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Integrable zero-range potentials in a plane.

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Abstract. We examine general statements in the Wronskian representation of Darboux transformations for plane zero-range potentials. Such expressions naturally contain scattering problem solution. We also apply Abel theorem to Wronskians for differential equations and link it to chain equations for Darboux transforms to fix conditions for further development of the underlying distribution concept. Moutard transformations give a convenient insight into the problem that allows one to formulate general assertions and give complete description of a point potential in a limit of long waves.

1. Introduction

Solvable models play a basic role as in classical/quantum mechanics [1] as in electrodynamics [2, 3] and statistical physics [4]. In this rather poor set of models, a significant place is occupied by the ones based on Zero-Range Potentials (ZRP), whose history started from the seminal work of E. Fermi [5]. Their mathematical significance in the context of extension theory was established in [6], see the review [7] and consideration in arbitrary dimensions [3]. The book [8] covers as multicenter applications of the theory of ZRPs as its multichannels generalizations. Very recent paper [9] studies two integrable models: it consider ZRP as perturbations of harmonic oscillator. Interesting applications in solid state physics can be found in [10].

Integrability concept is directly related to Darboux transformations (DT) theory [11, 12]. The zero range potentials were first considered by means of this theory in [13]. The approach introduces so-called generalized ZRPs (compared to spherically symmetrical - partial s-wave ZRP). Its distributional nature was examined and extended to two-dimensional space in [14]. An important development of the theory from a point of view of DT or dressing of a many-center potential and applications to electron-molecular scattering problems can be found in [15]. Dressing of ZRP also gives useful finite range potentials that could be widely applied (see e.g. [16]). To explain the idea, let us address the one-dimensional spectral problem at $x \in (-\infty, \infty)$

$$-\psi_{xx} + u\psi = \lambda\psi, \tag{1}$$

which is covariant with respect to DT

$$\psi_1 = (\partial - \sigma)\psi \tag{2}$$

if σ is a solution of the Riccati equation

$$\sigma' + \sigma^2 - \kappa^2 = u. \tag{3}$$

Substitution $\sigma = (\ln \phi)' = \phi'/\phi$ links it to the solution of the spectral problem (1) with $\lambda = -\kappa^2$. It defines the dressed potential

$$u_1 = u - 2\sigma'. \tag{4}$$

Consider a dressing of the ZRP represented by the distribution $-\alpha\delta(x)$, whose presence is equivalent to boundary condition

$$\lim_{\epsilon\to 0}[\psi(\epsilon) - \psi(-\epsilon)] = -\alpha\psi(0). \tag{5}$$

We should rely upon linear independent generalized solutions of the problem (1) with the ZRP:

$$\psi_1 = \sin(kxsgn(x) + \eta), \quad \psi_2 = \sin(kx), \qquad \lambda = k^2,\ k\cot\eta = -\alpha, \tag{6}$$

whose derivatives are also distributions. For $\alpha > 0$, the chosen potential implies one bound state with eigenfunction

$$\phi = \exp[-\alpha xsgnx].$$

A dressing with such prop function deletes the bound state and leads to the ZRP

$$u_1 = \alpha\delta(x).$$

The boundary condition (5) also changes. This simple example explains how the algorithm enlarges the space of integrable potentials. Other combinations of eigenfunctions generate ZRP with different properties.

Separation of variables on three-dimensional Schrödinger equation with spherical symmetry gives a radial differential equation for partial waves $\psi_l(r) = \frac{1}{r}\phi_l(r)$. Precisely, the function $\phi_l(r)$ satisfies the equation

$$\left[-\frac{1}{2}\frac{d^2}{dr^2} + \frac{l(l+1)}{2r^2} + u_l - E\right]\phi_l(r) = 0. \tag{7}$$

