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Local properties of the solution set of the operator equation in Banach spaces in a neighbourhood of a bifurcation point

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Abstract: In this work we study the problem of the existence of bifurcation in the solution set of the equation $F(x, \lambda) = 0$, where $F\colon X \times R^k \to Y$ is a C^2-smooth operator, X and Y are Banach spaces such that $X \subset Y$. Moreover, there is given a scalar product $\langle \cdot, \cdot \rangle\colon Y \times Y \to R^1$ that is continuous with respect to the norms in X and Y. We show that under some conditions there is bifurcation at a point $(0, \lambda_0) \in X \times R^k$ and we describe the solution set of the studied equation in a small neighbourhood of this point.

Keywords: bifurcation, finite-dimensional reduction, Fredholm operator, implicit function, variational gradient
MSC (2000): 34K18

1 Introduction

Let X and Y be real Banach spaces and $F\colon X \times R^k \to Y$ be a continuous map. Suppose that the equation

$$F(x, \lambda) = 0, \tag{1}$$

where $x \in X$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in R^k$, possesses the trivial family of solutions

$$\Lambda = \{(0, \lambda) \in X \times R^k \colon \lambda \in R^k\}.$$

A point (x, λ) such that $F(x, \lambda) = 0$ and $x \neq 0$ is called a nontrivial solution of (1). Bifurcation theory is concerned in part with the existence of nontrivial solutions of (1)

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in a small neighbourhood of Λ. A point $(0, \lambda_0) \in \Lambda$ is called a bifurcation point of (1) if every neighbourhood of $(0, \lambda_0)$ contains a nontrivial solution of (1).

Methods of bifurcation theory are often applied in mathematical physics. Let us mention some applications to mechanics of elastic constructions and hydromechanics. In [3] the buckling of a thin elastic plate subject to arbitrary forces and stresses along its boundary is studied by the use of a perturbation theory and a variational method. In [6] to describe a deformation of the minimal interface of two fluids in a vertical tube in a gravitational field one applies a method based on the Crandall-Rabinowitz bifurcation theorem and representation theory. In [9] the buckling of a thin elastic rectangular plate simply supported on sides is studied numerically. In [14] the forms of equilibrium of a thin elastic circular plate lying on an elastic foundation and simply supported along its boundary are investigated via a finite-dimensional reduction and the Krasnosielski bifurcation theorem. Finally, in [16] the buckling of a homogeneous finite beam clamped at the edges to an elastic foundation is studied by a method of a key function due to Sapronov.

Assume that F is C^1-smooth. For every $\lambda \in R^k$, let $F'_x(0, \lambda)\colon X \to Y$ denote the Fréchet derivative of F with respect to x at $(0, \lambda)$. Let $N(\lambda) = \ker F'_x(0, \lambda)$ and $R(\lambda) = \operatorname{im} F'_x(0, \lambda)$. It is easily seen that if $F'_x(0, \lambda_0)\colon X \to Y$ is a Fredholm operator of index zero then a necessary condition for $(0, \lambda_0)$ to be a bifurcation point of (1) is

$$\dim N(\lambda_0) > 0.$$

In this paper we investigate bifurcation at $(0, \lambda_0)$ when X is a linear subspace of Y, there is given a scalar product $\langle \cdot, \cdot \rangle\colon Y \times Y \to R^1$ that is continuous with respect to the norms in X and Y, and F is a C^p-smooth map ($p \geq 2$) that satisfies the following conditions:

(I_1) $F(0, \lambda) = 0$ for every $\lambda \in R^k$,
(I_2) $\dim N(\lambda_0) = 1$,
(I_3) $N(\lambda_0) \perp R(\lambda_0)$,
(I_4) $F'_x(0, \lambda_0)\colon X \to Y$ is a Fredholm operator of index 0.

Our aim is to prove a theorem on bifurcation at $(0, \lambda_0)$ and a local structure of a solution set of equation (1) in a neighbourhood of a bifurcation point. We apply a kind of finite-dimensional reduction of Liapunov-Schmidt type and the implicit function theorem. We are motivated by applications in mathematical physics [6], [14], [16] in which the problems under considerations (see above) are described by (1) with F that satisfies (I_1)–(I_4) and is a variational gradient. The main results of this work are Theorem 3.7 and its variational version: Conclusion 3.10. Theorem 3.7 is an analogue of the Crandall-Rabinowitz bifurcation theorem (see [17], [21]). However, our theorem is formulated in terms of a finite-dimensional reduction and in a variational case it seems to be easier to apply. Conclusion 3.10 is well adapted to a class of nonlinear problems of elasticity described by the von Kármán equations with one or a few parameters (see [4], [15], [16]) in the case when the linearization space is one-dimensional. An example is given in Section 4.

