Inseparability criteria based on matrices of moments

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Inseparability criteria for continuous and discrete bipartite quantum states based on moments of annihilation and creation operators are studied by developing the idea of Shchukin-Vogel criterion [Phys. Rev. Lett. **95**, 230502 (2005)]. If a state is separable, then the corresponding matrix of moments is separable too. Thus, we derive generalized criteria based on the separability properties of the matrix of moments. In particular, a criterion based on realignment of moments in the matrix is proposed as an analog of the standard realignment criterion for density matrices. Other inseparability inequalities are obtained by applying positive maps to the matrix of moments. Usefulness of the Shchukin-Vogel criterion to describe bipartite-entanglement of more than two modes is demonstrated: we obtain various three-mode inseparability criteria, including some previously known ones, which were originally derived from the Cauchy-Schwarz inequality.

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I. INTRODUCTION

In recent years, the study of continuous-variable (CV) systems from the point of view of quantum information has attracted much interest, stimulated by experimental progress (see [1,2] and references therein). In particular, the theory of quantum entanglement for CV systems has been considerably developed, including the derivation by Shchukin and Vogel [3] of a powerful inseparability criterion of bipartite harmonic quantum states based on partial transposition (PT) [4,5], the so-called positive partial transposition (PPT) criterion. The PPT criterion says that a separable state remains positive under partial transposition, therefore a nonpositive-partial-transposition (NPT) state must be entangled. Shchukin and Vogel have demonstrated that their criterion includes, as special cases, other well-known criteria of entanglement in two-mode CV systems, including those derived by Simon [6], Duan *et al.* [7], Mancini [8], Raymer *et al.* [9], Agarwal and Biswas [10], Hillery and Zubairy [11]. Thus, the Shchukin-Vogel (SV) criterion can be considered a breakthrough result, which shows a common basis of many inseparability criteria for continuous variables (in particular, the results of Duan *et al.* [7] seemed previously to be entirely independent of partial transposition). Another advantage of the SV criterion should be noted: it is given in terms of creation-operator and annihilation-operator moments, which are measurable in standard homodyne correlation experiments [12] (for recent reviews on entanglement detection see Refs. [13,14]).

Despite the evident progresses (see also [13–18] and references therein), the theory of quantum entanglement for CV systems can be considered less developed than the theory for discrete, finite-dimensional systems [13]. In the latter case, powerful inseparability criteria based on positive maps (see [13,19] and references therein) and linear contractions [20–23] (or permutations of the indices of density matrix [24]) have been studied as generalizations of the standard PPT criterion [4,5]. Inspired by these tools available to study discrete-variable entanglement, we propose a generalization of the Shchukin-Vogel CV approach.

Shchukin and Vogel [3] recognized a deep link between the property of positivity under the operation of PT of a two-mode density operator ρ, and the positivity under PT of the corresponding matrix of moments. In the present work, we obtain a more general relationship between the separability properties of the density operator and of the matrix of moments. Namely, we show that if a state is separable, then a suitably designed matrix of moments is separable too. This will allow us to apply all known separability criteria (not only the PPT one) to the matrix of moments rather than directly to the density matrix. For the sake of clarity, we will analyze explicitly mainly the bipartite two-mode case; anyway, the results can be extended to the multimode (see Sec. VII) and multipartite case.

As the objectives of the paper are of wide range, let us first specify the main goal and results of the paper. We analyze the Shchukin-Vogel inseparability criterion for matrices of moments from a new perspective useful for generalizations along the lines of the standard inseparability criteria for density matrices. More specifically, we emphasize the fact that separability is preserved by the mapping from states to matrices of moments. This more general approach leads us to propose entanglement criteria based on realignment and positive maps, which lead to inequalities directly applicable in experimental tests of entanglement.

In particular, in Sec. II, we present a general idea of separability criteria based on matrices of moments. In Sec. III, we review the Shchukin-Vogel criterion. In Secs. IV and V, we present our generalizations of the SV criterion based on the separability properties of the matrix of moments of creation and annihilation operators by referring to contraction maps (e.g., realignment) and positive maps (e.g., those of Kossakowski, Choi and Breuer). A few examples illustrating the applicability of the criteria are shown. In Sec. VI, we discuss detection of entanglement by expressing the entries of the density matrix in terms of the moments. In Sec. VII, we briefly discuss the use of the criteria to analyze bipartite-entanglement of more than two modes. Finally, we give our conclusions.

II. SEPARABILITY OF STATES AND MATRICES OF MOMENTS

Consider two modes A and B with associated annihilation and creation operators a and $a^\dagger$ for A and b and $b^\dagger$ for B. Shchukin and Vogel showed that a Hermitian operator $X=X^{AB}$ is nonnegative if and only if for any operator $f=f^{AB}$ whose normally-ordered form exists, i.e.,

$$f=\sum_{k_1,k_2,l_1,l_2=0}^{+\infty} c_{k_1k_2l_1l_2}a^{\dagger k_1}a^{k_2}b^{\dagger l_1}b^{l_2}, \tag{1}$$

it holds $\langle f^\dagger f\rangle_X \equiv \mathrm{Tr}\{f^\dagger f X\}\geq 0$.

Let us consider the operators

$$f_i \equiv f_k^A f_l^B, \tag{2}$$

with $f_k^A \equiv a^{\dagger k_1}a^{k_2}$ and $f_l^B \equiv b^{\dagger l_1}b^{l_2}$. Here i is the unique natural number associated with a double multi-index $(\mathbf{k},\mathbf{l})$, with $\mathbf{k}=(k_1,k_2)$, $\mathbf{l}=(l_1,l_2)$. Furthermore, the multi-indices $\mathbf{k}$ and $\mathbf{l}$ are associated with unique natural numbers $k\leftrightarrow(k_1,k_2)$ and $l\leftrightarrow(l_1,l_2)$. Any operator f whose normally form exists can thus be written as $f=\Sigma_i c_i f_i$. If we further define the matrix $M(X)=[M_{ij}(X)]$, whose elements are given by

$$M_{ij}(X)\equiv\langle f_i^\dagger f_j\rangle_X=\mathrm{Tr}\{f_i^\dagger f_j X\}, \tag{3}$$

we have

Lemma 1. An operator X is positive semidefinite ($X\geq 0$) if and only if $M(X)$ is positive semidefinite [3].

Indeed, X is positive semidefinite if and only if $\langle f^\dagger f\rangle_X\geq 0$ for all $f=\Sigma_i c_i f_i$, i.e., if and only if $\Sigma_{ij}c_i^* c_j M_{ij}(X)\geq 0$ for all possible $(c_i)_i=(c_1,c_2,\ldots)$. In turn, this implies that $X\geq 0$ if and only if $M(X)=[M_{ij}(X)]$ is a positive semidefinite (infinite) matrix. We will refer to correlation matrices as $M(X)$ as to the *matrices of moments*.

