}$$

It is valid starting from some distance from the transducer and does not depend on the amplitude at it [12]. In view of the vorticity being the independent on z, mode $\tilde{\psi}_4$ from the set of (9), the geometry of the problem becomes two-dimensional. The vortex flow develops in the (x,y) plane and must satisfy the equation $\boldsymbol{\nabla}\mathbf{U}=0$. There is an infinitely number of vortices of this kind. In the quasi-plane geometry, the velocity components may be taken, among others, in the form:

$$U_x=\frac{\partial}{\partial y}F(x^2/\mu+y^2,t),\qquad U_y=-\sqrt{\mu}\frac{\partial}{\partial x}F(x^2/\mu+y^2,t), \tag{23}$$

where F is any smooth function.

To satisfy the condition on the boundary (in the plane of the transducer) $y = 0$: $U_y = 0$, the vortex flow is taken in the following form:

$$\begin{aligned} U_x &= U_0 \exp(-2y) T(x/\sqrt{\mu}, t), \\ U_y &= 0.5 U_0 \big(\exp(-2y) - 1\big) \sqrt{\mu} \partial T(x/\sqrt{\mu}, t)/\partial x, \end{aligned} \tag{24}$$

where T is a smooth function. In the calculations we use two different functions corresponding to different types of the vortex flow:

$$\begin{aligned} T_1(x/\sqrt{\mu}, t) &= \exp(-x^2/\mu - t^2), \\ T_2(x/\sqrt{\mu}, t) &= x \cdot \exp(-x^2/\mu - t^2). \end{aligned} \tag{25}$$

Note that the functions T_1, T_2 achieve a maximum at $t = 0$, so that the vortex flow exists at negative t, though quickly decreases. An appropriate shift t_0 may be chosen to get the local maximum at $t = t_0$.

The results of numerical calculations of the scattered acoustic pressure accordingly to the formula (21) are presented in the figures below. Both sets of calculations refer to values of $\mu = 0.01$ and $\beta = 0.01$. The leading acoustic pressure has the form (22).

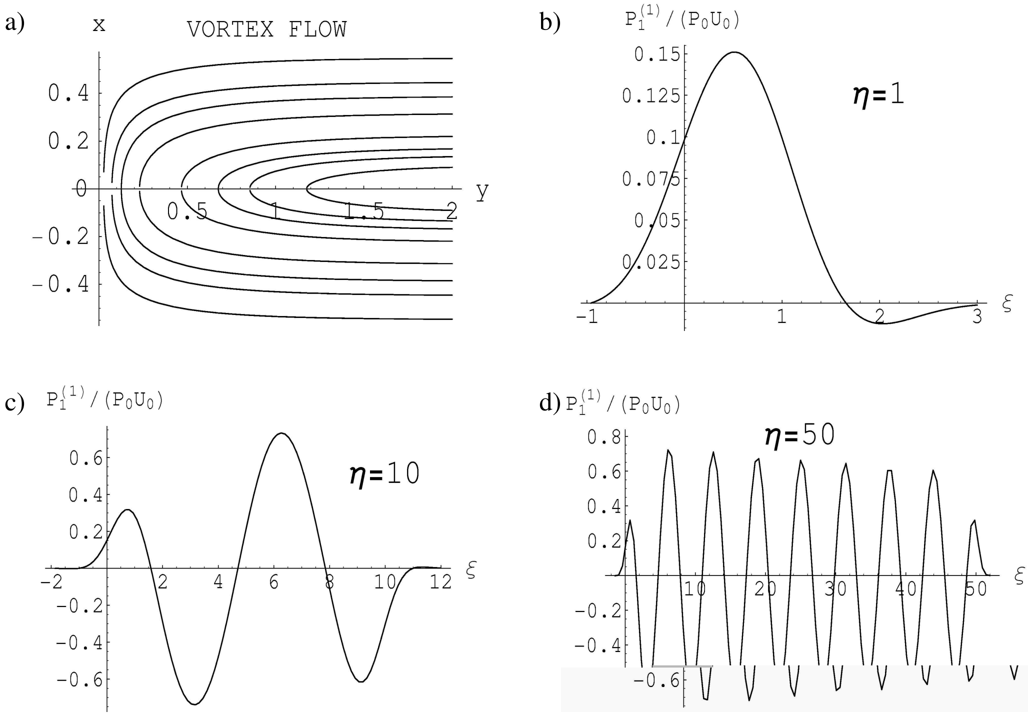

Fig. 1. a) Stationary streamlines of the scattering vortex flow in the plane (x, y). The direction of flow is clockwise; b), c), d) a ratio of the scattered pressure $P_1^{(1)}$ and a product of the amplitudes of the dominant beam pressure and vorticity P_0U_0 as a function of ξ at different $\eta = y - t$.

Figure 1 demonstrates a scattered acoustic pressure for vortex flow given by (24) with $T = T_1(x/\sqrt{\mu}, t)$ from (25). The curves are related to $x = X = \sqrt{0.5\mu/(\mu+1)}$. The scattered pressure achieves a maximum at $x = \pm X$. The figure shows streamlines of the vortex flow and ratios of the scattered pressure $P_1^{(1)}$ and a product of the amplitudes of the dominant beam pressure and vorticity P_0U_0 as a function of ξ for different η.