The paper is divided into four sections. In Section 2 we introduce some notions and we briefly sketch a scheme of finite-dimensional reduction. Section 3 is devoted to the study of bifurcation and local properties of the solution set of (1) near a bifurcation point. In Section 4 some applications of our results are indicated.

In practice it suffices to suppose that F is defined in a neighbourhood of $(0, \lambda_0)$ in $X \times R^k$, but we want to omit inessential details.

2 Finite-dimensional reduction

In this section we describe a kind of a finite-dimensional reduction of the Liapunov-Schmidt type. The scheme we present is adapted from [21] (see also [10], [11], [17], [20]).

From now on we assume that $X \subset Y$ are real Banach spaces with a scalar product $\langle \cdot, \cdot \rangle \colon Y \times Y \to R^1$ that is continuous with respect to the norms in X and Y. The norms in X and Y can be defined independently of the scalar product $\langle \cdot, \cdot \rangle$, and the norm in X does not have to be induced by the norm in Y. In particular, X and Y with $\langle \cdot, \cdot \rangle$ may be Hilbert spaces. Let $F \colon X \times R^k \to Y$ be a C^p-smooth map, where $p \geq 1$, satisfying conditions: (I_1), (I_3), (I_4) and
(I_2') $\dim N(\lambda_0) = n \neq 0$.

The aim is to show that under the above assumptions the problem of bifurcation for equation (1) at the point $(0, \lambda_0) \in X \times R^k$ is reducible to the problem of bifurcation for the equation $\varphi(\xi, \lambda) = 0$ with a certain map $\varphi \colon S \subset R^n \times R^k \to R^n$ at the point $(0, \lambda_0) \in R^n \times R^k$. The reader may find the proofs of the propositions given below in [13] and [15].

Proposition 2.1. For every $\lambda \in R^k$ the following equality holds:

$$Y = R(\lambda) \oplus N(\lambda). \tag{2}$$

Let $G \colon X \times R^n \times R^k \to Y$ be a map defined by

$$G(x, \xi, \lambda) = F(x, \lambda) + \sum_{i=1}^{n} (\xi_i - \langle x, e_i \rangle) e_i, \tag{3}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $\{e_1, e_2, \ldots, e_n\}$ is a fixed orthonormal base of $N(\lambda_0)$.

Proposition 2.2. The operator $G_x'(0, 0, \lambda_0) \colon X \to Y$ is an isomorphism.

It is easily seen that G is C^p-smooth. From the implicit function theorem it follows that there exist two open sets $U \subset X$ and $S \subset R^n \times R^k$ such that $0 \in U$, $(0, \lambda_0) \in S$ and the solution set of the equation

$$G(x, \xi, \lambda) = 0 \tag{4}$$

in $U \times S$ is a graph of a certain C^p-smooth function $x \colon S \to U$ such that $x(0, \lambda_0) = 0$. Moreover, it is obvious that $x(0, \lambda) = 0$ for all $(0, \lambda) \in S$, because $G(0, 0, \lambda) = 0$. Let

$\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)\colon S \to R^n$ be defined by coordinates as follows:

$$\varphi_i(\xi, \lambda) = \xi_i - \langle x(\xi, \lambda), e_i \rangle, \; i = 1, \ldots, n. \tag{5}$$

Proposition 2.3. $(0, \lambda_0) \in \Lambda$ is a bifurcation point of equation (1) if and only if $(0, \lambda_0) \in S$ is a bifurcation point of equation

$$\varphi(\xi, \lambda) = 0. \tag{6}$$

3 Theorem on bifurcation

In this section our main results are stated and proved.

Let $F\colon X \times R^k \to Y$ be a C^p-smooth map, $p \geq 2$, satisfying conditions (I_1)-(I_4) (see p. 562). Fix $e \in N(\lambda_0)$ such that $\langle e, e \rangle = 1$ and denote $\lambda_0 = (\lambda_{01}, \lambda_{02}, \ldots, \lambda_{0k})$. We will describe the solution set of (1) in terms of the finite-dimensional reduction. Notice that now in the formulas of maps G and φ there are $n = 1$ and $e_1 = e$. Differentiating the equality $G(x(\xi, \lambda), \xi, \lambda) = 0$ with respect to ξ at $(0, \lambda_0)$ we obtain

$$F'_x(0, \lambda_0) x'_\xi(0, \lambda_0) + (1 - \langle x'_\xi(0, \lambda_0), e \rangle) e = 0.$$

From this and (I_3) it follows that $x'_\xi(0, \lambda_0) = e$.

