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Interval incidence graph coloring☆

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ABSTRACT

In this paper we introduce a concept of *interval incidence coloring* of graphs and survey its general properties including lower and upper bounds on the number of colors. Our main focus is to determine the exact value of the *interval incidence coloring number* χ_{ii} for selected classes of graphs, i.e. paths, cycles, stars, wheels, fans, necklaces, complete graphs and complete k-partite graphs. We also study the complexity of the interval incidence coloring problem for subcubic graphs for which we show that the problem of determining whether $\chi_{ii} \leq 4$ can be solved in polynomial time whereas $\chi_{ii} \leq 5$ is $\mathcal{NP}$-complete.

1. Introduction

1.1. Problem definition

In the following we consider solely simple, nonempty connected graphs with the use of the standard notation of graph theory. For a given simple graph $G = (V, E)$, we define an *incidence* as a pair (v, e), where vertex $v \in V$ is one of the end-points of edge $e \in E$. The set of all incidences of G will be denoted[1] by I, where $I := \{(v, e): v \in V \wedge e \in E \wedge v \in e\}$ and $v \in e$ means that v is one of the ends of e.[2] We say that two incidences (v, e) and (w, f) are *adjacent* if and only if one of the following holds: (1) $v = w$ and $e \neq f$; (2) $e = f$ and $v \neq w$; (3) $e = \{v, w\}, f = \{w, u\}$ and $v \neq u$.

By an *incidence coloring* of G we mean a function $c: I \to \mathbb{N}$ such that $c(v, e) \neq c(w, f)$ for any adjacent incidences (v, e) and (w, f). The *incidence coloring number* of G, denoted by χ_i, is the smallest number of colors in an incidence coloring of G. The incidence coloring has been well-studied [7–9] and arises from the directed star arboricity problem [1,2,15], in which one wants to partition a set of arcs into the smallest number of forests of directed stars.

A finite nonempty set $A \subseteq \mathbb{N}$ is an *interval* if and only if it contains all integers between $\min A$ and $\max A$. For a given incidence coloring c of graph G and $v \in V$ let $A_c(v) := \{c(v, e): v \in e \wedge e \in E\}$. By an *interval incidence coloring* of graph G we mean an incidence coloring c of G such that for each vertex $v \in V$ the set $A_c(v)$ is an interval. By an interval incidence k-coloring we mean an interval incidence coloring using colors from the set $\{1, \ldots, k\}$. Interval incidence coloring is a new concept arising from a well-studied model of interval edge-coloring [4,11,14], which can be applied e.g. to the open-shop scheduling problem [12,13]. In [16] the authors introduced the concept of interval incidence coloring that models a message passing flow in networks, and in [17] the authors studied applications in one-multicast transmission per vertex model in multifiber WDM networks.

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1 To simplify notation, we write I instead of $I(G)$ whenever G is clear from the context. The same rule applies to other parameters of G appearing in the paper.

2 In our definition of a graph, an edge is a set built of its ends.

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The *interval incidence coloring number* of G, denoted by χ_{ii}, is the smallest number of colors in an interval incidence coloring of G. In this paper we study the value of χ_{ii} for some classes of graphs, its bounds as well as analyze the computational complexity of the problem of determining this number.

1.2. Multicasting communication in a multifiber WDM all-optical star network

The motivation for the present paper comes from the multicasting communication in a multifiber WDM all-optical star network, which was studied in [3,5,6]. We assume that the set of n vertices V is connected to the central vertex (star network) by at most p optical parallel fibers (multifiber). The central all-optical transmitter transforms each arriving signal to the same wavelength. Each vertex $v \in V$ has to send at most q multicasts to some other vertices $S_1(v), \ldots, S_q(v)$ ($S_i(v) \subset V$). The transmission through the central vertex uses WDM (*wavelength-division multiplexing*), i.e. different signals may be simultaneously sent through the same fiber but on different wavelengths.

The first step of the multicast transmission from vertex v to $S_i(v)$ is to send a message through a fiber to the central vertex on a set of wavelengths. In the next step, the central vertex redirects the message to each vertex of $S_i(v)$ using one of these wavelengths. The goal is to minimize the total number of wavelengths used in the simultaneous transmission of all multicasts in the network. This problem can be modeled by arc coloring of labeled (multi)digraph with certain special requirements on the set of colors [3,6].

Following [3], we define a formal model and introduce a general (p, q)-WAM problem in optical star networks. Every vertex from the set V of n vertices is connected to the central vertex with p optical fibers. A simultaneous transmission of all multicasts in this network can be modeled by a (multi)digraph D with vertex set V and with labeled arc sets going out from v which correspond to multicasts to vertex sets $S_1(v), \ldots, S_{q_v}(v)$ ($q_v \le q$), i.e. all outgoing arcs from v in set $A_i(v) = \{vw: w \in S_i(v)\}$ are labeled with i, for $i = 1, \ldots, q_v$.

We define a *p-fiber k-coloring* as a function assigning to each arc of digraph D a color from the set $\{1, \ldots, k\}$ such that for each vertex v and every color a, we have $inarc(v, a) + outlab(v, a) \le p$, where $inarc(v, a)$ denotes the number of arcs entering v and colored with a, and $outlab(v, a) = |\{i: \exists e \in A_i(v)$ and where e is colored with $a\}|$, i.e. the number of different labels of arcs outgoing v and colored with a.

Since the central vertex redirects every arriving signal to the same wavelength and different signals may be sent at the same time through the same fiber but with different wavelengths, the problem of simultaneous transmission of all multicasts in p-fiber network with the minimum number of wavelengths is equivalent to p-fiber coloring of arcs of digraph D with the minimum number of colors. Hence, we define the decision version of the problem of wavelength assignment of q-multicasts in p-fiber (optical) star networks as follows.

> The (p, q)-WAM problem: given a digraph D with at most q labels on arcs and an integer k; is there a p-fiber k-coloring of digraph D?

In the paper we consider the $(1, 1)$-WAM problem, where a fiber is unique and each vertex sends only one multicast. In this case, p-fiber coloring of multidigraph reduces to the coloring of arcs of digraph satisfying the following condition: any arc entering a vertex and any other arc at the same vertex (entering or outgoing) have different colors. This boils down to the problem of partitioning of a set of arcs into the smallest number of forests of directed stars, i.e. the previously-mentioned problem of directed star arboricity.

Let us focus our attention on the case of symmetrical communication, where every transmission from v to w implies the transmission from w and v. In this case the digraph modeling the communication between vertices is a simple graph and this problem can be reduced to the incidence coloring of graphs [7–9]. Moreover, we assume that the set of colors of incoming arcs at a vertex forms an interval. This corresponds to having consecutive wavelengths on the link between the central vertex and the destination vertex and it seems to be important for traffic grooming in WDM networks, where wavelengths could be groomed into wavebands [19].