The covariance of this equation with respect to DT

$$\psi_{l1} = \frac{1}{\sqrt{k^2+\kappa_l^2}}\left(\frac{d}{dr} - \sigma_l\right)\phi_l(r) \tag{8}$$

combines two solutions of the spectral problem (7) with $E = k^2$ and $E_l = -\kappa_l^2$, so that

$$\sigma_l = \ln'\phi_l(r, i\kappa_l). \tag{9}$$

The transformation was considered in [13, 15] for the quantum problem of scattering on integrable potential, by applying N-th order DT and taking spherical Hankel functions with specific parameters κ_m as prop functions. The potentials are expressed in terms of Wronskians, which go to Vandermond determinants at infinity. Zero logarithmic derivative of the Wronskians corresponds to generalized ZRP's.

The construction is similar to the one given in this article (see Section 2), where the Wronskians are computed and the solution of the scattering problem is given in explicit form. For example, in the case of l=1, the direct application of Rayleigh's formulas for Bessel functions yields

$$\phi_{1,i} = \sqrt{\frac{\pi}{2\kappa_i r}}H_{\frac{3}{2}}(\kappa_i r) = \frac{i}{\kappa_i^2 r^2}e^{i\kappa_i r}(i\kappa_i r - 1). \tag{10}$$

The Wronskian determinant in this case, by direct evaluation, is equal to:

$$W = e^{ir(\kappa_3+\kappa_2+\kappa_1)}(\kappa_2-\kappa_3)(\kappa_1-\kappa_3)(\kappa_1-\kappa_2)\frac{\kappa_1\kappa_2+\kappa_1\kappa_3+\kappa_2\kappa_3 - ir\kappa_1\kappa_2\kappa_3}{r\kappa_1^2\kappa_2^2\kappa_3^2}. \tag{11}$$

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Its derivative $ln_{rr}W$ is zero if $\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3 = 0$, and ZRP hence follows.

Section 2 is devoted to novel results: explicit form of the boundary conditions in 2d, analog of the three-dimensional generalized ZRP, is derived by the iterated DT. One can compare resulting formulas from [3] with those derived here.

Section 3 contains general formalism, based on separated variables, Abel theorem for Wronskians and DT chain equations that allow to avoid direct differentiations in some physically significant cases, including novel ones.

In the last Section, a Moutard transformation (MT) is applied to new alternative construction of ZRP for the wave, Helmholtz and Schrödinger equations in two dimensions. Iterated MT may produce locally dense set of such ZRP [14].

2. Two-dimensional ZRP for cylindrical symmetry case

To develop the method described in the introduction and show the dressing origin and generalizations of the results for abundant two-dimensional problems of, e.g., [17], let us consider the Schrödinger equation in atomic units for xy plane

$$-\frac{1}{2}(\frac{\partial^2\psi}{\partial^2 x} + \frac{\partial^2\psi}{\partial^2 y}) + U(\vec{r})\psi = E\psi. \tag{12}$$

As in the three-dimensional case [13], by reducing to a one-dimensional problem via separation of variables, ZRP may be introduced whose explicit form depends on the problem symmetry. Equation (12) for $U = 0$ is equivalent to the Helmholtz equation $(-\triangle + k^2)\psi(\vec{\rho}) = 0$, $E = 2k^2$, which, like its inhomogeneous counterpart $(-\triangle + u(\rho) + k^2)\psi(\vec{\rho}) = 0$ where $u = 2U$, finds application in electrodynamics. For a cylindric symmetry, going to polar coordinates $x = \rho\cos\phi, y = \rho\sin\phi$, we have

$$\Delta = \frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + \frac{1}{\rho^2}\frac{d^2}{d\phi^2}. \tag{13}$$