Theorem 3.1. There exist open sets $V_0 \subset X$ and $V \subset R^k$ such that $(0, \lambda_0) \in V_0 \times V$ and for every $(x, \lambda) \in V_0 \times V$ we have $F(x, \lambda) = 0$ if and only if $(\langle x, e \rangle, \lambda) \in S$ and $x = x(\langle x, e \rangle, \lambda)$.

Proof 3.2. Suppose contrary to our claim, that there are no open sets $V_0 \subset X$ and $V \subset R^k$ with the above properties. Then for every $n \in N$ there exists $(x_n, \lambda_n) \in X \times R^k$ such that $||x_n||_X \leq \frac{1}{n}$, $|\lambda_n - \lambda_0| \leq \frac{1}{n}$ and one of the following conditions is satisfied:
1. $F(x_n, \lambda_n) = 0$ and $(\langle x_n, e \rangle, \lambda_n) \notin S$,
2. $F(x_n, \lambda_n) = 0$, $(\langle x_n, e \rangle, \lambda_n) \in S$ and $x_n \neq x(\langle x_n, e \rangle, \lambda_n)$,
3. $F(x_n, \lambda_n) \neq 0$, $(\langle x_n, e \rangle, \lambda_n) \in S$ and $x_n = x(\langle x_n, e \rangle, \lambda_n)$.

If $(\langle x_n, e \rangle, \lambda_n) \in S$ and $x_n = x(\langle x_n, e \rangle, \lambda_n)$ then $F(x_n, \lambda_n) = F(x(\langle x_n, e \rangle, \lambda_n), \lambda_n) + (\langle x_n, e \rangle - \langle x(\langle x_n, e \rangle, \lambda_n), e \rangle) e = G(x(\langle x_n, e \rangle, \lambda_n), \langle x_n, e \rangle, \lambda_n) = 0$.
Since $x_n \to 0$ in X, there exists $n_0 \in N$ such that $x_n \in U$ for every $n \geq n_0$. If for some $n \geq n_0$ we have $F(x_n, \lambda_n) = 0$ and $(\langle x_n, e \rangle, \lambda_n) \in S$ then $0 = F(x_n, \lambda_n) + (\langle x_n, e \rangle - \langle x_n, e \rangle) e = G(x_n, \langle x_n, e \rangle, \lambda_n)$, and so $x_n = x(\langle x_n, e \rangle, \lambda_n)$.
Since $(\langle x_n, e \rangle, \lambda_n) \to (0, \lambda_0) \in S$ there exists $n_1 \in N$ such that $(\langle x_n, e \rangle, \lambda_n) \in S$ for every $n \geq n_1$ — a contradiction.

The equality $\langle G(x(\xi, \lambda), \xi, \lambda), e \rangle = 0$ implies

$$\varphi(\xi, \lambda) = -\langle F(x(\xi, \lambda), \lambda), e \rangle. \tag{7}$$

From (7) we obtain

$$\varphi'_\xi(\xi, \lambda) = -\langle F'_x(x(\xi, \lambda), \lambda) x'_\xi(\xi, \lambda), e \rangle,$$

and hence $\varphi'_\xi(0, \lambda_0) = 0$. Moreover, since $\varphi(0, \lambda) = 0$ for every $(0, \lambda) \in S$ we have $\varphi^{(m)}_{\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_m}}(0, \lambda_0) = 0$ for all $i_1, i_2, \ldots, i_m \in \{1, 2, \ldots, k\}$ and $m \in N$. In order to get our main result we have to assume that there is $i \in \{1, 2, \ldots, k\}$ such that $\varphi''_{\xi\lambda_i}(0, \lambda_0) \neq 0$. There is no loss of generality if we assume

(I_5) $\varphi''_{\xi\lambda_k}(0, \lambda_0) \neq 0$.

From now on, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k) \in R^k$, $\lambda' = (\lambda_1, \lambda_2, \ldots, \lambda_{k-1}) \in R^{k-1}$ we will write $\lambda = (\lambda', \lambda_k)$.

Proposition 3.3. There exist open sets $\Omega_0 \subset R^1 \times R^{k-1}$ and $\Omega \subset R^1$ such that $(0, \lambda'_0) \in \Omega_0$, $\lambda_{0k} \in \Omega$ and there exists a C^p-smooth map $f\colon \Omega_0 \to \Omega$ that satisfies the following conditions:

(1) $f(0, \lambda'_0) = \lambda_{0k}$,

(2) for every $(\xi, \lambda') \in \Omega_0$ and $\lambda_k \in \Omega$ we have $\varphi(\xi, \lambda', \lambda_k) = 0$ if and only if $\xi = 0$ or $\lambda_k = f(\xi, \lambda')$.