1.3. Our results

In [18] the authors studied the problem of interval incidence coloring for subcubic bipartite graphs and trees, showing polynomial time algorithms for these classes. Moreover, they have shown that for bipartite graphs with $\Delta = 4$ the interval incidence 5-coloring is easy and 6-coloring is hard ($\mathcal{NP}$-complete).

In this paper we study the problem of interval incidence coloring for different classes of graphs, i.e. paths, cycles, stars, wheels, fans, necklaces, complete graphs and complete k-partite graphs. We focus our attention on bounding or determining the exact value of χ_{ii}. We also study the complexity of the interval incidence coloring problem for subcubic graphs for which we show that the problem of determining whether $\chi_{ii} \le 4$ is easy, and $\chi_{ii} \le 5$ is $\mathcal{NP}$-complete.

2. Bounds on χ_{ii}

In this section we construct certain lower and upper bounds on the interval incidence coloring number. Note that $\chi_i \le \chi_{ii}$, hence any lower bound for χ_i is a lower bound for χ_{ii}.

Proposition 1. *For any nonempty graph G we have*

$$\Delta + 1 \le \chi_i \le \chi_{ii} \le \chi\Delta.$$

Proof. Let $v \in V$ be a vertex of degree Δ and $u \in V$ be its neighbor. The left-hand side inequality holds because for any incidence coloring of G colors assigned to incidences from the set $\{(v, e): v \in e \wedge e \in E\} \cup \{(u, \{u, v\})\}$ must be different, and this set has $\Delta + 1$ elements.

To prove the right-hand side inequality, we divide the vertex set into χ independent sets denoted by $I_1, I_2, \ldots, I_\chi$. We create a coloring c from I to $\mathbb{N}$ in the following way: for any $v \in I_i$ let us assign colors to incidences at vertex v (i.e. of form (v, e)) in such a way that $A_c(v) = \{(i-1)\cdot\Delta + 1, \ldots, (i-1)\cdot\Delta + \deg v\}$. Let w be any neighbor of v, hence $w \in I_j$ for some $j \ne i$. From $|(i-j)\cdot\Delta| \ge \Delta$ we have $A_c(v) \cap A_c(w) = \emptyset$, hence c is an interval incidence coloring of G. Moreover, $c(I) = \bigcup_{v\in V} A_c(v) \subseteq \{1, 2, \ldots, \Delta\chi\}$, which completes the proof. □

The proof of the above theorem implies also that every nonempty graph has at least one interval incidence coloring, which is not true in case of classical interval edge coloring [14].

2.1. Lower bound for χ_{ii}

In this section we construct a lower bound for χ_{ii} (Theorem 1) which is to be used as a main tool for proving the exact values of χ_{ii} for different classes of graphs.

Proposition 2. *Let c be any interval incidence coloring of G and let v and w be adjacent vertices in G connected by edge e. If* $c(v, e) < c(w, e)$ *then* $\min A_c(v) \le c(v, e) < \min A_c(w)$ *and* $\max A_c(v) < c(w, e) \le \max A_c(w)$.

Proof. Since every incidence at vertex w (i.e. of form (w, f)) is adjacent to (v, e), we have $c(v, e) \notin A_c(w)$. $A_c(w)$ is an interval of integers, hence $c(v, e) < \min A_c(w)$. Analogously, $\max A_c(v) < c(w, e)$. □

Let c be an interval incidence coloring of a nonempty graph G. We say that vertex $v \in V$ is *minimal* (*maximal*) if and only if $\min A_c(v) = \min c(I)$ ($\max A_c(v) = \max c(I)$). We say that vertex $v \in V$ is *locally minimal* (*locally maximal*) if and only if $\min A_c(v) < \min A_c(w)$ ($\max A_c(v) > \max A_c(w)$) for every $w \in N(v)$. Observe that any interval incidence coloring has at least one locally minimal vertex. Moreover, we can recolor incidences at locally minimal (locally maximal) vertex v in such a way that $\min c(I) = \min A_c(v)$ ($\max c(I) = \max A_c(v)$).

By Proposition 2 we have the following.

Proposition 3. *Let c be any interval incidence coloring of G. Vertex v is locally minimal (locally maximal) if and only if* $(c(v, \{v, w\})) < c(w, \{v, w\})$ $(c(v, \{v, w\}) > c(w, \{v, w\}))$ *for every* $w \in N(v)$. □

Let $N(v) = \{v_1, \ldots, v_{\deg(v)}\}$ and assume that $\deg(v_1) \ge \deg(v_2) \ge \cdots \ge \deg(v_d)$. For this order of neighbors let us define a function:

$$f(v) = \max\{j + \deg(v_j): j = 1, \ldots, \deg(v)\}.$$

Lemma 4. *Let c be any interval incidence coloring of graph G. For any locally minimal (locally maximal) vertex* $v \in V$ *we have* $|c(I)| \ge f(v)$.

Proof. Let v be a locally minimal vertex. Let $i \in \{1, \ldots, \deg(v)\}$ and let $w_i \in N(v)$ be a vertex such that $c(v, \{v, w_i\}) = \min A_c(v) - 1 + i$. By Proposition 3 we have $c(v, \{v, w_i\}) < c(w_i, \{v, w_i\})$. By Proposition 2 we have $c(v, \{v, w_i\}) < \min A_c(w_i)$, hence $|c(I)| \ge c(v, \{v, w_i\}) - \min A_c(v) + 1 + |A_c(w_i)| = i + \deg(w_i)$, thus we get $|c(I)| \ge \max\{i + \deg(w_i): i = 1, \ldots, \deg(v)\}$. One can easily prove that $\max\{i + \deg(w_i): i = 1, \ldots, \deg(v)\} \ge f(v)$. The thesis holds analogously for locally maximal vertices. □

Lemma 5. *Let c be any interval incidence coloring of graph G. Let* $v \in V$ *be a vertex that is neither locally minimal nor locally maximal, then:*

1. $|c(I)| \ge \deg(v) + 2$,
2. $|c(I)| \ge \min\{\deg(u) + \deg(w): u, w \in N(v) \wedge u \ne w\}$.

Proof. (1) Since vertex v is neither locally minimal nor locally maximal, then $\min c(I) < \min A_c(v)$ and $\max A_c(v) < \max c(I)$, hence $|c(I)| \ge \deg(v) + 2$.