For any density operator ρ^{AB}, from Lemma 1 we have that the corresponding matrix of moments $M(\rho^{AB})$ is positive semidefinite. For a factorized state $\rho^{AB}=\rho^A\otimes\rho^B$ we have

$$\begin{aligned}
M_{ij}(\rho^A\otimes\rho^B)&\\
&=\mathrm{Tr}\{f_i^\dagger f_j \rho^A\otimes\rho^B\}\\
&=\mathrm{Tr}\{(a^{\dagger k_1}a^{k_2})^\dagger(a^{\dagger k_1'}a^{k_2'})(b^{\dagger l_1}b^{l_2})^\dagger(b^{\dagger l_1'}b^{l_2'})\rho^A\otimes\rho^B\}\\
&=\mathrm{Tr}\{(a^{\dagger k_1}a^{k_2})^\dagger(a^{\dagger k_1'}a^{k_2'})\rho^A\}\mathrm{Tr}\{(b^{\dagger l_1}b^{l_2})^\dagger(b^{\dagger l_1'}b^{l_2'})\rho^B\}\\
&=\mathrm{Tr}\{(f_k^A)^\dagger f_{k'}^A\rho^A\}\mathrm{Tr}\{(f_l^B)^\dagger f_{l'}^B\rho^B\}\\
&=M_{kk'}^A(\rho^A)M_{ll'}^B(\rho^B),
\end{aligned} \tag{4}$$

where $M_{kk'}^A(\rho^A)\equiv\mathrm{Tr}\{(f_k^A)^\dagger f_{k'}^A\rho^A\}$, so that $M^A(\rho^A)=[M_{kk'}^A(\rho^A)]$ is the matrix of moments of subsystem A in state ρ^A (and similarly for B).

A matrix of moments uniquely defines a state, i.e., if $M(\rho)=M(\sigma)$ then $\rho=\sigma$. This is immediately proven by considering that if $M(\rho)=M(\sigma)$ then $\mathrm{Tr}\{(\rho-\sigma)f^\dagger f\}=0$ for all fs.

We introduce explicitly formal (infinite) bases [25] $|k\rangle\equiv|\mathbf{k}\rangle$ and $|l\rangle\equiv|\mathbf{l}\rangle$, in which we express the matrices of moments,

$$M(\rho)=\sum_{kk'll'}M_{kl,k'l'}(\rho)|k\rangle\langle k'|\otimes|l\rangle\langle l'|. \tag{5}$$

Taking into account the one-to-one correspondence between matrices of moments and states and Eq. (4), we conclude

Proposition 1. A state is separable, $\rho=\Sigma_i p_i\rho_i^A\otimes\rho_i^B$, $p_i\geq 0$, $\Sigma_i p_i=1$, if and only if the corresponding matrix of moments is also separable, i.e., $M(\rho)=\Sigma_i p_i M^A(\rho_i^A)\otimes M_i^B(\rho_i^A)$ with $M^A(\rho^A)=\Sigma_{kk'}M_{kk'}^A(\rho^A)|k\rangle\langle k'|$ and analogously for $M^B(\rho^B)$.

Notice that the local matrices of moments $M^{A(B)}(\rho_i^{A(B)})$ in the Proposition are physical, i.e., can consistently be interpreted as related to a local state. Thus, one has to take into account the subtle point that a matrix of moments could be separable in terms of generic positive matrices, but not in terms of physical local matrices of moments. Such a point does not arise when studying the entanglement of a density matrix: in that case, any convex decomposition in tensor products of positive matrices is automatically a good physical separable decomposition. Therefore, it might be that no method based on the study of separability properties of matrix of moments, can distinguish all entangled states.

III. PARTIAL TRANSPOSITION AND SHCHUKIN-VOGEL CRITERION

Let us now recall the Shchukin-Vogel reasoning [3]. Let us first define the operation of partial transposition. Given a density operator

$$\rho=\sum_{k,l,k',l'}\rho_{kl,k'l'}|kl\rangle\langle k'l'| \tag{6}$$

in some fixed basis (say in Fock basis), where $\rho_{klk'l'}=\langle kl|\rho|k'l'\rangle$, its partial transposition (with respect to subsystem B) is

$$\rho^\Gamma=\sum_{k,l,k',l'}\rho_{kl,k'l'}|kl'\rangle\langle k'l|. \tag{7}$$

Transposition is a positive but not completely positive [26] linear map which is well defined also in an infinite-dimensional setting. Positivity of ρ^Γ is a necessary condition for separability of ρ [4,5]. We rederive explicitly the relation between the matrix of moments of ρ and the one of the partially transposed state ρ^Γ,

$$\begin{aligned}
M_{kl,k'l'}(\rho^\Gamma)&=\mathrm{Tr}[(a^{\dagger k_1}a^{k_2})^\dagger(a^{\dagger k_1'}a^{k_2'})(b^{\dagger l_1}b^{l_2})^\dagger(b^{\dagger l_1'}b^{l_2'})\rho^\Gamma]\\
&=\mathrm{Tr}\{(a^{\dagger k_1}a^{k_2})^\dagger(a^{\dagger k_1'}a^{k_2'})[(b^{\dagger l_1}b^{l_2})^\dagger(b^{\dagger l_1'}b^{l_2'})]^T\rho\}\\
&=\mathrm{Tr}\{(a^{\dagger k_1}a^{k_2})^\dagger(a^{\dagger k_1'}a^{k_2'})(b^{\dagger l_1'}b^{l_2'})^\dagger(b^{\dagger l_1}b^{l_2})\rho\}\\
&=M_{kl',k'l}(\rho),
\end{aligned} \tag{8}$$

following from the property $b^T=b^\dagger$. Therefore, the matrix of moments of the partially transposed state corresponds to the partial transpositions of the matrix of moments of the state. Moreover, considering Lemma 1, we have

Criterion 1. (Shchukin-Vogel [3]) A bipartite quantum state ρ is NPT if and only if $M(\rho^{\Gamma})=[M(\rho)]^{\Gamma}$ is NPT.

Considering the remarks following Proposition 1 it is noteworthy that analyzing the partial transposition of the matrix of moments we are able to conclude about the PPT or NPT property of the states. In particular, this means that the only possible entangled states, for which the analysis of the separability properties of the corresponding matrix of moments is not enough to reveal their entanglement, are PPT bound entangled states [27,28].

Given Criterion 1, there is still the problem of analyzing the positivity of $[M(\rho)]^{\Gamma}$. Since the matrix of moments is infinite, one necessarily focuses on submatrices. Let us define $M_N(\rho^{\Gamma})$ to be the submatrix corresponding to the first N rows and columns of $M(\rho^{\Gamma})$. According to the original work by Shchukin and Vogel [3], a bipartite quantum state would be NPT if and only if there exists an N such that $\det M_N(\rho^{\Gamma})<0$. As shown in [29], this is not correct, since the sign of all leading principal minors, i.e., of $\det M_N(\rho^{\Gamma})$, for all $N\geq 1$, does not characterize completely the (semi-)positivity of matrices of moments which are singular. For any (possibly infinite) matrix $\mathcal{M}$, let $\mathcal{M}_{\mathbf{r}}$, $\mathbf{r}=(r_1,\ldots,r_N)$ denote the $N\times N$ principal submatrix which is obtained by deleting all rows and columns except the ones labeled by $r_1,\ldots,r_N$. By applying Sylvester's criterion (see, e.g., [30]) we find [29]

Criterion 2. A bipartite state ρ is NPT if and only if there exists a negative principal minor, i.e., $\det[M(\rho^{\Gamma})]_{\mathbf{r}}<0$ for some $\mathbf{r}\equiv(r_1,\ldots,r_N)$ with $1\leq r_1<r_2<\ldots<r_N$.