Proof 3.4. Let $\psi\colon S \to R^1$ be a function defined by

$$\psi(\xi, \lambda) = \int_0^1 \varphi'_\xi(t\xi, \lambda) dt. \tag{8}$$

Observe that we have

$$\varphi(\xi, \lambda) = \xi\psi(\xi, \lambda). \tag{9}$$

Hence $\varphi(\xi, \lambda) = 0$ only if $\xi = 0$ or $\psi(\xi, \lambda) = 0$. From (8) and (I_5) it follows that $\psi(0, \lambda_0) = \varphi'_\xi(0, \lambda_0) = 0$ and $\psi'_{\lambda_k}(0, \lambda_0) = \varphi''_{\xi\lambda_k}(0, \lambda_0) \neq 0$. Applying the implicit function theorem we get the desired claim.

Let $B_r(\lambda'_0)$ denote a ball in R^{k-1} of radius r centered at λ'_0, and $B_\delta(0)$ a ball in X of radius δ centered at 0.

Theorem 3.5. Let $f\colon \Omega_0 \to \Omega$ be a function of Proposition 3.3 and $r > 0$ be a number such that $(-r, r) \times B_r(\lambda'_0) \subset \Omega_0$. There exist open sets $\tilde{V}_0 \subset X$ and $\tilde{V} \subset B_r(\lambda'_0) \times \Omega$ such that $(0, \lambda_0) \in \tilde{V}_0 \times \tilde{V}$ and for every $(x, \lambda) \in \tilde{V}_0 \times \tilde{V}$ we have $F(x, \lambda) = 0$ if and only if $x = 0$ or there exists $\xi \in (-r, r)$ such that $\lambda_k = f(\xi, \lambda')$ and $x = x(\xi, \lambda', f(\xi, \lambda'))$.

Proof 3.6. There exists $\delta \in (0, r)$ such that for every $x \in X$ if $||x||_X < \delta$ then $|\langle x, e\rangle| < r$. Let $\tilde{V}_0 = V_0 \cap B_\delta(0)$ and $\tilde{V} = V \cap (B_r(\lambda'_0) \times \Omega)$, where $V_0 \subset X$ and $V \subset R^k$ are open sets of Theorem 3.1. Take $(x, \lambda) \in \tilde{V}_0 \times \tilde{V}$.

($\Rightarrow$) By Theorem 3.1, if $F(x, \lambda) = 0$ then $(\langle x, e\rangle, \lambda) \in S$ and $x = x(\langle x, e\rangle, \lambda)$, which gives $\varphi(\langle x, e\rangle, \lambda) = 0$. From Proposition 3.3 it follows that $\langle x, e\rangle = 0$ or $\lambda_k = f(\langle x, e\rangle, \lambda')$. If $\langle x, e\rangle = 0$ then $x = x(0, \lambda) = 0$. If $\lambda_k = f(\langle x, e\rangle, \lambda')$ then $x = x(\langle x, e\rangle, \lambda', f(\langle x, e\rangle, \lambda'))$.

($\Leftarrow$) Assume now that $x = 0$ or there exists $\xi \in (-r, r)$ such that $\lambda_k = f(\xi, \lambda')$ and $x = x(\xi, \lambda', f(\xi, \lambda'))$. In the first case, $F(x, \lambda) = F(0, \lambda) = 0$. In the second case, by Proposition 3.3, we have $\varphi(\xi, \lambda) = 0$, and hence $F(x, \lambda) = F(x, \lambda) + \varphi(\xi, \lambda)e = F(x, \lambda) + (\xi - \langle x(\xi, \lambda), e\rangle)e = F(x, \lambda) + (\xi - \langle x, e\rangle)e = G(x, \xi, \lambda) = 0$.

We are now in a position to prove our main result.

Theorem 3.7. Under assumptions (I_1)-(I_5), the solution set of equation (1) in a certain neighbourhood of $(0, \lambda_0) \in \Lambda$ is the union of two sets: Λ and Ξ. The set Ξ is given by

$$\Xi = \{(\hat{x}(\xi, \lambda'), \lambda', f(\xi, \lambda')) \colon |\xi| < r, |\lambda' - \lambda_0'| < r\},$$