Figure 2 relates to vortex flow with $T = T_2(x/\sqrt{\mu}, t)$ from (25). Though streamlines are symmetric analogously to the previous case, the vertical velocity changes sign while going from the upper to the lower half-space: the direction of flow is clockwise in the upper half-space ($x > 0$), and it is counterclockwise in the lower one ($x < 0$). The illustrations are related to $x = 0$. In both the examples, the streamlines are stationary due to the similar dependence of the velocity components on time accordingly to (24).

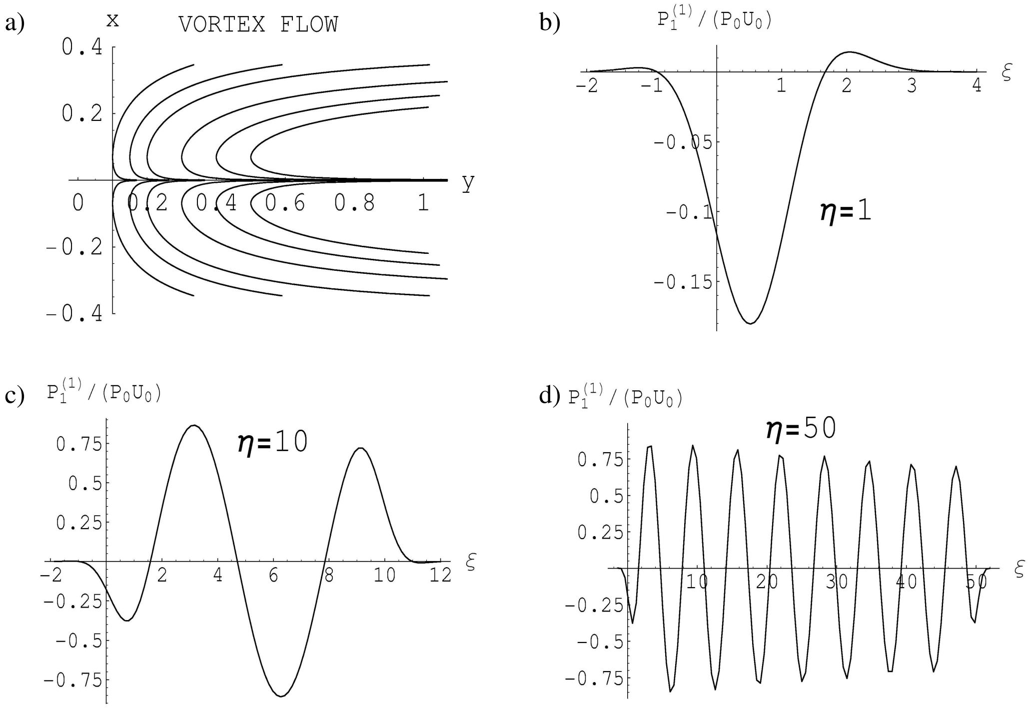

Fig. 2. a) Stationary streamlines of the scattering vortex flow in the plane (x, y). The direction of flow is clockwise in the upper half-space ($x > 0$), and it is counterclockwise in the lower one ($x < 0$); b), c), d) a ratio of the scattered pressure $P_1^{(1)}$ and a product of the amplitudes of the dominant beam pressure and vorticity P_0U_0 as a function of ξ at different $\eta = y - t$.

The analysis shows that while η increases, the quantity $P_1^{(1)}/(P_0U_0)$ oscillates from approximately -0.7 and 0.7. The numerical examples are rather demonstrative in view of the difficulties of double integration of the governing equation (the second from (20)).

The initial waveform $p_1^{(0)}$ must satisfy the KZK equation, that complicates the problem essentially.

5. Conclusions

Interaction of the pressure waves with the non-acoustic types of flow, and in particular, the scattering of the acoustic waves on the vortices, belong to the most important problems of fluid dynamics. The problem may be subdivided into specific problems: 1) the definition of the acoustic and non-acoustic motions, that may be a complex problem itself in a nonuniform, affected by external forces media; 2) the deriving of equations governing the dynamics of the progressive beam on the base on the projecting of the overall system of conservative laws into the specific evolution equations for every mode accounting nonlinear interactions with itself and other modes; 3) the consequent analytical or numerical solution.

The advantage of the projecting is mostly in the deriving of governing evolution equations including the first order derivative with respect to time instead of second order equations, where the acoustic branches are not distinguished. The projecting allows to separate the equations for every branch of acoustic or vortex modes. Equation (12) accounts for the distortions of the rightwards progressive beam, in contrast to Eq. (16) which is responsible for distortions of the total acoustic pressure. The approximate solution is of a great difficulty because of the presence of the vortex flow supposes the going out of one dimension. Therefore, an equation to be solved is of the KZK type in the quasi-plane geometry with the nonlinear right-hand side including integro-differential operators. Calculations in accordance to formulae (20) and (21) need double integration with the proper boundary conditions. In the frames of the method, a scattered wave induced by the vortex-sound interaction is considered. The vortex flow itself may be a result of a loss in momentum of acoustic dominative beam. The proper vortex flow should be evaluated in accordance with the equations governing the acoustic streaming [13].

Approximate governing equations depend on the initial amplitudes of sound and the vortex flow: it is of importance, which kinds of nonlinear terms should be kept in the source part. In the present paper, the vortex and sound amplitudes are of the same order. Numerical estimations show that the ratio of amplitudes of the scattered and initial pressure waves tends approximately to 0.7 with the growth of time. In general, a vortex flow does not satisfy the links (23) or (24), it is solenoidal but depends on the concrete problem to be solved.

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