The separation of variables $\exp[i\nu\phi]R$ yields R either as the solution $R(\rho) = Y_v(k\rho), ...$ of the Bessel equation

$$[\frac{d^2}{d(k\rho)^2} + \frac{1}{k\rho}\frac{d}{d\,(k\rho)} + 1 - \frac{\nu^2}{(k\rho)^2}]R = 0, \tag{14}$$

if $E = k^2 > 0$, or, when $E = \kappa^2 < 0$, as the solution of the modified Bessel equation

$$-[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{\nu^2}{\rho^2}]R = -\kappa^2 R, \tag{15}$$

with the feature of the singular behaviour of $R = K_v(\kappa z)$ at $\rho = 0$. The special case of $E = 0$ and analogue separation of variables $\exp[i\nu\phi]R$ yields

$$[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{\nu^2}{\rho^2}]R(\rho) = 0. \tag{16}$$

It is the Euler equation with the basic solutions $R = x^{\pm\nu}$.

Let us apply the DT technique to these equations following the scheme of [13, 15]. For the sake of convenience in the application of Crum formulas (see again e.g. [13]), let us transform the equations via $R = \rho^{-1/2}a\,(\rho)$. This gives

$$-\frac{\partial^2 a\,(\rho)}{\partial(k\rho)^2} + \frac{-1+\nu^2}{4(k\rho)^2}a\,(\rho) - a\,(\rho) = 0. \tag{17}$$

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Hence, for a seed solution $a_\nu(\rho) = \rho^{1/2} R(k\rho)$ and a prop functions $c_m(\rho) = \rho^{1/2} R(\kappa_m \rho)$, the N-fold transform of $a = \rho^{1/2} R$ is identified by means of the dressed potential

$$u^{[N]}(\rho) = u(\rho) - 2\frac{d^2}{d\rho^2} \ln W(\rho^{1/2} R_1(\rho), ..., \rho^{1/2} R_N(\rho)), \tag{18}$$

and the dressed solution

$$S_\nu(\rho)^{[N]} = \frac{W(\rho^{1/2} R_1(\rho), ..., \rho^{1/2} R_N(\rho), \rho^{1/2} S_\nu(\rho))}{\rho^{1/2} W(\rho^{1/2} R_1(\rho), ..., \rho^{1/2} R_N(\rho))}. \tag{19}$$

Having in mind the scattering problem application as it is explained in the Introduction (see also [13]), it is convenient to use Hankel functions $H_\nu^{(1,2)}(\kappa_i \rho) = J_v(z) \pm i Y_\nu(z)$, and express the general solution of the radial equation as

$$\psi_\nu = C(H_\nu^{(1)}(k\rho) s_\nu + H_\nu^{(2)}(k\rho)), \tag{20}$$

s_ν being a scattering matrix. The convenient choice for the dressing scheme is

$$S_\nu = \sqrt{\frac{\pi}{2k}} J_v(k\rho), \qquad R_m = \sqrt{\frac{\pi}{2\kappa_m}} H_\nu^{(1)}(\kappa_m \rho). \tag{21}$$

Substitution of asymptotes at infinity (for integer ν),

$$J_v(z) \sim \left[\sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right]_{z=k\rho} = \frac{\sqrt{2}}{\sqrt{\pi k\rho}} \left[\cos\left(k\rho - \frac{1}{2}\pi\nu - \frac{1}{4}\pi\right)\right], \tag{22}$$

$$Y_\nu(z) = \frac{J_v(z)\cos(\nu\pi) - J_{-v}(z)}{\sin(\nu\pi)} \sim \sqrt{\frac{2}{\pi k\rho}} \sin\left(k\rho - \frac{\nu\pi}{2} - \frac{\pi}{4}\right),$$

into (19) yields the solution

$$S^{[N]}(\rho) \sim \frac{C}{\sqrt{\rho}} \left(\frac{V(\kappa_1, ..., \kappa_N, ik)}{V(\kappa_1, ..., \kappa_N)} \exp[ik\rho] - \frac{V(\kappa_1, ..., \kappa_N, -ik)}{V(\kappa_1, ..., \kappa_N)} \exp[-ik\rho] \right), \tag{23}$$

which solves the scattering problem on the potentials of the form (18). The dressed zero potential (18) goes to

$$u^{[N]}(\rho) = -2\frac{d^2}{d\rho^2} \ln(V(\kappa_1, ..., \kappa_N) \exp[i \sum \kappa_m \rho]), \tag{24}$$

which has also zero value, hence defining ZRP.