where $\hat{x}$ and f are C^p-smooth functions such that $\hat{x}(0, \lambda_0') = 0$, $f(0, \lambda_0') = \lambda_{0k}$, $\hat{x}'_\xi(0, \lambda_0') = e$, $f'_\xi(0, \lambda_0') = -\frac{1}{2}\frac{\varphi''_{\xi\xi}(0,\lambda_0)}{\varphi''_{\xi\lambda_k}(0,\lambda_0)}$, $\hat{x}'_{\lambda_s}(0, \lambda_0') = 0$ and $f'_{\lambda_s}(0, \lambda_0') = -\frac{\varphi''_{\xi\lambda_s}(0,\lambda_0)}{\varphi''_{\xi\lambda_k}(0,\lambda_0)}$ for every $s \in \{1, 2, \ldots, k-1\}$.
Moreover, the intersection of Λ and Ξ in a sufficiently small neighbourhood of $(0, \lambda_0)$ can be parametrized as follows

$$I_{\Lambda,\Xi} = \{(0, \lambda', f(\hat{\xi}(\lambda'), \lambda')) \colon |\lambda' - \lambda_0'| < \varrho\}$$

where $0 < \varrho \leq r$ and $\hat{\xi}$ is a C^p-smooth function such that $\hat{\xi}(\lambda_0') = 0$ and $\hat{\xi}'_{\lambda_s}(\lambda_0') = 0$ for every $s \in \{1, 2, \ldots, k-1\}$, which gives that $(0, \lambda_0)$ is a bifurcation point of (1).

Proof 3.8. Let $f \colon \Omega_0 \to \Omega$ be a function of Proposition 3.3. Fix $r > 0$ such that $(-r, r) \times B_r(\lambda_0') \subset \Omega_0$. Let $\hat{x} \colon (-r, r) \times B_r(\lambda_0') \to X$ be given by $\hat{x}(\xi, \lambda') = x(\xi, \lambda', f(\xi, \lambda'))$. Then $f(0, \lambda_0') = \lambda_{0k}$ and $\hat{x}(0, \lambda_0') = x(0, \lambda_0) = 0$. Differentiating $\hat{x}$ we get $\hat{x}'_\xi(0, \lambda_0') = e$ and $\hat{x}'_{\lambda_s}(0, \lambda_0') = 0$ for every $s \in \{1, 2, \ldots, k-1\}$. Moreover, differentiating the equality $\psi(\xi, \lambda', f(\xi, \lambda')) = 0$ we obtain $f'_\xi(0, \lambda_0') = -\frac{\psi'_\xi(0,\lambda_0)}{\psi'_{\lambda_k}(0,\lambda_0)} = -\frac{1}{2}\frac{\varphi''_{\xi\xi}(0,\lambda_0)}{\varphi''_{\xi\lambda_k}(0,\lambda_0)}$ and $f'_{\lambda_s}(0, \lambda_0') = -\frac{\psi'_{\lambda_s}(0,\lambda_0)}{\psi'_{\lambda_k}(0,\lambda_0)} = -\frac{\varphi''_{\xi\lambda_s}(0,\lambda_0)}{\varphi''_{\xi\lambda_k}(0,\lambda_0)}$ for every $s \in \{1, 2, \ldots, k-1\}$. From Theorem 3.5 it follows that there exist open sets $\tilde{V}_0 \subset X$ and $\tilde{V} \subset B_r(\lambda_0') \times \Omega$ such that $(0, \lambda_0) \in \tilde{V}_0 \times \tilde{V}$ and $\{(x, \lambda) \in \tilde{V}_0 \times \tilde{V} \colon F(x, \lambda) = 0\} = \{(x, \lambda) \in \tilde{V}_0 \times \tilde{V} : x = 0\} \cup \{(x, \lambda) \in \tilde{V}_0 \times \tilde{V} : \exists_{\xi \in (-r,r)}\, x = x(\xi, \lambda', f(\xi, \lambda')) \wedge \lambda_k = f(\xi, \lambda')\} = (\Lambda \cup \Xi) \cap \tilde{V}_0 \times \tilde{V}$. A point $(x, \lambda) \in \Lambda \cap \Xi$ only if it satisfies the following system

$$\begin{cases} x = \hat{x}(\xi, \lambda'), \\ \lambda_k = f(\xi, \lambda'),\ \xi \in (-r, r),\ \lambda' \in B_r(\lambda_0'), \\ x = 0. \end{cases}$$