Let us define $U(v) = \{u \in N(v): c(v, \{v, u\}) > c(u, \{v, u\})\}$ and $W(v) = \{w \in N(v): c(v, \{v, w\}) < c(w, \{v, w\})\}$. Observe that $U(v) \cap W(v) = \emptyset$ and $U(v) \cup W(v) = N(v)$. By Proposition 3 we have $U(v) \ne \emptyset$ and $W(v) \ne \emptyset$. Let $U_c(v) = \{c(v, \{v, u\}): u \in U(v)\}$ and $W_c(v) = \{c(v, \{v, w\}): w \in W(v)\}$, then $U_c(v) \cup W_c(v) = A_c(v)$.

(2) It is easy to see that $\min U_c(v) - \max W_c(v) \le 1$, otherwise $\min U_c(v) \ge 2 + \max W_c(v) > 1 + \max W_c(v)$, which contradicts that $A_c(v)$ is an interval. Let $u \in U(v)$ and $w \in W(v)$ be vertices such that $c(v, \{v, u\}) = \min U_c(v)$ and $c(v, \{v, w\}) = \max W_c(v)$. By Proposition 2 we have $\max A_c(u) < c(v, \{v, u\})$ and $c(v, \{v, w\}) < \min A_c(w)$, thus $\max A_c(u) \le c(v, \{v, u\}) - 1 \le c(v, \{v, w\}) < \min A_c(w)$, hence we get $|c(I)| \ge |A_c(u)| + |A_c(w)| \ge \min\{\deg(u) + \deg(w): u, w \in N(v) \wedge u \ne w\}$. □

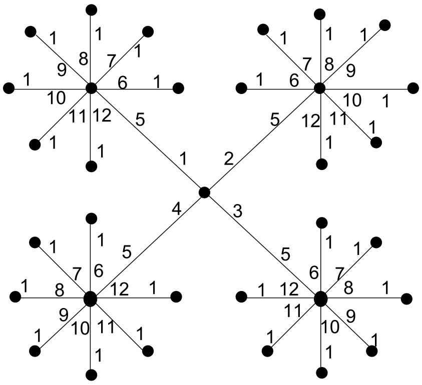

Fig. 1. The interval incidence 12-coloring of $S(8, 4)$.

As a consequence of the above properties and lemmas, we get a theorem which is a very useful tool for proving lower bounds on the interval incidence coloring number for different classes of graphs.

Theorem 1. *For any connected graph G we have*

$$\chi_{ii}(G) \geq \max\{\min_{v \in V} f(v), \max_{v \in V} \min\{f(v), g(v)\}\},$$

where

$$g(v) = \max\{\deg(v) + 2, \min\{\deg(u) + \deg(w) \colon u, w \in N(v) \wedge u \neq w\}\}.$$

Proof. Let c be any minimal interval incidence coloring of graph G. There is at least one minimally vertex, therefore by Lemma 4 we have $\chi_{ii} \geq \min_{v \in V} f(v)$. Now, take any vertex $v \in V(G)$. If v is locally minimal or locally maximal, then by Lemma 4 we have $\chi_{ii} \geq f(v)$. Otherwise, if v is not locally minimal nor locally maximal, then by Lemma 5 we have $\chi_{ii} \geq g(v)$. Thus we get $\chi_{ii} \geq \max_{v \in V} \min\{f(v), g(v)\}$, which completes the proof. $\square$

Corollary 6. *For any connected graph G with vertex v of degree $k \geq 1$ such that any of its neighbors has a degree at least $d \geq k$ we have $\chi_{ii} \geq k + d$.*

Proof. Observe that $f(v) \geq k + d$ and $g(v) \geq \min\{\deg(u) + \deg(w) \colon u, w \in N(v) \wedge u \neq w\} \geq 2d$, hence by Theorem 1 we get $\chi_{ii} \geq k + d$. $\square$

Corollary 7. *For any regular graph G we have $\chi_{ii}(G) \geq 2\Delta$.* $\square$

Theorem 2. *For any graph G we have $\chi_{ii} \geq 2(\omega - 1)$, where ω is the clique number of G.*

Proof. By definition of ω, K_ω is a subgraph of G. For any minimal coloring c of G (i.e. using χ_{ii} colors from the set $\{1, 2, \ldots, \chi_{ii}\}$), there is a vertex $v \in V(K_\omega)$ such that $\min A_c(v) < \min A_c(u)$ for every $u \in V(K_\omega) \setminus \{v\}$, hence by Proposition 2 $V(K_\omega) \setminus \{v\} \subset W(v)$, where $W(v) = \{w \in N(v) \colon c(v, \{v, w\}) < c(w, \{v, w\})\}$. Let $q = |W(v)|$ and take any $i \in \{1, \ldots, q\}$ and let $w_i \in W(v)$ be a vertex such that $c(v, \{v, w_i\}) = \min W_c(v) - 1 + l_i$, where $l_1 < l_2 < \cdots < l_q$. Observe that $l_i \geq i$. By Proposition 2 we have $c(v, \{v, w_i\}) < \min A_c(w_i)$, hence $|c(I)| \geq c(v, \{v, w_i\}) - \min W_c(v) + 1 + |A_c(w_i)| = l_i + \deg(w_i)$, hence $c(I) \geq \max\{l_i + \deg(w_i) \colon i = 1, \ldots, q\} \geq 2(\omega - 1)$. $\square$

3. Note on the hereditary property

For a given graph G let us define the problem $IIC_{SUB}(G)$ as follows: is there a subgraph H of G such that $\chi_{ii}(H) > \chi_{ii}(G)$? Surprisingly, for some trees the answer to this question is *yes*.

We construct the following family of trees $S(d, k)$, for $d \geq k$. The central vertex of $S(d, k)$ is a vertex of degree k, all its neighbors have a degree d and all other vertices are leaves. It is easy to observe that $\Delta(S(d, k)) = d$ and for $k > 1$ there are exactly $k(d - 1)$ leaves. Fig. 1 presents the case of $S(d, k)$ where $d = 8$ and $k = 4$.

Lemma 8. $\chi_{ii}(S(d, k)) = d + k$ *for any integer $0 < k \leq d$.*

Proof. By Corollary 6 we have $\chi_{ii}(S(d, k)) \geq d+k$. Let v be the central vertex of $S(d, k)$ and let $v_1, v_2, \ldots, v_k$ be its neighbors. Let $c \colon I \to \mathbb{N}$ be any function that satisfies the following conditions:

- $c(v, \{v, v_i\}) = i$ and $c(v_i, \{v_i, v\}) = k + 1$ for $i = 1, \ldots, k$;
- $c(u, \{u, v_i\}) = 1$ for each leaf u that is adjacent to v_i;
- $A_c(v_i) = \{k + 1, \ldots, d + k\}$ for $i = 1, \ldots, k$.