Let us consider asymptotes at zero, having in mind $Y_0(z) \sim \frac{2}{\pi} \ln z$, and restrict ourselves to the case $\nu = n > 0$; then $Y_n(z) \sim \frac{2^n (n-1)!}{\pi} z^{-n}$ and $J_n(z) \sim \frac{1}{2^n n!} z^n$.

The boundary condition which defines ZRP is extracted from the asymptotic solutions monomials behaviour

$$\partial^{2n}(z^n \psi_n)|_{z=0} = \frac{1}{2^n n!} = (z^n \psi_n)|_{z=0} = \alpha_n \frac{2^n (n-1)!}{\pi}, \tag{25}$$

where α_n is the partial scattering length and κ_m are roots of the algebraic equation $k^N = i\alpha_n$, that guarantee the relation $s_n = \prod_{m=1}^{m=N} \frac{ik - \kappa_m}{ik + \kappa_m}$.

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3. General remarks: one-dimensional case.

For a problem with symmetry, after separation of variables one is led to a one-dimensional description. The integrable potentials are effectively generated in terms of the Wronskian representation naturally appearing within the DT theory [11, 20]. To evaluate the Wronskians it is convenient to use Abel theorem [18]:

Theorem 1 *Let* $\psi_i(x) \in C^n, \partial = \partial/\partial x$. *If*

$$(p_0\partial^N + p_1\partial^{N-1} + ...)\psi_i(x) = 0,$$

the Wronskian derivative is given by

$$W'(\psi_1, ..., \psi_N) = \frac{p_1}{p_0}W. \tag{26}$$

Consider now the second order Schrödinger operator $L = -\partial^2 + u = (-\partial + \sigma)(\partial + \sigma)$ for a one-dimensional problem based on Equation (1), no matter what interval is of interest. The iterated Darboux transform (in a slightly different notation compared to Introduction)

$$\psi_{i+1} = (\partial - \sigma_i)\psi_i = \left(\partial - \frac{\varphi_i'}{\varphi_i}\right)\psi = \left(\varphi_i\partial\varphi_i^{-1}\right)\psi_i, \tag{27}$$

$\psi_i \in R_\lambda, \varphi_i \in R_{\mu_i}$, combines eigenspaces of L. For potentials one has

$$u_{i+1} = u_i - 2\sigma_i', \tag{28}$$

if the so-called Miura constraint

$$\sigma_i' + {\sigma_i}^2 + \mu_i = u_i \tag{29}$$

holds. The combining of DT (28) and Miura link (29) yields the chain equation (e.g. in [20])

$$(\sigma_i + \sigma_{i+1})' = \sigma_i^2 - \sigma_{i+1}^2 + \mu_i - \mu_{i+1}. \tag{30}$$

N-fold DT [11, 20] have the property:

$$(\partial - \sigma_N) \cdot ... \cdot (\partial - \sigma_1)\phi_i = 0,$$

where $\phi_i \in R_{\mu_i}$. Expanding the l.h.s. yields

$$\left(\partial^N - \sum_{i=1}^{i=N}\sigma_i\partial^{N-1} + ...\right)\phi_i = 0,$$