Focusing on the principal submatrix $[M(\rho)]_{\mathbf{r}}$, is equivalent to considering a matrix given by moments $M_{ij}(\rho)=\mathrm{Tr}\{f_i^{\dagger}f_j\rho\}$ only for some specific operators f_i. In turn, this amounts to study positivity of ρ (or ρ^{Γ}, when we consider $[M(\rho^{\Gamma})]_{\mathbf{r}}$) only with respect to a subclass of operators $f^{\dagger}f$ (see the proof of Lemma 1), i.e., with $f=\sum_{i=1}^{N}c_{r_i}f_{r_i}$. Hereafter, if not otherwise specified, we slightly abuse notation and denote by $f=(f_{r_1},f_{r_2},\ldots,f_{r_N})$ a subclass of the class of operators [Eq. (2)]. Let $M_f(\rho^{\Gamma})\equiv[M(\rho^{\Gamma})]_{\mathbf{r}}$ with $f=(f_{r_1},f_{r_2},\ldots,f_{r_N})$ denote the principal submatrix corresponding to $\mathbf{r}=(r_1,r_2,\ldots,r_N)$. Criterion 2 can then equivalently be rewritten as

Criterion 3. A bipartite state ρ is NPT if and only if there exists f such that $\det M_f(\rho^{\Gamma})$ is negative.

More compactly,

$$\rho \text{ is PPT} \Leftrightarrow \forall f: \quad \det M_f(\rho^{\Gamma})\geq 0,$$

$$\rho \text{ is NPT} \Leftrightarrow \exists f: \quad \det M_f(\rho^{\Gamma})<0. \tag{9}$$

Notice that in general $M_f(\rho^{\Gamma})\neq[M_f(\rho)]^{\Gamma}$, i.e., the operation of partial transposition and the choice of a principal submatrix do not commute. The criterion requires to consider submatrices of the partially transposed matrix of moments, i.e., $M_f(\rho^{\Gamma})$, not to take submatrices of the matrix of moments and study their partial transposition. Nonetheless, also considering the partial transposition of a submatrix of the matrix of moments is a test for separability, if the submatrix is chosen in the right way [see Sec. IV, in particular Eq. (17)].

On the other hand, for any f (i.e., for any $\mathbf{r}$), the moments which constitute the entries of $M_f(\rho^{\Gamma})$ and $M_f(\rho)$, when both expressed with respect to ρ, are simply related by Hermitian conjugation of the mode b.

IV. INSEPARABILITY CRITERIA VIA REORDERING OF MATRICES OF MOMENTS

In this section, we will be interested in studying the separability properties of the matrix of moments through a reordering of its elements. Indeed, apart from partial transposition, there are other entanglement criteria based on such reorderings. In the bipartite setting, the only nontrivial one which is also independent of partial transposition is realignment. For a state ρ as in Eq. (6), the realigned state reads

$$\rho^{R}=\sum_{k,l,k',l'}\rho_{kl,k'l'}|kk'\rangle\langle ll'|. \tag{10}$$

In a finite-dimensional setting, necessary conditions for separability can be formulated as $\|\rho^{\Gamma}\|\leq 1$ [4] and $\|\rho^{R}\|\leq 1$ [20,21], where $\|A\|=\mathrm{Tr}\{\sqrt{A^{\dagger}A}\}$ is the trace norm of A. The converse statements, $\|\rho^{\Gamma}\|>1$ and $\|\rho^{R}\|>1$, are therefore sufficient conditions for the state to be entangled. It is worth noting that $\|\rho^{\Gamma}\|\leq 1$, contrary to the realignment criterion, is also a sufficient condition for separability for 2×2 and 2×3 systems [5].

We have seen how the partial transposition of the matrix of moments corresponds to the matrix of moments of the partially transposed state, leading to the SV criterion. It is immediate to define a realigned matrix of moments following Eq. (10). Unfortunately, there is no simple relation between the realigned matrix of moments and the realigned state. More importantly, partial transposition and realignment, while both corresponding to a reordering of the elements of a matrix, appear to be on a different footing as regards their applicability in an infinite-dimensional setting. Indeed, the partial transposition criterion can be stated as a condition on positivity of the partially transposed state or matrix of moments, besides a condition on the corresponding trace norm. On the other hand, the realignment condition can be expressed only in the latter way, so that it is not suited to study the separability properties of a non-normalized (and non-normalizable) infinite matrix, e.g., in the case of the matrix of moments. To circumvent such an issue, in the following we will analyze separability properties of properly truncated matrix of moments, opening the possibility to deploy the power of the techniques developed for finite-dimensional systems. We remark that such a "truncation approach" could also be applied directly to CV density matrices, as it was done, for example, in [17], but in this work we focus on the matrices of moments. One of the main reasons is that, as already remarked about SV criterion, moments are measurable in standard homodyne correlation experiments.

In the SV approach, one typically refers directly to the total infinite matrix of moments $M(\rho^{\Gamma})$ (see Criterion 1), studying positivity of its principal minors (see Criterion 2). Instead, we propose to first truncate the matrix of moments $M(\rho)$, and then analyze with different criteria the separability of the truncated matrix of moments. Indeed, truncation is equivalent to focusing on (some) submatrix. The submatrix must be chosen correctly, avoiding the introduction of artifact entanglement by the truncation. The truncated matrix is positive and, once normalized, can be considered a legitimate state of an effective bi- or multipartite finite-dimensional system. Explicitly, consider subsets of indices

$$I_A = \{k^{(1)}, \ldots, k^{(d_A)}\} \leftrightarrow \{\mathbf{k}^{(1)}, \ldots, \mathbf{k}^{(d_A)}\},$$

$$I_B = \{l^{(1)}, \ldots, l^{(d_B)}\} \leftrightarrow \{\mathbf{l}^{(1)}, \ldots, \mathbf{l}^{(d_B)}\}$$

and the corresponding projectors $P_A = \Sigma_{k\in I_A}|k\rangle\langle k|$ and $P_B = \Sigma_{l\in I_B}|l\rangle\langle l|$. Then we can define a finite-dimensional matrix

$$M_{I_A I_B}(\rho) = (P_A \otimes P_B) M(\rho)(P_A \otimes P_B) \tag{11}$$

and we have that $M_{I_A I_B}(\rho)/\mathrm{Tr}\{M_{I_A I_B}(\rho)\}$ is a well-defined state (positive and with trace equal to one) for a $d_A \otimes d_B$ system, which is separable if the starting state ρ is separable. Indeed, according to Proposition 1, if ρ is separable then $M(\rho)$ is separable too; moreover, a further local projection cannot induce the creation of entanglement.

As we noted at the end of Sec. III, any choice of a principal submatrix can be described as considering a specific class f of operators, i.e., a restricted set of products of annihilation and creation operators in normal order. Now, we are interested in the classes of operators corresponding to the choice of I_A and I_B. This means we will always consider only tensor-product classes of operators,

$$\begin{aligned}\tilde{f} &= f^A \otimes f^B \\ &= (a^{\dagger k_1^{(1)}} a^{k_2^{(1)}}, \ldots, a^{\dagger k_1^{(d_A)}} a^{k_2^{(d_A)}}) \otimes (b^{\dagger l_1^{(1)}} b^{l_2^{(1)}}, \ldots, b^{\dagger l_1^{(d_B)}} b^{l_2^{(d_B)}}) \\ &= (a^{\dagger k_1^{(1)}} a^{k_2^{(1)}} b^{\dagger l_1^{(1)}} b^{l_2^{(1)}}, \ldots).\end{aligned} \tag{12}$$

With the help of this notation, a truncated matrix of moments will be denoted in the following as

$$M_{\tilde{f}}(\rho) \equiv \sum_{\substack{k,k'\in I_A \\ l,l'\in I_B}} M_{kl,k'l'}(\rho)|kl\rangle\langle k'l'| \tag{13}$$

for an operator class $\tilde{f}$, which is given by a tensor product of classes (as marked by tilde).