Since $\hat{x}(0, \lambda_0') = 0$ and $\hat{x}'_\xi(0, \lambda_0') = e \neq 0$, there exist: $0 < \varrho \leq r$, an open set $B \subset (-r, r)$ such that $0 \in B$ and a C^p-smooth function $\hat{\xi} \colon B_\varrho(\lambda_0') \to B$ such that $\hat{\xi}(\lambda_0') = 0$ and for all $(\xi, \lambda') \in B \times B_\varrho(\lambda_0')$ we have $\hat{x}(\xi, \lambda') = 0$ only if $\xi = \hat{\xi}(\lambda')$. Differentiating the equality $\hat{x}(\hat{\xi}(\lambda'), \lambda') = 0$ we receive $\hat{x}'_\xi(\hat{\xi}(\lambda'), \lambda')\hat{\xi}'_{\lambda_s}(\lambda') + \hat{x}'_{\lambda_s}(\hat{\xi}(\lambda'), \lambda') = 0$ for every $s \in \{1, 2, \ldots, k-1\}$, and hence $\hat{\xi}'_{\lambda_s}(\lambda_0') = 0$. Summarizing $I_{\Lambda,\Xi} \subset \Lambda \cap \Xi$ and in a sufficiently small neighbourhood of $(0, \lambda_0)$ the intersection $\Lambda \cap \Xi$ is equal to $I_{\Lambda,\Xi}$.

Conclusion 3.9. Assume that (I_1)-(I_5) hold and $k = 2$. Then the solution set of (1) in a small neighbourhood of $(0, \lambda_0) \in \Lambda$ is the union of two surfaces: Λ and Ξ. The surface Ξ can be parametrized as follows

$$\Xi = \{(\hat{x}(\xi, \lambda_1), \lambda_1, f(\xi, \lambda_1)) \colon (\xi, \lambda_1) \in (-r, r) \times (\lambda_{01} - r, \lambda_{01} + r)\},$$

where $\hat{x}\colon (-r,r)\times(\lambda_{01}-r,\lambda_{01}+r)\to X$ and $f\colon (-r,r)\times(\lambda_{01}-r,\lambda_{01}+r)\to R^1$ are C^p-smooth functions such that $\hat{x}(0,\lambda_{01})=0$, $f(0,\lambda_{01})=\lambda_{02}$, $\hat{x}'_\xi(0,\lambda_{01})=e$, $\hat{x}'_{\lambda_1}(0,\lambda_{01})=0$, $f'_\xi(0,\lambda_{01})=-\frac{1}{2}\frac{\varphi''_{\xi\xi}(0,\lambda_{01})}{\varphi''_{\xi\lambda_2}(0,\lambda_{01})}$ and $f'_{\lambda_1}(0,\lambda_{01})=-\frac{\varphi''_{\xi\lambda_1}(0,\lambda_{01})}{\varphi''_{\xi\lambda_2}(0,\lambda_{01})}$. In a sufficiently small neighbourhood of $(0,\lambda_0)$ the surfaces Λ and Ξ intersect only along the curve

$$I_{\Lambda,\Xi}=\{(0,\lambda_1,f(\hat{\xi}(\lambda_1),\lambda_1))\colon \lambda_1\in(\lambda_{01}-\varrho,\lambda_{01}+\varrho)\},$$

where $0<\varrho\le r$ and $\hat{\xi}\colon(\lambda_{01}-\varrho,\lambda_{01}+\varrho)\to(-r,r)$ is a C^p-smooth function such that $\hat{\xi}(\lambda_{01})=\hat{\xi}'(\lambda_{01})=0$, and hence $(0,\lambda_0)$ is a bifurcation point of (1).

Let us consider the following condition:
(I'_3) $F\colon X\times R^k\to Y$ is a variational gradient of a certain functional $E\colon X\times R^k\to R^1$ with respect to the scalar product $\langle\cdot,\cdot\rangle$, i.e. for all $x,y\in X$ and $\lambda\in R^k$

$$E'_x(x,\lambda)y=\langle F(x,\lambda),y\rangle.$$

It is evident that (I'_3) implies (I_3). Furthermore, by formula (7) we obtain

$$\varphi''_{\xi\lambda_s}(0,\lambda_0)=-E^{(3)}_{xx\lambda_s}(0,\lambda_0)(e,e,1) \tag{10}$$

for $s\in\{1,2,\dots,k\}$. From this it follows that if F satisfies (I'_3) then (I_5) can be replaced by the equivalent condition:
(I'_5) $E^{(3)}_{xx\lambda_k}(0,\lambda_0)(e,e,1)\neq 0$.
By (7) we also obtain

$$\varphi''_{\xi\xi}(0,\lambda_0)=-E^{(3)}_{xxx}(0,\lambda_0)(e,e,e). \tag{11}$$

Summarizing, in a variational case we have the following result.