It is easy to notice that c is an interval incidence coloring of $S(d, k)$, hence we get $\chi_{ii}(S(d, k)) = d + k$. $\square$

Let us denote by $S^i(d,k)$ a graph $S(d,k)$ with a path of length i attached to the central vertex. For instance, $S^1(1,1)$ is a path P_3 and $S^2(2,2) = S(2,3)$.

Lemma 9. *$\chi_{ii}(S^i(d,d)) \leq 2d-1$ for any integers $d \geq 4$ and $i \geq 1$.*

Proof. Let $p = \lfloor \frac{d}{2} \rfloor$ and $q = \lceil \frac{d}{2} \rceil$, obviously $p+q=d$. Let v be the central vertex of $S^i(d,d)$ and $v_1, v_2, \ldots, v_d$ be its neighbors of degree d. Let u be a vertex adjacent to v that lies on the attached path (of length i). Let $c\colon I \to \mathbb{N}$ be any function that satisfies the following conditions:

- $c(v,\{v,v_i\}) = d+i$ and $c(v_i,\{v_i,v\}) = 1$ for $i=1,\ldots,p$;
- $c(v,\{v,v_{p+i}\}) = d-i$ and $c(v_i,\{v_{p+i},v\}) = 2d-1$ for $i=1,\ldots,q$;
- $c(v,\{v,u\}) = d$ and $c(u,\{u,v\}) = 1$;
- $c(w,\{w,v_i\}) = 2d-1$ for each leaf w that is adjacent to v_i for $i=1,\ldots,p$;
- $c(w,\{w,v_{p+i}\}) = 1$ for each leaf w that is adjacent to v_{p+i} for $i=1,\ldots,q$;
- $A_c(v_i) = \{1,\ldots,d\}$ for $i=1,\ldots,p$;
- $A_c(v_{p+i}) = \{d,\ldots,2d-1\}$ for $i=1,\ldots,q$.

Now, extend c to incidences of the attached path using colors 1, 2, 3, 4. It is easy to notice that this is a proper interval incidence coloring of $S^i(d,k)$ using $2d-1$ colors. □

Observe that $\Delta(S^i(d,d)) = d+1$ and $|V(S^i(d,d))| = d^2+1+i$. Since for any $d>0$ and $i>0$ graph $S(d,d)$ is a proper induced subgraph of $S^i(d,d)$, by Lemmas 8 and 9 we have the following.

Theorem 3. *The problem $IIC_{SUB}(T)$ has the positive answer for some trees. Moreover,*

- *for every $k \geq 5$ there is a tree T with $\Delta(T) = k$ such that $IIC_{SUB}(T) = yes$.*
- *for every $n \geq 18$ there is n-vertex tree T such that $IIC_{SUB}(T) = yes$.*

An interesting question that arises at that point is whether there exist small graphs for which $IIC_{SUB} = yes$. Another interesting problem is verifying IIC_{SUB} in polynomial time on trees.

4. Exact values of χ_{ii} for selected classes of graphs

In the present section we give exact values of χ_{ii} or construct exact (polynomial time) algorithms for selected classes of graphs.

4.1. Paths, cycles and stars

Observe that $\chi_{ii}(P_2) = 2$ and $\chi_{ii}(P_3) = \chi_{ii}(P_4) = 3$. If G is a path with at least 5 vertices or a cycle, then by Corollary 6 we have $\chi_{ii}(G) \geq 4$. Obviously, there is an interval incidence 4-coloring of G.

Proposition 10. *For any path or cycle we have $\chi_{ii} \leq \Delta+2$. Moreover,*

- *if G is a path of the length at most 4, then $\chi_{ii}(G) = \Delta+1$,*
- *if G is a path with at least 5 vertices or a cycle, then $\chi_{ii}(G) = \Delta+2$.* □

Proposition 11. *$\chi_{ii} = \Delta+1$ for any star with at least two vertices.* □

4.2. Wheels, fans and necklaces

Let $n \geq 3$. By wheel W_n we denote a graph constructed by adding a new vertex to a cycle C_n and linking it to each vertex from C_n. Let v be the central vertex of W_n and $v_1,\ldots,v_n$ be its neighbors, such that v_i and v_{i+1} are adjacent.

Proposition 12. *$\chi_{ii}(W_{2k}) = \Delta+2 = 2k+2$, for $k \geq 2$.*

Proof. We have $g(v) \geq \Delta+2$ and $f(v) = \Delta+3$, hence by Theorem 1 we get $\chi_{ii} \geq \min\{g(v), f(v)\} = \Delta+2$. Now, we construct interval incidence $(\Delta+2)$-coloring as follows: $A_c(v) = \{2,3,\ldots,\Delta+1\}$, $A_c(v_{2i}) = \{1,2,3\}$, $A_c(v_{2i-1}) = \{\Delta,\Delta+1,\Delta+2\}$ for $i=1,\ldots,k$. While coloring the incidences, we must respect the following conditions: $c(v,\{v,v_{2i}\}) \geq 4$, $c(v,\{v,v_{2i-1}\}) \leq \Delta-1$, $c(v_{2i},\{v,v_{2i}\}) = 1$ and $c(v_{2i-1},\{v,v_{2i-1}\}) = \Delta+2$. A sample 10-coloring of W_8 is presented in Fig. 2. □

Proposition 13. *$\chi_{ii}(W_{2k+1}) = \Delta+3$, for $k \geq 1$.*

Proof. We have $g(v) \geq \Delta+2$ and $f(v) = \Delta+3$, hence by Theorem 1 we have $\chi_{ii} \geq \min\{g(v), f(v)\} = \Delta+2$. Now, assume that coloring c is an interval incidence $(\Delta+2)$-coloring of W_{2k+1}, $\Delta = 2k+1$. If v is minimal or maximal, then by Lemma 4 we have a contradiction $|c(I)| \geq f(v) = \Delta+3$. Hence, $A_c(v) = \{2,3,\ldots,\Delta+1\}$. Observe that $c(v_i,\{v,v_i\}) \in \{1,\Delta+2\}$ for any $i=1,\ldots,2k+1$, which implies that every v_i is minimal or maximal. Since the outer cycle is odd and no two minimal (no two maximal) vertices are adjacent, we have a contradiction. We construct $(\Delta+3)$-coloring as