Elements of matrix [Eq. (13)] can be reordered to get entanglement criteria in full analogy to those based on reordering of the density matrix elements. Thus, we formally apply to $M_{\tilde{f}}(\rho)$ the "partial transposition"

$$(M_{\tilde{f}}(\rho))^{\Gamma} = \sum_{k,l,k',l'} M_{klk'l'}(\rho)|k'l\rangle\langle kl'|, \tag{14}$$

and the "realignment"

$$(M_{\tilde{f}}(\rho))^{R} = \sum_{k,l,k',l'} M_{klk'l'}(\rho)|kk'\rangle\langle ll'|, \tag{15}$$

in complete analogy to Eqs. (7) and (10). Let us define the normalized trace norms

$$\nu^{\Gamma}_{\tilde{f}}(\rho) \equiv \frac{\|(M_{\tilde{f}}(\rho))^{\Gamma}\|}{\mathrm{Tr}\{M_{\tilde{f}}(\rho)\}}, \quad \nu^{R}_{\tilde{f}}(\rho) \equiv \frac{\|(M_{\tilde{f}}(\rho))^{R}\|}{\mathrm{Tr}\{M_{\tilde{f}}(\rho)\}}. \tag{16}$$

It is worth noting that, because of the tensor-product structure of $\tilde{f}$, we have

$$(M_{\tilde{f}}(\rho))^{\Gamma} = M_{\tilde{f}}(\rho^{\Gamma}) \tag{17}$$

for all $\tilde{f}$ and all ρ.

The SV criterion can now be equivalently formulated as

Criterion 4. A bipartite state ρ is NPT if and only if there exists a tensor-product class $\tilde{f}$, given by Eq. (12), such that $M_{\tilde{f}}(\rho^{\Gamma})$ is not positive or, equivalently, $\nu^{\Gamma}_{\tilde{f}}(\rho) > 1$.

The Rudolph-Chen-Wu [20,21] realignment criterion for density matrices, can be generalized straightforwardly for the matrices of moments as follows:

Criterion 5. A bipartite quantum state ρ is inseparable if there exists $\tilde{f}$, such that $(M_{\tilde{f}}(\rho))^R$ has trace norm $\|(M_{\tilde{f}}(\rho))^R\|$ greater than $\mathrm{Tr}\{M_{\tilde{f}}(\rho)\}$.

More compactly,

$$\rho \text{ is separable} \Rightarrow \forall \tilde{f}: \quad \nu^{R}_{\tilde{f}}(\rho) \le 1,$$

$$\rho \text{ is inseparable} \Leftarrow \exists \tilde{f}: \quad \nu^{R}_{\tilde{f}}(\rho) > 1. \tag{18}$$

In principle, the criterion (18) based on the realignment of the matrix of moments is inequivalent to the SV criterion based on PT, similarly as, for finite-dimensional density matrices, the Peres-Horodecki criterion is not equivalent to the Rudolph-Chen-Wu criterion.

Exemplary applications of partial transposition and realignment

Let us give a few examples of application of the inseparability criteria based on PT and realignment of matrices of moments. We recall that $[M_{\tilde{f}}(\rho)]^{\Gamma} = M_{\tilde{f}}(\rho^{\Gamma})$ for a tensor-product $\tilde{f}$.

Example 1. To detect the entanglement of the singlet state $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, one can choose $\tilde{f} = (1,a)\otimes(1,b) \equiv (1,a,b,ab)$ yielding the following matrix of moments $M_{\tilde{f}}(\rho) \equiv [M_{ij}] = [\langle \tilde{f}_i^{\dagger}\tilde{f}_j\rangle]$,

$$M_{\tilde{f}}(\rho) = \begin{bmatrix} 1 & \langle a\rangle & \langle b\rangle & \langle ab\rangle \\ \langle a^{\dagger}\rangle & \langle N_a\rangle & \langle a^{\dagger}b\rangle & \langle N_a b\rangle \\ \langle b^{\dagger}\rangle & \langle ab^{\dagger}\rangle & \langle N_b\rangle & \langle aN_b\rangle \\ \langle a^{\dagger}b^{\dagger}\rangle & \langle N_a b^{\dagger}\rangle & \langle a^{\dagger}N_b\rangle & \langle N_a N_b\rangle \end{bmatrix}, \tag{19}$$

where $\rho = |\psi\rangle\langle\psi|$, and $N_a = a^{\dagger}a$, $N_b = b^{\dagger}b$ are the number operators. The only nonzero terms of Eq. (19) for the singlet state are: $M_{11}=1$, $M_{22}=M_{33}=-M_{23}=-M_{32}=1/2$. Elements of $[M_{ij}]$ can be reordered, according to Eqs. (14) and (15), to get $(M_{\tilde{f}}(\rho))^{\Gamma}$ and $(M_{\tilde{f}}(\rho))^{R}$ equal to

$$\begin{bmatrix} M_{11} & M_{21} & M_{13} & M_{23} \\ M_{12} & M_{22} & M_{14} & M_{24} \\ M_{31} & M_{41} & M_{33} & M_{43} \\ M_{32} & M_{42} & M_{34} & M_{44} \end{bmatrix}, \begin{bmatrix} M_{11} & M_{12} & M_{21} & M_{22} \\ M_{13} & M_{14} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{41} & M_{42} \\ M_{33} & M_{34} & M_{43} & M_{44} \end{bmatrix}, \tag{20}$$

respectively. Thus, for the singlet state one gets the trace norms, defined by Eq. (16), greater than 1, i.e., $\nu_{\tilde{f}}^{\Gamma}=\nu_{\tilde{f}}^{R}=(1+\sqrt{2})/2$, as well as negative $\det M_{\tilde{f}}(\rho^{\Gamma})=-1/16$ and $\min\operatorname{eig} M_{\tilde{f}}(\rho^{\Gamma})=(1-\sqrt{2})/2$. It is seen that both the PT and realignment based criteria detect the entanglement of the singlet state. It is worth noting that one could analyze just the submatrix of the first matrix of Eq. (20) corresponding to $\mathbf{r}=(1,4)$. This amounts to considering, in the standard SV approach, $M_f(\rho^{\Gamma})$ with $f=(1,ab)$. Then one gets

$$M_f(\rho^{\Gamma}) = \begin{bmatrix} 1 & \langle ab^{\dagger}\rangle \\ \langle a^{\dagger}b\rangle & \langle N_a N_b\rangle \end{bmatrix}, \tag{21}$$

from which the Hillery-Zubairy criterion of entanglement follows [11]:

$$\det M_f(\rho^{\Gamma}) = \langle N_a N_b\rangle - |\langle ab^{\dagger}\rangle|^2 < 0. \tag{22}$$

For our state, one gets $M_f(\rho^{\Gamma})=[1,-1/2;-1/2,0]$, which results in $\det M_f(\rho^{\Gamma})=-1/4$.

Example 2. The realignment-based and PT-based criteria can also detect the entanglement of partially entangled states. To show this, let us analyze the state $|\psi\rangle=\frac{1}{\sqrt{3}}(|00\rangle+|01\rangle+|10\rangle)$ for which negativity is equal to 2/3. By choosing $\tilde{f}$ the same as in Example 1, one gets

$$M_{\tilde{f}}(\rho) = \frac{1}{3}\begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{23}$$

which implies $\nu_{\tilde{f}}^{\Gamma}=\nu_{\tilde{f}}^{R}=1.1891>1$ [as well as $\det M_{\tilde{f}}(\rho^{\Gamma})=-1/81<0$]. Thus, the entanglement of the state can be detected by both criteria. As in Example 1, we can use the submatrix of moments $M_f(\rho^{\Gamma})=[1,1/3;1/3,0]$, given by Eq. (21) [or, which is the same, the submatrix $[M_{\tilde{f}}(\rho^{\Gamma})]_{\mathbf{r}}$ of the partially transposed $M_{\tilde{f}}(\rho)$ of Eq. (23), for $\mathbf{r}=(1,4)$], which also has negative determinant (equal to $-1/9$) and minimum eigenvalue, given by $(3-\sqrt{13})/6\approx-0.1$.