Conclusion 3.10. Under assumptions: (I_1), (I_2), (I'_3), (I_4) and (I'_5), the solution set of equation (1) in a certain neighbourhood of $(0,\lambda_0)\in\Lambda$ is the union of two sets: Λ and Ξ. The set Ξ is given by

$$\Xi=\{(\hat{x}(\xi,\lambda'),\lambda',f(\xi,\lambda'))\colon |\xi|<r,|\lambda'-\lambda'_0|<r\},$$

where $\hat{x}$ and f are C^p-smooth functions such that $\hat{x}(0,\lambda'_0)=0$, $f(0,\lambda'_0)=\lambda_{0k}$, $\hat{x}'_\xi(0,\lambda'_0)=e$, $f'_\xi(0,\lambda'_0)=-\frac{1}{2}\frac{E^{(3)}_{xxx}(0,\lambda_0)(e,e,e)}{E^{(3)}_{xx\lambda_k}(0,\lambda_0)(e,e,1)}$, $\hat{x}'_{\lambda_s}(0,\lambda'_0)=0$ and $f'_{\lambda_s}(0,\lambda'_0)=-\frac{E^{(3)}_{xx\lambda_s}(0,\lambda_0)(e,e,1)}{E^{(3)}_{xx\lambda_k}(0,\lambda_0)(e,e,1)}$ for every $s\in\{1,2,\dots,k-1\}$.
Moreover, the intersection of Λ and Ξ in a sufficiently small neighbourhood of $(0,\lambda_0)$ can be parametrized as follows

$$I_{\Lambda,\Xi}=\{(0,\lambda',f(\hat{\xi}(\lambda'),\lambda'))\colon |\lambda'-\lambda'_0|<\varrho\}$$

where $0<\varrho\le r$ and $\hat{\xi}$ is a C^p-smooth function such that $\hat{\xi}(\lambda'_0)=0$ and $\hat{\xi}'_{\lambda_s}(\lambda'_0)=0$ for every $s\in\{1,2,\dots,k-1\}$, which gives that $(0,\lambda_0)$ is a bifurcation point of (1).

4 Applications

It is obvious that if we assume that F is a map from a small neighbourhood of the point $(0,\lambda_0)$ in $X\times R^k$ to Y, our results remain true. After this remark we are ready to give an example of application of Conclusion 3.10 to mathematical physics. All the results of Section 4 were proved either in [12] or [15]. However, to make this exposition self-sufficient we give the main ideas of the proofs.

For every $m\in N$ and $\mu\in(0,1)$, let $C^{m,\mu}(\bar D)$ denote the real Hölder space of functions defined on $D=\{(u,v)\in R^2\colon u^2+v^2<1\}$ with the standard norm

$$||x;C^{m,\mu}(\bar D)|| = \max_{|\alpha|\le m}\sup\{|D^\alpha x(u,v)|\colon (u,v)\in D\} + \max_{|\alpha|\le m}\sup\left\{\frac{|D^\alpha x(u,v)-D^\alpha x(\bar u,\bar v)|}{|(u-\bar u, v-\bar v)|^\mu}\colon (u,v),(\bar u,\bar v)\in D, (u,v)\neq(\bar u,\bar v)\right\},$$

where $D^\alpha x=\frac{\partial^{|\alpha|}x}{\partial^{\alpha_1}u\partial^{\alpha_2}v}$, $\alpha=(\alpha_1,\alpha_2)\in N_0\times N_0$, $N_0=N\cup\{0\}$ and $|\alpha|=\alpha_1+\alpha_2$. It is well-known that $C^{m,\mu}(\bar D)$ is a Banach space (see [1]). Let

- $C^{4,\mu}_{0,0}(\bar D)=\{f\in C^{4,\mu}(\bar D)\colon \Delta f|_{\partial D}=f|_{\partial D}=0\}$,
- $C^{2,\mu}_{0}(\bar D)=\{f\in C^{2,\mu}(\bar D)\colon f|_{\partial D}=0\}$,
- $X=C^{4,\mu}_{0,0}(\bar D)\times C^{4,\mu}_{0,0}(\bar D)$,
- $Y=C^{0,\mu}(\bar D)\times C^{0,\mu}(\bar D)$.