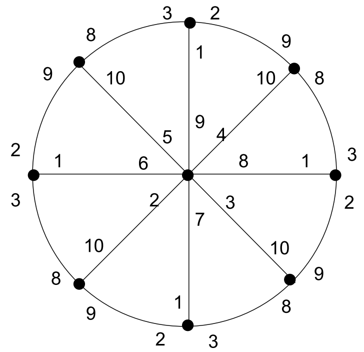

Fig. 2. The minimal interval incidence coloring of W_8.

follows: $c(v, \{v, v_i\}) = i$, $c(v_i, \{v, v_i\}) = \Delta + 1$ for every $i = 1, \dots, \Delta$, $c(v_\Delta, \{v_1, v_\Delta\}) = \Delta + 2$, $c(v_\Delta, \{v_{\Delta-1}, v_\Delta\}) = \Delta + 3$, $c(v_{\Delta-1}, \{v_{\Delta-1}, v_\Delta\}) = \Delta$. At last, $c(v_{2i-1}, \{v_{2i-1}, v_{2i}\}) = \Delta - 1$, $c(v_{2i}, \{v_{2i-1}, v_{2i}\}) = \Delta + 2$, for $i = 1, \dots, k$ and $c(v_{2i-1}, \{v_{2i-2}, v_{2i-1}\}) = \Delta$, $c(v_{2i-2}, \{v_{2i-2}, v_{2i-1}\}) = \Delta + 3$, for $i = 2, \dots, k$. □

By fan F_n we denote a graph obtained from a wheel W_n by removing one edge from the outer cycle. Let v be the central vertex of F_n and $v_1, \dots, v_n$ be its neighbors, such that v_i and v_{i+1} are adjacent. One can observe that, by removing one edge from a wheel W_{2k}, it is easy to obtain an interval incidence $(\Delta + 2)$-coloring of F_{2k}. For F_{2k+1} one can easily construct a coloring as follows: $c(v, \{v, v_{2i-1}\}) = i + 1$, $c(v_{2i-1}, \{v, v_{2i-1}\}) = \Delta + 2$, $c(v, \{v, v_{2i}\}) = \Delta + 2 - i$, $c(v_{2i}, \{v, v_{2i}\}) = 1$, for $i = 1, \dots, k$, $A_c(v_1) = A_c(v_{2k+1}) = \{\Delta + 1, \Delta + 2\}$, $A_c(v_{2i}) = \{1, 2, 3\}$, $A_c(v_{2i+1}) = \{\Delta, \Delta + 1, \Delta + 2\}$. Thus we get

Proposition 14. *$\chi_{ii}(F_k) = \Delta + 2$, for $k \geq 1$.* □

By θ_m ($m \geq 3$) we denote the class of graphs such that each graph consists of m paths (P_i, $i \geq 3$) of any length joined to common endpoints. We say that G is a *necklace* if $G \in \theta_m$ for some $m \geq 3$.

Proposition 15. *If G is a necklace then $\chi_{ii}(G) = \Delta + 2$.*

Proof. By Theorem 1 we have $\chi_{ii} \geq \Delta + 2$. We construct a $(\Delta + 2)$-coloring of any necklace. Let a and b be the two common endpoints of $m = \Delta$ paths. Let $A_c(a) = A_c(b) = \{1, \dots, \Delta\}$ and take any coloring that assigns different colors at incidences at a and b connected with a path. Since $m \geq 3$, we can color each path with colors from $\{1, 2, \Delta, \Delta + 1, \Delta + 2\}$. If the length of a path is odd (including endpoints a and b) or is at least 4, one can easily color incidences on this path. If there are exactly two vertices on a path joining a and b, it suffices to color this path with colors Δ, $\Delta + 1$, $\Delta + 2$, because one of the colors of incidences at a or b (and joined with a path) differs from Δ. □

4.3. *Complete k-partite graphs*

Graph $G = (V, E)$ is complete k-partite if its vertex set can be partitioned into k disjoint independent sets $V_1, \dots, V_k$ and for every $v \in V_i$, $u \in V_j$ and for every $i, j = 1, \dots, k (i \neq j)$ there exists an edge between v and u. We denote this graph by $K_{p_1, p_2, \dots, p_k}$, where $|V_i| = p_i$ and $p_1 + \dots + p_k = n$. If $k = n$, this is a complete graph K_n.

Proposition 16. $\chi_{ii}(K_n) = 2\Delta$.

Proof. Let $v_1, \dots, v_k$ be the vertices of graph K_n. Let $c(v_i, \{v_i, v_{(i+j) \bmod n}\})$ be $i + j$ for $i = 1, \dots, n - 1$ and $j = 1, \dots, n - 1$. It is easy to check that c is an interval incidence coloring. □

Proposition 17. $\chi_{ii}(K_{p_1, p_2, \dots, p_k}) = 2n - \max\{p_i + p_j : i, j = 1, \dots, k \wedge i \neq j\}$.

Proof. Let c be an interval incidence χ_{ii}-coloring and let v be a minimal vertex in c. Let u be a neighbor of v such that $c(v, \{u, v\}) = \max A_c(v)$. Thus $A_c(u) \cap A_c(v) = \emptyset$ and $\chi_{ii} \geq |A_c(u)| + |A_c(v)| = \deg u + \deg v \geq 2n - \max\{p_i + p_j : i, j = 1, \dots, k \wedge i \neq j\}$.

Now, let us construct an interval incidence coloring of $K_{p_1, p_2, \dots, p_k}$ that uses exactly $2n - \max\{p_i + p_j : i, j = 1, \dots, k \wedge i \neq j\}$ colors. Without loss of generality we can assume that $p_1 \leq p_2 \leq \dots \leq p_k$ and $V_i = \{v_1^i, v_2^i, \dots, v_{p_i}^i\}$. Let

$$c(v_j^i, \{v_j^i, v_s^r\}) = \begin{cases} p_1 + p_2 + \dots + p_{r-1} + s & r > i, \\ n + p_1 + p_2 + \dots + p_{r-1} + s & r < i. \end{cases}$$

We have $A_c(v_j^i) = \{p_1 + \dots + p_i + 1, \dots, p_1 + \dots + p_k = n, n + 1, \dots, n + p_1 + \dots + p_{i-1}\}$ and $|A_c(v_j^i)| = p_1 + \dots + p_{i-1} + p_{i+1} + \dots + p_k = n - p_i = \deg(v_j^i)$. Assuming that $r < i$, we have $c(v_j^i, \{v_j^i, v_s^r\}) = n + p_1 + p_2 + \dots + p_{r-1} + s > n + p_1 + p_2 + \dots + p_{r-1} = \max A_c(v_s^r)$. If $r > i$, then $c(v_j^i, \{v_j^i, v_s^r\}) = p_1 + p_2 + \dots + p_{r-1} + s < \min A_c(v_j^i)$, hence c is an interval incidence coloring. □

5. Interval incidence *k*-coloring for subcubic graphs: 4 is easy, 5 is hard

In this section we focus on the interval incidence k-coloring problem for subcubic graphs, i.e. we prove that 4-coloring of subcubic graphs is easy (i.e. the existence of such coloring is verifiable in polynomial time) and 5-coloring of subcubic graphs is hard ($\mathcal{NP}$-complete).