Example 3. The realignment-based criterion is sensitive also for some infinite-dimensional entangled states, as can be shown on the example of superpositions of coherent states, referred to as the two-mode Schrödinger cat states,

$$|\psi'\rangle = \mathcal{N}'(|\alpha,-\beta\rangle - |-\alpha,\beta\rangle),$$

$$|\psi''\rangle = \mathcal{N}''(|\alpha,\beta\rangle - |-\alpha,-\beta\rangle),$$

which are normalized by functions $\mathcal{N}'$ and $\mathcal{N}''$ of the complex amplitudes α and β. As actually shown in [3], the entanglement of $|\psi''\rangle$ (but also of $|\psi'\rangle$) can be detected by the standard SV criterion for $f=(1,b,ab)$, for which one gets a negative determinant $\det M_f(\rho^{\Gamma})$. The realignment-based criterion applied to the factorized $\tilde{f}=(1,a)\otimes(1,b)$ is also sensitive enough to detect entanglement of both states $|\psi'\rangle$ and $|\psi''\rangle$. E.g., for both states with $\alpha=0.3$ and $\beta=0.2$, one gets the trace norms for realignment and PT greater than one, i.e., $\nu_{\tilde{f}}^{R}=1.1666$ and $\nu_{\tilde{f}}^{\Gamma}=1.1783$. Note again that by analyzing determinant or minimum eigenvalue of submatrix $[M_{\tilde{f}}(\rho^{\Gamma})]_{\mathbf{r}}$ for $\mathbf{r}=(1,4)$, given by Eq. (21), one can detect entanglement of the state by handling less moments.

V. POSITIVE MAPS ACTING ON MATRICES OF MOMENTS

In this section we generalize the SV criterion by applying the theory of positive maps (see reviews [13,19]).

The standard criterion of separability for states which is based on positive maps says the following [4,5]: a bipartite state ρ is separable if and only if every positive linear map Λ acting partially (say on the second subsystem only) transforms ρ into a new matrix with nonnegative spectrum, i.e.,

$$(\mathrm{id}_A \otimes \Lambda_B)[\rho^{AB}] \geq 0. \tag{24}$$

(For brevity, the system-identifying superscripts are usually omitted.) Therefore, if the partial action of a positive map on a state of a composite system spoils the positivity of the state, then the state must be entangled. Obviously, the Peres-Horodecki PPT criterion can be formulated as Eq. (24), with $\Lambda=T$ being the transposition operation. On the other hand, we note that realignment is not a positive map, and the related criterion involves the evaluation of the trace norm of the realigned state, which is in general not even Hermitian.

One direction of the separability criterion based on positive maps can be applied in the space of matrices of moments to conclude that the starting state is entangled. Indeed, the reasoning at the base of the partial map criterion does not require any normalization and regards only the property of positivity. More explicitly:

Criterion 6. Let Λ be a linear map preserving positivity of (infinite) matrices, and let $M(\rho)$ be a separable matrix of moments, i.e., $M(\rho)=\Sigma_n p_n M_n(\rho^A)\otimes M_n(\rho^B)$ with $p_n\geq 0$. Then the (infinite) matrix resulting from the partial action of Λ, i.e., $(\mathrm{id}\otimes\Lambda)[M(\rho)]=\Sigma_n p_n M_n(\rho^A)\otimes\Lambda[M_n(\rho^B)]$, is also positive.

Therefore, if we are given a matrix of moments $M(\rho)$ for two modes and a positive map Λ and we find that $(\mathrm{id}\otimes\Lambda)[M(\rho)]$ is not positive, then we conclude that the matrix of moments as well as the starting state is not separable.

If there were a mapping between positive linear maps on states and positive linear maps on the corresponding matrices of moments, we could perhaps derive a general theorem of the Shchukin-Vogel type. Unfortunately such a connection, if existing at all, does not seem to be immediate. Transposition appears in this sense to be very special, since transposition of states translates simply into transposition of matrices of moments. Here, we will limit ourselves to the application of partial maps to truncated matrices of moments, so that we have the following:

Criterion 7. If, for some $\tilde{f}$, there is a positive linear map Λ such that $(\mathrm{id}\otimes\Lambda)[M_{\tilde{f}}(\rho)]$ is not positive, then ρ is entangled.

This Criterion is a direct consequence of the observation at the basis of Proposition 1 and Criterion 6. Essentially, if one constructs a (sub)matrix of moments that preserves the separable structure of a state, and finds that the matrix of moments is entangled (using any arbitrary criterion, in this case linear maps), then one knows that the state was entangled. We remark that we are only able to establish a sufficient condition for entanglement (alternatively, a necessary condition for separability), contrary to the analogous theorem for density matrices by Horodecki *et al.* [5], which says that there always is a map able to detect the entanglement.

We remark that in the case of transposition, which is defined for any dimension, the application of the map to the matrix of moments is equivalent to considering the matrix of moments of the partially transposed state. Therefore it is possible to directly focus on submatrices of the form $M_f(\rho^\Gamma)$. On the other hand, in general, we may consider maps whose action is defined on finite dimensions: consequently, we have to first take (properly chosen) submatrices $M_{\tilde{f}}(\rho)$, and only then act partially on them to obtain $M'_{\tilde{f}}=(\mathrm{id}\otimes\Lambda)[M_{\tilde{f}}(\rho)]$. This does not exclude that, after the action of the map, we may consider the positivity of an even smaller submatrix $(M'_{\tilde{f}})_{\mathbf{r}}$ of the partially transformed submatrix of moments.

For example, one can apply nondecomposable [31] maps to try to detect the entanglement of PPT entangled states. Classes of such maps were constructed for arbitrary finite dimension $N\geq 3$, e.g., by Kossakowski [32], Ha [33], and recently by Yu and Liu [34], Breuer [35], and Hall [36].

We are not able to provide examples of PPT bound entangled states, the entanglement of which is detected by applying positive maps on submatrices of moments, but the existence of such examples is not excluded. Furthermore, we stress that it may happen that a detection method based on an indecomposable map is able to detect more efficiently the entanglement of an NPT state than PT itself, e.g., it may be sufficient to consider smaller submatrices of moments. In any case, through the application of various indecomposable maps one can easily generate criteria for separability that are possibly independent from those obtained from PT. Indeed, as an important application of the proposed method we stress that it enables a simple derivation of interesting inseparability inequalities, e.g.,

$$2(\langle N_a N_b\rangle + \langle N_a^2 N_b\rangle) < |\langle N_a b\rangle - \langle a^\dagger b\rangle|^2, \tag{25}$$

which corresponds to the condition on the determinant of Eq. (36) obtained in the next subsection.

Exemplary applications of positive maps

The proposed method can be summarized as follows: first truncate the matrix of moments, i.e., $M\to M_{\tilde{f}}$, then apply a positive map, i.e., $M_{\tilde{f}}\to M'_{\tilde{f}}$, and check the positivity of the partially transformed submatrix of moments $M'_{\tilde{f}}$. In turn, this amounts to considering positivity of submatrices $(M'_{\tilde{f}})_{\mathbf{r}}$, or, by virtue of Sylvester's criterion, to checking positivity of determinants $\det(M'_{\tilde{f}})_{\mathbf{r}}$. Thus, one can say that submatrices of partially transformed submatrices are considered.

Here, we give a few examples of application of our inseparability criteria based on some specific classes of positive maps applied to matrices of moments.