The application of the Crum result and the Abel theorem, Eq. (26), gives

$$u[N] = u - 2\ln_{xx} W[N] = u - 2\left(\frac{W'[N]}{W[N]}\right)' = u - 2\,(\frac{p_1}{p_0})' = u + 2\sum_{i=1}^{i=N}\sigma_i', \tag{31}$$

because $p_0 = 1,\;\; p_1 = -\sum_{i=1}^{i=N}\sigma_i$. This gives the necessary condition of the extra ZRP

$$\sum_{i=1}^{i=N}\sigma_i = const, \tag{32}$$

and an expression for the Wronskian

$$W = \exp[-\int \sum_{i=1}^{i=N} \sigma_i dx]. \tag{33}$$

Further, by insertion of $\sigma_i = \phi_i'/\phi_i = (\ln\phi_i)'$ into (33), one obtains

$$W = \exp[-\int \sum_{i=1}^{i=N} \ln_x \phi_i dx] = \exp[-\ln \prod_{i=1}^{i=N} \phi_i] = \prod_{i=1}^{i=N} \phi_i^{-1}. \tag{34}$$

N-fold application of (28) thus leads to

$$\sum_{i=1}^{i=N} \sigma_i = \frac{1}{2}\int (u_{N+1} - u_1)dx + C. \tag{35}$$

This formula results in a constant Wronskian via (33) and may be applied in scattering or eigenvalue problems if $u_i = u_0$ or $u_{N+1} - u_1 = 0$. For the cases studied in the previous section, $u_0 = \frac{-1+\nu^2}{4(k\rho)^2}, \rho \in [0,\infty)$ and $u_0 = \frac{l(l+1)}{2r^2}$, $r \in [0,\infty)$ [13].

There are many interesting applications for an operator

$$u_0 = -\frac{d}{dx}x\frac{d}{dx} + (\frac{x}{4} + \frac{m^2}{x}), \tag{36}$$

arising after separation of variables in the three-dimensional Shrödinger operator [19]. We can supply it with extra zero-range potential. The variable x here is directly linked with parabolic coordinates, and solutions describe Stark effect and particles with spin via Pauli equation. The ZRP construction is based on the scheme presented here and solutions of the correspondent equation with u_0, given in the Fock textbook [19].

In the alternative language of "superpotentials" σ_i, we base ourselves on Eq. (30). Developing it in a sequence of equalities

$$\begin{gathered}(\sigma_{i+1} + \sigma_{i+2})' = \sigma_{i+1}^2 - \sigma_{i+2}^2 + \mu_{i+1} - \mu_{i+2}, \\ \dots\dots\end{gathered} \tag{37}$$

a relation convenient for evaluation of Wronskian via (33) can be derived:

$$(\sigma_1 + 2\sigma_2 + ...2\sigma_N + \sigma_{N+1})' = \sigma_1^2 - \sigma_{N+1}^2 + \mu_1 - \mu_{N+1}. \tag{38}$$

As a test, this reproduces (35) if the link (29) is taken into account. Notice that the relevant case of periodic closure (details in e.g. [20]) $\sigma_{N+1} = \sigma_1$, i.e. $(\ln\phi_{N+1})' = (\ln\phi_1)'$ and hence $\mu_1 - \mu_{N+1} = 0$, is concerned with the constraint

$$\sum_{i=1}^{i=N} \sigma_i' = 0, \tag{39}$$

that again gives ZRP as extra potential. The solutions of the closed chain (30) on $x \in (-\infty,\infty)$ provide finite-gap potentials and the corresponding eigenfunctions of continuous spectrum [21]. A combination of finite-gap potential with ZRP on the axis $x \in (-\infty,\infty)$, implemented directly by (5), leads to eigen functions similar to (6). It is obtained by direct application of N-fold DT in Wronskian form similar to (19). The results generalize [9] for the case of delta-perturbation of periodic finite-gap potential. The result may also be obtained as a limit with respect to soliton dressing on finite-gap background [21]. Another option, defined through the relations $\sigma_{N+1} = \sigma_1 + C$ and $\sigma_1^2 - \sigma_{N+1}^2 = -2\sigma_1 C - C^2$, furnishes

$$\sum_{i=1}^{i=N} \sigma_i' = \frac{\mu_1 - \mu_N}{2} - \sigma_1 C - \frac{C^2}{2}. \tag{40}$$

Plugging this into (33), one has

$$W = \phi_1^C \exp[-(\frac{\mu_1 - \mu_N}{2} - \frac{C^2}{2})x], \tag{41}$$

i.e

$$-2\ln_{xx} W = \mu_N - \mu_1 + 2C\sigma_1 + C^2 \tag{42}$$

that gives simple expression of either iterated potential or Wronskians.