5.1. 4-coloring of subcubic graphs

Let G be a subcubic graph, and $V_i \subset V(G)$ be the set of vertices of degree i.

Lemma 18. *If* $\chi_{ii}(G) = 4$ *then*

(i) *each vertex* v *of degree* 3 *has at most one neighbor of degree* 3*,*
(ii) *each vertex* v *of degree* 3 *has at least one neighbor of degree* 1*,*
(iii) *no two vertices of degree* 3 *have a common neighbor of degree* 2*.*

Proof. Suppose $\chi_{ii}(G) = 4$ and let c be any 4-coloring of G. The property (i) follows from the fact that in the set $\{1, 2, 3, 4\}$ one can find only two intervals of length 3, i.e. $\{1, 2, 3\}$ and $\{2, 3, 4\}$, and in each of them there is only one element not belonging to the second one. In order to prove the property (ii) observe that if $A_c(v) = \{1, 2, 3\}$ ($A_c(v) = \{2, 3, 4\}$), then vertex u for which the incidence $(v, \{v, u\})$ is colored with 3 (with 2, respectively) must be a leaf. Let us move to prove the property (iii). For $v \neq u$ let $\deg(v) = \deg(u) = 3$. Suppose conversely that there exists a vertex x of degree 2 that is a common neighbor of v and u. Obviously, $c(x, \{x, v\}) \notin \{2, 3\}$ and $c(x, \{x, u\}) \notin \{2, 3\}$. As $\deg(x) = 2$, $A_c(x)$ is $\{1, 2\}$ or $\{2, 3\}$ or $\{3, 4\}$, hence $2 \in A_c(x)$ or $3 \in A_c(x)$, a contradiction. □

Let G^* be a (multi)graph with $V(G^*) = V_3 \cup X$, where X is a set of vertices representing some paths in G, and the edge set defined as follows: two vertices u and v from V_3 are adjacent if and only if $\{u, v\} \in E(G)$ or there is a path of the length 3 or 5 consisting only of vertices from V_2 that connects u and v in G. Observe that if there is such a path from v to v, then we have a loop (cycle of the length 1) at vertex v. If there is a path of length 4 between u and v ($u, v \in V_3$) consisting only of vertices from V_2, then we add vertex $x \in X$ replacing this path and connect x with both u and v.

By Lemma 18(ii), if $\chi_{ii}(G) = 4$, then the induced subgraph $G[V \setminus V_1]$ is composed of paths and cycles, hence G^* is composed of paths and cycles.

Lemma 19. *If* $\chi_{ii}(G) = 4$ *then*

(iv) *graph* G^* *has no cycle of odd length.*

Proof. Suppose $\chi_{ii} = 4$ and let c be any 4-coloring of G. Observe that any vertex $v \in V_3$ has $A_c(v) = \{1, 2, 3\}$ or $\{2, 3, 4\}$. Suppose conversely that G^* has an odd cycle. This implies one of the following:

(1) there are vertices $u, v \in V_3$ adjacent in G^* (possible $u = v$) such that $A_c(u) = A_c(v)$,
(2) there are vertices $u, v \in V_3$ adjacent in G^* to a common neighbor $x \in X$ such that $A_c(u) \neq A_c(v)$.

(1): Assume that $A_c(u) = A_c(v) = \{1, 2, 3\}$. In this case, u and v are joined with a path of the length 3 or 5 in G, hence there are two vertices x_u and x_v of degree 2 (in G) such that x_u is adjacent to u, x_v is adjacent to v, and $A_c(x_u) = A_c(x_v) = \{3, 4\}$. In the first case (path of the length 3) x_u and x_v are adjacent, a contradiction. In the second case (path of the length 5), there are two other adjacent vertices of degree 2 (in G) y_u and y_v such that y_u is adjacent to x_u, y_v is adjacent to x_v, and $A_c(x_u) = A_c(x_v) = \{1, 2\}$, a contradiction.

(2): Assume that $A_c(u) = \{1, 2, 3\}$ and $A_c(v) = \{2, 3, 4\}$. In this case, u and v are joined with a path of the length 4 in G, hence there are three vertices x_u, x_v and y of degree 2 (in G) such that x_u is adjacent to u, x_v is adjacent to v, y is adjacent to both x_u and x_v, and $A_c(x_u) = \{3, 4\}$, $A_c(x_v) = \{1, 2\}$, $A_c(y) = \{2, 3\}$, a contradiction. □

Lemma 20. *If* G *satisfies the properties* (i)–(iii) *of* Lemma 18 *and property* (iv) *of* Lemma 19*, then* $\chi_{ii}(G) = 4$*.*

Proof. Suppose that subcubic graph G satisfies the properties (i)–(iv). Since G satisfies the property (iv), we can construct a 2-coloring of G^* (with colors $a < b$). As G satisfies the properties (i)–(iii), this coloring can be extended in a greedy manner to a 3-coloring p of the whole graph G (possibly, using the third color $c > b$) in such a way that:

- each vertex of degree 1 is assigned color a (if its neighbor is colored with b or c) or color b (if its neighbor is colored with a),
- if color c is used at vertex v of degree 2, then the distance from v to any vertex of degree 3 is at least 3,
- there is at most one vertex colored with c on every path between vertices of degree 3 consisting of vertices from V_2 only.

Now, we construct incidence coloring q of graph G. Let us assign a set of colors to each vertex v as follows:

- if $\deg(v) = 1$, then $A_q(v) = \{1\}$ (if $p(v) = a$) or $A_q(v) = \{4\}$ (if $p(v) = b$),
- if $\deg(v) = 2$, then $A_q(v) = \{1, 2\}$ (if $p(v) = a$), $A_q(v) = \{3, 4\}$ (if $p(v) = b$) or $A_q(v) = \{2, 3\}$ (if $p(v) = c$),
- if $\deg(v) = 3$, then $A_q(v) = \{1, 2, 3\}$ (if $p(v) = a$) or $A_q(v) = \{2, 3, 4\}$ (if $p(v) = b$).