Kossakowski and Choi maps

The Kossakowski class of positive maps transforms matrices $A=[A_{ij}]_{N\times N}$ in $\mathcal{C}^N$ onto matrices in the same space as follows [32]

$$\Lambda_K[A] = \frac{\mathbb{1}}{N}\mathrm{Tr}\,A + \frac{1}{N-1}g\cdot(Rx+\kappa y\,\mathrm{Tr}\,A), \tag{26}$$

where "$\cdot$" stands for the scalar product, $\kappa=\sqrt{(N-1)/N}$, $x=(x_i)_i$, $x_i=\mathrm{Tr}\{Ag_i\}$, and $g=(g_i)_i$ satisfying $g_i=g_i^*$, $\mathrm{Tr}\{g_ig_j\}=\delta_{ij}$, $\mathrm{Tr}\{g_i\}=0$ for $i,j=1,\ldots,N^2-1$. In our applications, we assume $y=0$, R to be rotations $R(\theta)\in\mathrm{SO}(N^2-1)$, and g_i to be generators of SU(N). Note that the Ha maps [33] do not belong to Eq. (26). In a special case for $A=[A_{ij}]_{3\times 3}$, the Kossakowski map is reduced to the Choi map [37],

$$\Lambda_{\mathrm{Choi}}[A] = -A + \mathrm{diag}\left(\begin{bmatrix}\alpha A_{11}+\beta A_{22}+\gamma A_{33},\\ \gamma A_{11}+\alpha A_{22}+\beta A_{33},\\ \beta A_{11}+\gamma A_{22}+\alpha A_{33}\end{bmatrix}\right), \tag{27}$$

which is positive if and only if $\alpha\geq 1$, $\alpha+\beta+\gamma\geq 3$ and $1\leq\alpha\leq 2\Rightarrow\beta\gamma\geq(2-\alpha)^2$, while decomposable if and only if $\alpha\geq 1$ and $1\leq\alpha\leq 3\Rightarrow\beta\gamma\geq(3-\alpha)^2/4$. We denote the resulting (un-normalized) matrix of moments shortly as

$$M'_{\tilde{f}}(\rho)\equiv(\mathrm{id}\otimes\Lambda_{\mathrm{Choi}})[M_{\tilde{f}}(\rho)]. \tag{28}$$

It is worth noting that some bound entangled states can be detected [22] by applying to ρ the Störmer map [38], which is a special case of the Choi map for $\alpha=2$, $\beta=0$, $\gamma=1$ and of Eq. (26) for $\theta=\pi/3$ and $N=3$.

Example. As an exemplary application of a positive map, let us apply the Störmer map to 9×9 matrix of moments $M'_{\tilde{f}}(\rho)$ for $\tilde{f}=(1,a,a)\otimes(1,b,b)$. Note that the chosen map is nondecomposable. For simplicity, we analyze only the submatrix $(M'_{\tilde{f}}(\rho))_{\mathbf{r}}$ for $\mathbf{r}=(2,3,7)$,

$$(M'_{\tilde{f}}(\rho))_{\mathbf{r}} = \begin{bmatrix} M_{11}+M_{22} & -M_{23} & -M_{27}\\ -M_{32} & M_{22}+M_{33} & -M_{37}\\ -M_{72} & -M_{73} & M_{77}+M_{99}\end{bmatrix} = \begin{bmatrix} 1+\langle N_a\rangle & -\langle N_a\rangle & -\langle a^\dagger b\rangle\\ -\langle N_a\rangle & 2\langle N_a\rangle & -\langle a^\dagger b\rangle\\ -\langle a^\dagger b\rangle^* & -\langle a^\dagger b\rangle^* & \langle N_aN_b\rangle+\langle N_b\rangle\end{bmatrix}, \tag{29}$$

where $M_{ij}=\langle\tilde{f}_i^\dagger\tilde{f}_j\rangle$ are elements of the original (not-transformed) matrix of moments, $M_{\tilde{f}}$. Matrix (29) for the singlet state is given by $\frac{1}{2}[3,-1,1;-1,2,1;1,1,1]$ having negative determinant (equal to $-1/4$), which reveals the entanglement of the state. Analogously, the entanglement of the partially entangled state $|\psi\rangle=\frac{1}{\sqrt{3}}(|00\rangle+|01\rangle+|10\rangle)$ can also be detected by Eq. (29), which is now reduced to

$[M'_{\tilde{f}}(\rho)]_{\mathbf{r}}=\frac{1}{3}[4,-1,-1;-1,2,-1;-1,-1,1]$ with negative determinant (equal to $-1/27$).

Breuer map

Our inseparability criterion for matrices of moments can also be based on the Breuer positive map defined in a space of even dimension $d\geq 4$ as follows [35]:

$$\Lambda_{\mathrm{Breuer}}[A]=\mathbb{1}\,\mathrm{Tr}\,A-A-\vartheta[A], \tag{30}$$

where $\vartheta[A]=UA^TU^\dagger$ can be interpreted as a time reversal transformation and is given by a skew-symmetric unitary matrix U. The latter can be constructed explicitly as $U=RDR^T$ in terms of [36],

$$D=\sum_{k=0}^{d/2-1} e^{i\phi_k}(|2k\rangle\langle 2k+1|-|2k+1\rangle\langle 2k|), \tag{31}$$

for any angles ϕ_k and arbitrary orthogonal matrix R. Although antisymmetric unitary matrices exist only in even-dimensional spaces, the Breuer map can be generalized for arbitrary dimensions (see, e.g., [36]). Thus, it is tempting to propose an analogous criterion by applying the Breuer map to a matrix of moments,

$$M''_{\tilde{f}}(\rho)\equiv(\mathrm{id}\otimes\Lambda_{\mathrm{Breuer}})[M_{\tilde{f}}(\rho)] \tag{32}$$

and checking positivity of the transformed matrix $M''_{\tilde{f}}(\rho)$. It is worth noting that the Breuer map is a special case of the Yu-Liu positive map [34], thus even more powerful and computationally simple inseparability criteria for density matrices [34–36] can also be applied for matrices of moments.

Example 1. To reveal entanglement of the singlet state, let us first analyze a matrix $M_{\tilde{f}}(\rho)$ of moments generated by some 16-element $\tilde{f}$. Antisymmetric unitary matrix U can, for example, be constructed as the antidiagonal matrix

$$U=\begin{bmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\end{bmatrix}. \tag{33}$$

Then, by applying the corresponding Breuer map, one can easily get, from Eq. (32), the transformed 16×16 matrix $M''_{\tilde{f}}(\rho)$ for arbitrary state ρ. This matrix reveals, for example, entanglement of the singlet state for various choices of $\tilde{f}$, e.g.: $\tilde{f}^{(1)}=(1,a,N_a,a^2)\otimes(1,b,N_b,b^2)$, $\tilde{f}^{(2)}=(1,a,N_a,1)\otimes(1,b,N_b,1)$, or even $\tilde{f}^{(3)}=(1,a,1,1)\otimes(1,b,1,1)$.

Note that $\tilde{f}^{(2)}$ and $\tilde{f}^{(3)}$ do not provide more information than $(1,a,N_a)\otimes(1,b,N_b)$ and $(1,a)\otimes(1,b)$, respectively. The matrices of moments corresponding to the former sets of operators contain redundant copies of the moments related to the latter sets, i.e., a repetition of an operator amounts to have a matrix of moments with repeated columns and rows. We considered such redundant sets of operators because Breuer criterion requires one of the subsystems to be at least four-dimensional, but at the same time we wanted to emphasize that is possible to detect (by means of Breuer's map) entanglement with fewer and fewer combinations of "independent" operators. We point out that $\tilde{f}^{(1)}$ provides for sure more information in general than $\tilde{f}^{(2)}$, and in turn the latter more than $\tilde{f}^{(3)}$.