Consider one more example of a chain condition, useful in the context of ZRP theory. Let us put $u_{i+1} = a_i u_i$, then

$$\begin{gathered} \sigma_i' = \tfrac{1}{2}(u_i - u_{i+1}) = \tfrac{1}{2}u_i(1 - a_i), \\ u_1 = a_0 u_0, \\ u_2 = a_1 u_1 = a_1 a_0 u_0, \\ \ldots \\ u_{N+1} = a_N u_N = \prod_{i=0}^{N} a_i u_0. \end{gathered} \tag{43}$$

Consider an example with $u_i = b_i/x^2$, which means

$$b_i = \prod_{j=0}^{i} a_j. \tag{44}$$

On the other hand, $\sigma_i' = \frac{b_i}{2x^2}(1-a_i)$, so that $\sigma_i = \int \frac{b_i}{2x^2}(1-a_i)dx = \frac{(a_i-1)b_i}{2x} + c_i$. Miura constraint (29) implies

$$\frac{b_i}{2x^2}(1 - a_i) + [\frac{(a_i - 1)b_i}{2x} + c_i]^2 + \mu_i = b_i/x^2. \tag{45}$$

Euler equation case (16) (zero energy in quantum mechanics, long wave limit in wave theory) is recovered for $\mu_i = c_i = 0$, or

$$\frac{b_i}{2}(1 - a_i) + [\frac{(a_i - 1)^2 b_i^2}{2^2}] - b_i = 0, (1 - a_i) + [\frac{(a_i^2 - 2a_i + 1)b_i}{2}] - 2 = 0. \tag{46}$$

By virtue of (44), then

$$(a_i^2 - 2a_i + 1)\prod_{j=0}^{i} a_j = 2(1 + a_i). \tag{47}$$

The condition is a recurrent algebraic equation for a_i; for instance, if $i = 1$ we have a cubic equation for a_1

$$(a_1^2 - 2a_1 + 1)a_1 a_0 = 2(1 + a_1).$$

In general Bessel function case ($Z_n(z), z = k\rho$ in 2d), a similar transformation changes the parameters l (3d) or ν (2d case). It is known that the case is concerned with recurrent formulas

$$Z_{n+1} = \left(-\frac{d}{dz} + \frac{n}{z}\right) Z_n, \qquad Z_{n-1} = \left(\frac{d}{dz} + \frac{n}{z}\right) Z_n, \tag{48}$$

leading to a set of iterated DTs

$$Z_0 = \prod_{i=1}^{i=n} \left(\frac{d}{dz} + \frac{i}{z}\right) Z_n \tag{49}$$

that link different potentials (u_0 in our notation).

4. General algorithm in 2d. Moutard transformation.

The Moutard equation

$$\psi_{\sigma\tau} + u(\sigma,\tau)\psi = 0 \tag{50}$$

(which has the obvious connection with the Schrödinger, wave and Helmholtz equations for complex σ, τ) is covariant with respect to the Moutard transformation. The transformed coefficient (potential in mathematical physics) is given by

$$u[1] = u - 2(\log\varphi)_{\sigma\tau} = -u + \varphi_\sigma\varphi_\tau/\varphi^2 \ . \tag{51}$$