Let $\{u, v\} \in E(G)$. From properties of graph G and coloring p we can distribute among incidences some colors from the above sets in the following manner:

- let $\deg(u) = 3$ and $\deg(v) = 3$, if $p(u) = a$ and $p(v) = b$, then $q(u, \{u, v\}) = 1$ and $q(v, \{v, u\}) = 4$,
- let $\deg(u) = 2$ and $\deg(v) = 3$, if $p(u) = a$ ($p(u) = b$) and $p(v) = b$ ($p(v) = a$), then $q(u, \{u, v\}) = 1$ ($q(u, \{u, v\}) = 4$) and $q(v, \{v, u\}) \geq 3$ ($q(v, \{v, u\}) \leq 2$),
- let $\deg(u) = 2$ and $\deg(v) = 2$, if $p(u) = a$ and $p(v) = b$, then $q(u, \{u, v\}) \leq 2$ and $q(v, \{v, u\}) \geq 3$,
- let $\deg(u) = 2$ and $\deg(v) = 2$, if $p(u) = a$ ($p(u) = b$) and $p(v) = c$, then $q(u, \{u, v\}) = 1$ ($q(u, \{u, v\}) = 4$) and $q(v, \{v, u\}) = 3$ ($q(v, \{v, u\}) = 2$),
- let $\deg(u) = 1$ and $\deg(v) \leq 3$, if $p(u) = a$ ($p(u) = b$) and $p(v) = b$ ($p(v) = a$), then $q(u, \{u, v\}) = 1$ ($q(u, \{u, v\}) = 4$) and $q(v, \{v, u\}) = 2$ or $q(v, \{v, u\}) = 3$.

It is easy to see that q is a proper interval incidence 4-coloring of graph G. $\square$

By Lemmas 18–20 the problem of interval incidence coloring using 4 colors for subcubic graphs is equivalent to verifying the properties (i)–(iv), hence we get the following.

Theorem 4. *For subcubic graphs the problem of interval incidence coloring with 4 colors can be solved in linear time.* $\square$

5.2. 5-coloring of subcubic graphs is $\mathcal{NP}$-complete

Let us denote by $\overline{3SAT}$ the restriction of the classical $3SAT$ problem, defined as follows: $\overline{3SAT}$ is the problem of satisfiability of a given CNF formula with 2 or 3 literals in each clause and satisfying the condition that for any variable x the total number of clauses with literals x or $\neg x$ is no more than 3. Moreover, we may assume that for each variable both x and $\neg x$ appear in the formula. This problem is known to be $\mathcal{NP}$-complete [10].

Theorem 5. *The problem of verifying whether $\chi_{ii} \leq 5$ is $\mathcal{NP}$-complete for subcubic graphs.*

Proof. We construct a polynomial time reduction from $\overline{3SAT}$ to the problem of interval incidence 5-coloring. For a given formula $\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ of $\overline{3SAT}$, we construct a subcubic graph $G(\phi)$ as follows:

- each clause $\{x \vee y\}$ is represented by gadget T_2 (Fig. 3),
- each clause $\{x \vee y \vee z\}$ is represented by gadget T_3 (Fig. 4),
- each literal is represented by vertex x or y in T_2, or x, y or z in T_3,
- for each variable x we join all literals x and $\neg x$ by a gadget:
  - F_2 (Fig. 5), if there is one instance of x and one instance of $\neg x$,
  - F_3 (Fig. 6), if there are two instances of x and one instance of $\neg x$, or conversely (one x and two $\neg x$).

Note that, in fact, we take isomorphic copies of graphs T_2, T_3 and F_2, F_3. Observe that $\Delta(G(\phi)) = 3$ and there is no vertex of degree 3 with three neighbors of degree 3. Let c be any interval incidence 5-coloring of $G(\phi)$ such that incidences at pendant vertices are colored with 1 or 5 (it is always possible to recolor any 5-coloring). We define auxiliary vertex labeling p of G with labels a, a' and b. For each vertex v we define $p(v)$ as follows: if $1 \in A_c(v)$, then $p(v) = a$, if $5 \in A_c(v)$, then $p(v) = a'$, otherwise $p(v) = b$. If $\deg(v) = 3$, then labels a, a', b correspond to sets $\{1, 2, 3\}$, $\{3, 4, 5\}$, $\{2, 3, 4\}$, respectively. Since there are no two adjacent vertices of degree two, it is easy to observe that any two adjacent vertices have different labels.

Claim 21. *Labeling p is a vertex coloring of $G(\phi)$ and the following holds:*

(i) *each vertex of degree ≤ 2 is assigned label a or a',*
(ii) *at most one neighbor of a vertex of degree ≥ 2 is assigned label b,*
(iii) *every cycle $x_1, \ldots, x_6$ has exactly two vertices labeled with b: x_1 and x_4, or x_3 and x_6.*

Proof. Obviously, the property (i) holds for pendant vertices. Since every vertex v of degree 2 has two neighbors of degree 3 and each color set of incidences at vertex of degree 3 contains color 3, then $3 \notin A_c(v)$, thus we get the property (i). Suppose conversely that two neighbors of vertex v ($\deg(v) \geq 2$) have label b, i.e. color set of incidences is $\{2, 3, 4\}$. Then $1 \in A_c(v)$ and $5 \in A_c(v)$, a contradiction. Thus the property (ii) holds. To prove the property (iii) observe that all vertices of the even cycle $x_1, \ldots, x_6$ are labeled with a and a', or exactly two of its vertices are labeled with b. By the property (i) the labels assigned to vertices x_2 and x_5 cannot be a and a' (respectively) and cannot be a' and a. By the property (ii) the labels cannot be both equal to b. Assume that x_2 is labeled with b, then by the property (ii) there is no other vertex in the cycle labeled with b, a contradiction. Hence the property (iii) follows. $\square$

Claim 22. *The following property holds for any variable x for which gadgets representing clauses containing x or $\neg x$ are joined with an appropriate gadget F_2 or F_3:*

(iv) *if any vertex x ($\neg x$) is labeled with b, then any vertex $\neg x$ (x) is labeled with a or a'.*

Proof. Assume that x has assigned label b, then by Claim 21(ii) vertex x_1 is labeled with a or a', hence by Claim 21(iii) both x_3 and x_6 are labeled with b, thus $\neg x$ cannot be labeled with b. Other cases can be considered analogously. $\square$

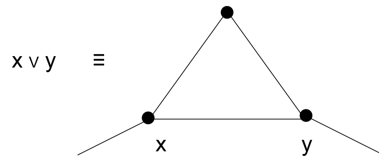

Fig. 3. Gadget T_2 for clause $\{x \vee y\}$.