The entanglement detection can be much simplified by analyzing the submatrix of $M''_{\tilde{f}}(\rho)$ corresponding, e.g., to $\mathbf{r}=(2,5)$,

$$[M''_{\tilde{f}}(\rho)]_{\mathbf{r}}=\begin{bmatrix} M_{11}+M_{44} & -M_{25}-M_{47}\\ -M^*_{25}-M^*_{47} & M_{66}+M_{77}\end{bmatrix}, \tag{34}$$

where, as usual, $M_{ij}=\langle\tilde{f}_i^\dagger\tilde{f}_j\rangle$ are elements of the original matrix $M_{\tilde{f}}(\rho)$. For $\tilde{f}=\tilde{f}^{(1)}$, matrix (34) reduces to

$$[M''_{\tilde{f}^{(1)}}(\rho)]_{\mathbf{r}}=\begin{bmatrix} 1+\langle a^{\dagger 2}a^2\rangle & -\langle a^\dagger b\rangle-\langle a^{\dagger 3}ab\rangle\\ -\langle a^\dagger b\rangle^*-\langle a^{\dagger 3}ab\rangle^* & \langle(1+N_a)N_aN_b\rangle\end{bmatrix}. \tag{35}$$

For the example of the singlet state, one gets $[M''_{\tilde{f}^{(1)}}(\rho)]_{\mathbf{r}}=[1,1/2;1/2,0]$, for which the determinant is $-1/4$. One can get even simpler criterion from Eq. (34) by choosing $\tilde{f}=\tilde{f}^{(2)}$,

$$[M''_{\tilde{f}^{(2)}}(\rho)]_{\mathbf{r}}=\begin{bmatrix} 2 & \langle N_ab\rangle-\langle a^\dagger b\rangle\\ \langle N_ab^\dagger\rangle-\langle ab^\dagger\rangle & \langle N_aN_b\rangle+\langle N_a^2N_b\rangle\end{bmatrix}. \tag{36}$$

Explicitly, for the singlet state, we have $\det[M''_{\tilde{f}^{(2)}}(\rho)]_{\mathbf{r}}=\det[2,1/2;1/2,0]=-1/4$. By contrast to $\tilde{f}^{(1)}$ and $\tilde{f}^{(2)}$, matrix (34) for $\tilde{f}=\tilde{f}^{(3)}$ is positive. Nevertheless entanglement can be revealed by choosing a larger submatrix of $M''_{\tilde{f}^{(3)}}(\rho)$ corresponding to $\mathbf{r}=(2,5,7,8)$, which results in

$$[M''_{\tilde{f}^{(3)}}(\rho)]_{\mathbf{r}}=\begin{bmatrix} 2 & x_- & 0 & x_+\\ x_-^* & z & y_+^* & 0\\ 0 & y_+ & 2\langle N_b\rangle & y_-\\ x_+^* & 0 & y_-^* & z\end{bmatrix}, \tag{37}$$

where $x_\pm=\pm\langle b\rangle-\langle a^\dagger b\rangle$, $y_\pm=\pm\langle aN_b\rangle-\langle N_b\rangle$, and $z=\langle(N_a+1)N_b\rangle$. For the singlet state, one again gets $\det[M''_{\tilde{f}^{(3)}}(\rho)]_{\mathbf{r}}=-1/4$.

It is not surprising that one has to change submatrix [i.e., Eq. (37) instead of Eq. (34)], because for $\tilde{f}^{(3)}$ less entries of the matrix $M_f(\rho)$ contain independent information (actually, only a 4×4 matrix [corresponding to $(1,a)\otimes(1,b)$] out of the larger 16×16 matrix (all the other entries are just repetitions)).

Example 2. To reveal the entanglement of the Bell state $|\psi\rangle=\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$, one can apply $\tilde{f}=\tilde{f}^{(1)}$ or $\tilde{f}^{(2)}$ and the Breuer map to be the same as in the former example. Here, one can choose submatrix $[M''_{\tilde{f}}(\rho)]_{\mathbf{r}}$ corresponding to $\mathbf{r}=(1,6,9)$, which reads as

$$\begin{bmatrix} M_{2,2}+M_{3,3} & -M_{1,6}-M_{3,8} & M_{2,10}+M_{3,11} \\ -M_{6,1}-M_{8,3} & M_{5,5}+M_{8,8} & -M_{6,9}-M_{8,11} \\ M_{10,2}+M_{11,3} & -M_{9,6}-M_{11,8} & M_{10,10}+M_{11,11} \end{bmatrix}. \quad (38)$$

For the analyzed Bell state, Eq. (38) yields $\det[M''_{\tilde{f}^{(1)}}(\rho)]_{\mathbf{r}}=\det[M''_{\tilde{f}^{(2)}}(\rho)]_{\mathbf{r}}=-1/4$ clearly demonstrating the entanglement.

Thus, it is seen how new inseparability inequalities, corresponding to $\det[M''_{\tilde{f}}(\rho)]_{\mathbf{r}}<0$, can be obtained by application of positive maps to matrices of moments.

VI. DETECTION OF BOUND ENTANGLEMENT OF FINITE-DIMENSIONAL STATES THROUGH ANALYSIS OF MOMENTS

The original SV criterion is based on partial transposition, thus it cannot reveal PPT bound entanglement. On the other hand, it is known that the standard realignment criterion applied directly to the density matrix can detect entanglement of some bound entangled states [20–24]. A question arises: can PPT bound entanglement be detected by our realignment-based generalized criterion? We have tested numerically some bound entangled states of dimensions 3×3 [27,39], 2×4 [27], $d\times d$ [40,41] as well as infinite [17,18], but we have not been able to detect entanglement by our generalized criterion.

All numerical simulations suggest that the norms of reordered $M_{\tilde{f}}$ satisfy the inequality $\nu_{\tilde{f}}^{\Gamma}\geq\nu_{\tilde{f}}^{R}$ or, equivalently, $\|(M_{\tilde{f}})^{\Gamma}\|\geq\|(M_{\tilde{f}})^{R}\|$. If this observation is true in general, then the described realignment-based criterion is useless in detecting PPT bound entanglement. Nevertheless, bound entanglement can be detected via moments with the help of the formula (see, e.g., [42]),

$$\langle m_1|\rho|m_2\rangle=\frac{1}{\sqrt{m_1!\,m_2!}}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\langle(a^\dagger)^{m_2+j}a^{m_1+j}\rangle, \quad (39)$$

which enables calculation of a given density matrix from moments of creation and annihilation operators. It is worth noting two properties: (i) the above sum is finite for finite-dimensional states (ii) Eq. (39) is not convergent for some states of the radiation field including thermal field with mean photon number ≥ 1. The formula readily generalizes for two-mode fields as

$$\langle m_1,n_1|\rho|m_2,n_2\rangle=\sum_{j,k=0}^{\infty}\frac{\langle(a^\dagger)^{m_2+j}a^{m_1+j}(b^\dagger)^{n_2+k}b^{n_1+k}\rangle}{(-1)^{j+k}j!\,k!\,\sqrt{m_1!\,n_1!\,m_2!\,n_2!}}. \quad (40)$$