ZRP chain is generated under condition

$$(\log\varphi)_{\sigma\tau} = 0, \tag{52}$$

or, equivalently, when $\log\varphi = \Phi(\sigma) + \Psi(\tau)$ with arbitrary Φ and Ψ. So, if φ and ψ are different solutions of (50), the solution $\psi[1]$ of the twin equation with $(\psi, u) \to (\psi[1], u[1])$ is

$$\psi[1] = \psi - \varphi\Omega(\varphi,\psi)/\Omega(\varphi,\varphi) \tag{53}$$

where Ω is the integral of the exact differential form

$$d\Omega = \varphi\psi_\sigma d\sigma + \psi\varphi_\tau d\tau. \tag{54}$$

We specify now results from [20] in the following convenient form:

Theorem 2 *The N-times iterated potential built on the zero background is*

$$u = 6(\ln\Delta)_{\sigma\tau} \ , \tag{55}$$

where $\Delta = \det[\Delta_{ik}]$ and,

$$\Delta_{ik} = \int d\Omega(\phi_k, \phi_i) + C_{ik} \ , \quad C_{ik} + C_{ki} = \phi_k(0)\phi_i(0) \ ,$$

This choice fix the constants of integration C_{ik}. The integrand is determined by

$$\Omega(\phi_k, \phi_i) = -2\int [\delta_1 d\sigma + \delta_2 d\tau] \ ,$$

with $\delta_1 = \phi_k\phi_{i\sigma}$ and $\delta_2 = \phi_{k\tau}\phi_i$.

Proof directly follows from results published in [20]. Investigation of asymptotes proves multikink structure of $(\ln\Delta)_x$ for the solutions

$$\phi_k = A_k \exp(a_k\sigma + c_k) + B_k \exp(b_k\tau), \tag{56}$$

which means a multisoliton content of u.

If we perform integration from $-\infty$, to σ, τ, for kinks with the choice of constants $a_i > 0$, $b_i > 0, c_k > 0$, we obtain

$$\Delta_{ik} = [\alpha_{ik} p_i p_k \exp(\chi_i + \chi_k) + p_i \exp(\chi_i) + \beta_{ik}] \exp[(b_i + b_k)\tau] + C_{ik}, \tag{57}$$

where $\chi_i = a_i\sigma - b_i\tau + c_i$, $\alpha_{ik} = \frac{a_i}{a_i+a_k}$, $\beta_{ik} = \frac{b_i}{b_i+b_k}$, $\xi_k = a_k\sigma + \phi_k$, $A_i/B_i = p_i$. Due to the impossibility to expand Δ_{ik} as a superposition of exponents with the opposite powers like for the multisoliton determinant representation for the KP equation, a special asymptotic calculation technique was elaborated in [20]. A zero limit of the parameters that corresponds to the width of solitons results in delta - functions that form a network in $\xi\eta$ plane.

There is a way to define ZRP based on direct map of the equation (1) (equivalent to (50) for complex variables), taking into account a symmetry and class of eventual singularities [14]. For example, integrating (1) over a disk S inside a circumference L of small radius ϵ and using Gauss theorem, one has an identity that may be considered as a definition of $\delta_2(\rho, \phi)$, if the rest terms exist in some class of ψ.

An important paper of E. Ganzha, on local completeness of iterated Moutard transformations [22], allows to built coordinate systems by means of the link with Ribokur transformations and solutions of the corresponding Lame equations. The generalization is straightforward if the coordinate curve is closed [14]. In a similar way, the consideration in a vicinity of arbitrary coordinate line may lead to a ZRP in the spirit of the theorem in this paper expressed by means of Equation (55).

5. Conclusion

Integrability means also that the ZRPs may be considered in the context of evolution by Lax pair, therefore serving as a basis for point particles motion description. The technical difference is in the accounting of the σ_t term in Miura link. Note, that Goursat transformation and its generalizations open a way to the definition of ZRP for matrix spectral problems, arisen, e.g., in Pauli equation [14]. Application of the theory conventionally makes use of the Green function of a correspondent problem [2], and in the 2d case this needs special consideration [17]. The theory of ZRP perturbation of periodic finite-gap potential may be useful in models of solid state physics that would account point defects of a crystal.

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