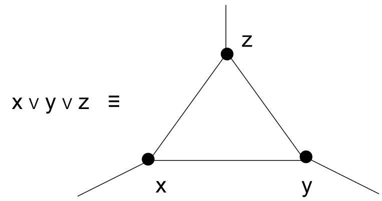

Fig. 4. Gadget T_3 for clause $\{x \vee y \vee z\}$.

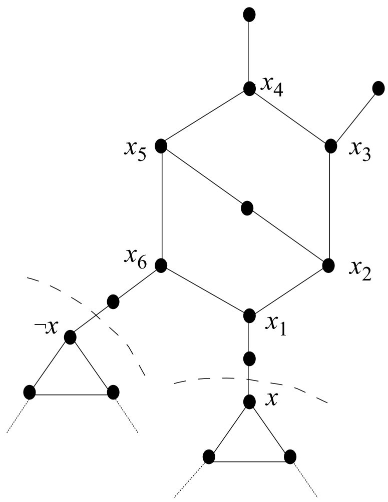

Fig. 5. Gadget F_2 connecting two literals: x and $\neg x$.

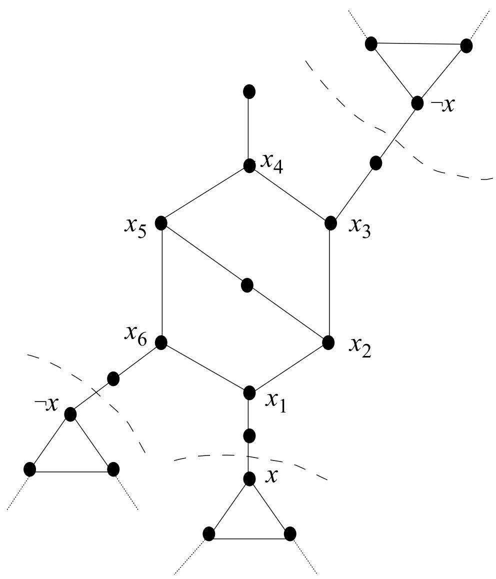

Fig. 6. Gadget F_3 connecting three literals: one x and two $\neg x$, or two x and one $\neg x$.

Since graphs T_2 and T_3 are triangles, by Claim 21(i) we have the following.

Claim 23. *In any gadget T_2 or T_3 labels assigned to vertices are different, and one of the vertices of degree 3 is labeled with b.* □

Now, let us demonstrate that having a proper labeling of vertices of $G(\phi)$ we can construct an interval incidence 5-coloring of $G(\phi)$.

Claim 24. *Consider any labeling p of graph $G(\phi)$ satisfying all properties from Claims 21 and 22, then there is an interval incidence 5-coloring of $G(\phi)$.*

Proof. We construct an interval incidence 5-coloring c satisfying the following property: for any vertex v we have $1 \in A_c(v)$ if and only if $p(v) = a$ and $5 \in A_c(v)$ if and only if $p(v) = a'$. By Claim 21(i) we can color the incidence at vertex of degree 1 with color 1 (if its label is equal to a) or with 5 (label is a'). Take any vertex v of degree 2 and assume that it is labeled with a (a' analogously). By Claim 21(ii) either both neighbors of v must be labeled with a' or one with a' and the other with b, hence it is easy to color both incidences at v with colors from $\{1, 2\}$. Now, observe that for each vertex v of degree 3 one of the following holds:

(1) v is equal to vertex of degree 3 in some gadget T_2 or T_3,
(2) v is equal to x_2 or x_5 in some gadget F_2 or F_3,
(3) v is equal to x_1, x_3, x_4 or x_6 in some gadget F_2 or F_3.

Let us assume that $p(v) = a'$ (analogously $p(v) = a$). By Claim 21(ii) the sequence of labels assigned to all neighbors of v is a, a, b or a, a, a, and by Claim 21(i) vertex u of degree 2 (adjacent to v) is colored with a. Hence, it is easy to color all incidences at vertex v with colors $\{3, 4, 5\}$, starting from $c(v, \{u, v\}) = 3$. Now, let us assume that $p(v) = b$, then v fulfills property (1) or (3), and assume that vertex u of degree 2 (outside a gadget T_2 and adjacent to v) is labeled with $p(u) = a$ ($p(u) = a'$ analogously). In the first case (1) sequence of labels assigned to all neighbors of v is obviously a, a, a'. In the second case (3) by Claim 21(iii) this sequence is equal to a, a, a', same as in the first case. Analogously, starting from $c(v, \{u, v\}) = 3$, it is easy to color all incidences at vertex v. □

Now, we will show that formula ϕ is satisfiable if and only if graph $G(\phi)$ admits an interval incidence 5-coloring.

($\Rightarrow$) Assume that formula ϕ is satisfiable and let w be an assignment of values TRUE and FALSE to each variable such that $w(\phi)$ is TRUE, i.e. each clause contains at least one literal x for which $w(x) =$ TRUE. We construct labeling p as follows: for each clause take exactly one literal with value TRUE and label the corresponding vertex in the graph T_2 or T_3 with b. Now, let us assume the case that variable x is contained in three clauses and literal x appears once, literal $\neg x$ appears twice (other cases analogously). Apply the following rules of labeling:

Case 1. If none of $\neg x$ is labeled with b, then label x_3 and x_6 with b.
Case 2. If one or two $\neg x$ are labeled with b, then label x_1 and x_4 with b.

Since the graph induced by unlabeled vertices is bipartite, we can label the remaining vertices with a or a'. It is easy to verify, that such a labeling p satisfies all properties from Claims 21 and 22. By Claim 24 there is an interval incidence 5-coloring of $G(\phi)$.

($\Leftarrow$) Assume that there is an interval incidence 5-coloring c of graph $G(\phi)$. For each vertex v we define $p(v)$ as follows: $p(v) = a$ if $1 \in A_c(v)$, and $p(v) = a'$, if $5 \in A_c(v)$, and $p(v) = b$, otherwise. From each clause corresponding to gadget T_2 or T_3 by Claim 23 we can choose a vertex labeled with b, and we put the TRUE value for literal corresponding to this vertex. By Claim 22 this assignment is legal, hence the value of the formula is TRUE. □

6. Final remarks and open problems

In this paper and in [18] the authors studied the properties of interval incidence chromatic number for many classes of graphs. We observed that the inequality $\chi_{ii} \leq 2\Delta$ holds for all these classes. Therefore we ask if it is true for arbitrary graphs.

In Section 3 we observed that subgraphs may have greater interval incidence coloring number than supergraphs. The smallest counterexample graph has $\Delta = 5$. It would be interesting to know if there is a graph with that property and smaller Δ.

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