Let us analyze a special case of Eq. (40) for two qubits. Single-qubit annihilation operator is simply the Pauli operator given by $a=\sigma^-=[0,1;0,0]$, which implies that there are only four nonzero terms in sum (40). We can explicitly write two-qubit density in terms of the moments as follows:

$$\rho=\begin{bmatrix} \langle\bar{N}_a\bar{N}_b\rangle & \langle\bar{N}_a b^\dagger\rangle, & \langle a^\dagger\bar{N}_b\rangle, & \langle a^\dagger b^\dagger\rangle \\ \langle\bar{N}_a b\rangle, & \langle\bar{N}_a N_b\rangle, & \langle a^\dagger b\rangle, & \langle a^\dagger N_b\rangle \\ \langle a\bar{N}_b\rangle, & \langle ab^\dagger\rangle, & \langle N_a\bar{N}_b\rangle, & \langle N_a b^\dagger\rangle \\ \langle ab\rangle, & \langle aN_b\rangle, & \langle N_a b\rangle, & \langle N_a N_b\rangle \end{bmatrix}, \quad (41)$$

where $\bar{N}_a=1-N_a$ and $\bar{N}_b=1-N_b$. Matrix (41) can be partially transposed and realigned. All principal minors of ρ^Γ are positive if and only if ρ is separable. The above simple example for 2×2 system was given to show the method only. To detect bound entanglement, one has to analyze at least 2×4 or 3×3 systems. For brevity, we will not present explicitly density matrices in terms of moments for these systems. Nevertheless, they can easily be constructed using Eq. (40) and then realigned, according to Eq. (10), to detect entanglement of some bound entangled states [20–22]. Finally, let us remark that there are drawbacks of the method: (i) it works if we know the dimension $d<\infty$ of a given state. (ii) Usually, it is simpler to directly reconstruct density matrix rather than to reconstruct it via moments.

VII. SIMPLE CONSTRUCTION OF MULTIMODE ENTANGLEMENT CRITERIA

The two-mode SV criterion can readily be applied in the analysis of bipartite-entanglement of m modes. For this purpose, one can define an m-mode normally ordered operator

$$f\equiv f(\{a_i\})=\sum_{\{n_i\}=0}^{\infty}\sum_{\{m_i\}=0}^{\infty}c(\{n_i,m_i\})\prod_{i=1}^{m}(a_i^{n_i})^\dagger a_i^{m_i}, \quad (42)$$

where for brevity we denote $\{n_i\}\equiv\{n_1,n_2,\ldots,n_m\}$, and similarly other expressions in curly brackets. As in the proof of Lemma 1, we have that an operator X is positive semidefinite if and only if $\mathrm{Tr}\{Xf^\dagger f\}\geq 0$ for every f as in Eq. (42). To analyze how mode a_j is entangled to all the other modes, it is enough to identify, in the reasoning followed in the previous sections, system A with the mode j and system B with all the other modes. Therefore we take $a=a_j$, while normally ordered powers $b^{\dagger l_1}b^{l_2}$ are substituted by normally ordered powers

$$a_1^{\dagger(k_1)_1}a_1^{(k_1)_2}\ldots a_{j-1}^{\dagger(k_{j-1})_1}a_{j-1}^{(k_{j-1})_2},$$

$$a_{j+1}^{\dagger(k_{j+1})_1}a_{j+1}^{(k_{j+1})_2}\ldots a_m^{\dagger(k_m)_1}a_m^{(k_m)_2}.$$

As in the two-mode setting, we may (and we will) analyze positivity of an operator X with respect to a restricted class of operators f, more specifically with only some coefficients $c(\{n_i,m_i\})$ that do not vanish. This corresponds to testing positivity of principal submatrices.

For example, we show that Eq. (9) implies the three-mode Hillery-Zubairy criterion [11] originally derived from the Cauchy-Schwarz inequality. By choosing $f=(1,abc)$ (we use the notation introduced in Sec. III), one gets $M_f(\rho^\Gamma)=[1,\langle a^\dagger bc\rangle;\langle ab^\dagger c^\dagger\rangle,\langle N_aN_bN_c\rangle]$, where $N_c=c^\dagger c$ and, analogously, N_a and N_b are the number operators. Imposing negativity of the determinant, one derives

$$\langle N_a N_b N_c\rangle < |\langle a^\dagger bc\rangle|^2, \tag{43}$$

which is the desired Hillery-Zubairy criterion [11], i.e., a sufficient condition for the state to be entangled. By restricting the above case to two modes (say $c=1$), one can choose $f=(1,ab)$, which leads the Hillery-Zubairy two-mode entanglement condition [11], given by Eq. (22), as already shown in [3]. By choosing a different function f, one can propose other Hillery-Zubairy-type three-mode criteria. For example, let us choose $f=(a,bc)$ then $M_f(\rho^\Gamma)=[\langle N_a\rangle,\langle abc\rangle;\langle abc\rangle^*,\langle N_bN_c\rangle]$, which results in a sufficient condition for the three-mode entanglement,

$$\langle N_a\rangle\langle N_b N_c\rangle < |\langle abc\rangle|^2. \tag{44}$$

In a special case, Eq. (44) is reduced to another two-mode entanglement condition of Hillery and Zubairy: $\langle N_a\rangle\langle N_b\rangle < |\langle ab\rangle|^2$, derived from the Cauchy-Schwarz inequality in [11].

VIII. CONCLUSIONS

We have studied inseparability criteria for bipartite quantum states, which are given in terms of the matrices of observable moments of creation and annihilation operators, therefore generalizing the analysis by Shchukin and Vogel. Indeed, we have suggested (also by means of examples) that all the techniques originally developed to detect "directly"—that is, by considering the physical density matrix—the entanglement of states, can be deployed at the level of the matrices of moments. In doing this there are advantages—e.g., by considering an appropriate submatrix of the matrix of moments one can apply techniques developed for finite dimensional system to detect the entanglement of infinite-dimensional systems—and disadvantages—e.g., while the separable structure of an entangled state is inherited by all properly constructed matrices of moments, it is not completely clear how the entanglement of the starting physical state gets encoded in the matrix of moments, and in some cases it may be difficult to choose the correct technique to detect it.

In particular, we have proposed a criterion based on realignment of elements of the moment matrices of special symmetry (i.e., corresponding to tensor product $\tilde{f}$s), as a generalization of the Rudolph-Chen-Wu realignment criterion applied for density matrices. Another reordering of elements of the moment matrices corresponds to the partial transposition as in the original SV criterion. We have proposed another criterion based on positive maps applied to appropriate submatrices of moments. We further observe that the formalism of matrices of moments can be certainly combined with the powerful criterion invented in the finite-dimensional setting by Doherty *et al.* [43], in the attempt to detect, e.g., the entanglement of continuous-variable systems. How powerful this combination can be is nonetheless not evident or easily predictable, and we leave it as an interesting open problem.

We have also discussed applications of the SV criteria to describe bipartite-entanglement of more than two modes. In particular, we have obtained the three-mode Hillery-Zubairy criteria originally derived from the Cauchy-Schwarz inequality, and derived new ones of the same type.

As regards the confidence in the certification of entanglement, if entanglement is verified within error bars for the matrix of moments (e.g., by considering the determinants of submatrices of the partially transposed matrix of moments as in the original SV criterion), then entanglement is certified for the physical state. This is true both in the case where error bars come from uncertainties in an experiment—from which the entries of the matrix of moments are obtained—or from numerical tools. We remark that here we are just considering certification of entanglement: in this paper we have not explored the relation between the degree of entanglement—as quantified by some entanglement measure—of the physical state and the degree of entanglement of the matrix of moments.

In conclusion, although it is an open question whether our criteria generalizing the Shchukin-Vogel idea are sensitive enough to detect bound entanglement, they enable to derive new classes of classical inequalities, which can be used for practical detection of quantum entanglement.

Note added. Recently, the SV criterion was thoroughly applied to the multipartite CV case in [44].

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