The norms in X and Y are defined by coordinates. That is as the maximum (or the sum) of norms of both coordinates of a given element. The function given by

$$\langle(x_1,x_2),(y_1,y_2)\rangle=\frac{1}{\pi}\iint_D(x_1y_1+x_2y_2)dudv$$

is a scalar product in Y, which is continuous with respect to the norms in X and Y. We define $F\colon X\times R^2_+\to Y$ as follows

$$F(x,\lambda)=(\Delta^2x_1-[x_1,x_2]+2\lambda_1\Delta x_1+\lambda_2x_1-\gamma x_1^3,-\Delta^2x_2-\frac{1}{2}[x_1,x_1]), \tag{12}$$

where $R_+=(0,+\infty)$, $x=(x_1,x_2)$, $\lambda=(\lambda_1,\lambda_2)$, γ is a positive constant and $[\cdot,\cdot]\colon X\to Y$ is given by

$$[x_1,x_2]=\frac{\partial^2x_1}{\partial u^2}\frac{\partial^2x_2}{\partial v^2}-2\frac{\partial^2x_1}{\partial u\partial v}\frac{\partial^2x_2}{\partial u\partial v}+\frac{\partial^2x_1}{\partial v^2}\frac{\partial^2x_2}{\partial u^2}.$$

The equation

$$F(x,\lambda)=0 \tag{13}$$

with F given by (12) is called the von Kárman equation for a thin circular elastic plate which lies on an elastic base and is uniformly radially compressed along its boundary. In mechanics x_1 is a deflection function, x_2 is a stress function, λ_1 is a value of a compressing force, λ_2 and γ are parameters of an elastic foundation. The solutions of (13) lying in a sufficiently small neighbourhood of the set of trivial solutions of (13) are called the forms of equilibrium of a plate. The map F is C^∞-smooth and an easy computation shows that for all $y = (y_1, y_2) \in X$

$$F'_x(x,\lambda)y = (\Delta^2 y_1 - [y_1, x_2] - [x_1, y_2] + 2\lambda_1 \Delta y_1 + \lambda_2 y_1 - 3\gamma x_1^2 y_1, -\Delta^2 y_2 - [x_1, y_1]). \quad (14)$$

Let $E\colon X \times R^2_+ \to R^1$ be given by

$$E(x,\lambda) = \frac{1}{2\pi}\iint_D \big((\Delta x_1)^2 - (\Delta x_2)^2 - [x_1, x_1]x_2\big)\, dudv + \frac{1}{2\pi}\iint_D \left(-2\lambda_1\left(\left(\frac{\partial x_1}{\partial u}\right)^2 + \left(\frac{\partial x_1}{\partial v}\right)^2\right) + \lambda_2 x_1^2 - \frac{1}{2}\gamma x_1^4\right) dudv. \quad (15)$$

E is easily seen to be C^∞-smooth.

Theorem 4.1 (see Th. 2.4 of [12]). The map F is a variational gradient of the functional E with respect to the scalar product $\langle \cdot, \cdot \rangle$.

Sketch of the proof 4.2. For all $x, y \in X$ and $\lambda \in R^2_+$, we have

$$\begin{aligned} E'_x(x,\lambda)y = \frac{d}{dt}E(x+ty,\lambda)|_{t=0} = &\frac{1}{\pi}\iint_D \Delta x_1 \Delta y_1 dudv - \frac{1}{\pi}\iint_D \Delta x_2 \Delta y_2 dudv \\ &-\frac{1}{\pi}\iint_D [x_1, y_1]x_2 dudv - \frac{1}{2\pi}\iint_D [x_1, x_1]y_2 dudv \\ &-\frac{1}{\pi}\iint_D 2\lambda_1\left(\frac{\partial x_1}{\partial u}\frac{\partial y_1}{\partial u} + \frac{\partial x_1}{\partial v}\frac{\partial y_1}{\partial v}\right) dudv \\ &+\frac{1}{\pi}\iint_D (\lambda_2 x_1 y_1 - \gamma x_1^3 y_1) dudv. \end{aligned}$$

Integrating by part we receive

$$\begin{aligned} \iint_D \Delta x_1 \Delta y_1 dudv &= \iint_D (\Delta^2 x_1) y_1 dudv, \\ \iint_D \Delta x_2 \Delta y_2 dudv &= \iint_D (\Delta^2 x_2) y_2 dudv, \\ \iint_D [x_1, y_1]x_2 dudv &= \iint_D [x_1, x_2]y_1 dudv \end{aligned}$$

and

$$\iint_D \left(\frac{\partial x_1}{\partial u}\frac{\partial y_1}{\partial u} + \frac{\partial x_1}{\partial v}\frac{\partial y_1}{\partial v}\right) dudv = -\iint_D (\Delta x_1) y_1 dudv.$$

Hence $E'_x(x,\lambda)y = \langle F(x,\lambda), y\rangle$, which completes the proof.

Theorem 4.3 (see Th. 2.2 of [12]). For every $\lambda \in R^2_+$, $F'_x(0,\lambda)\colon X \to Y$ is a Fredholm map of index 0.

Sketch of the proof 4.4. Fix $\lambda \in R^2_+$. By (14) we get

$$F'_x(0,\lambda)y = (\Delta^2 y_1 + 2\lambda_1 \Delta y_1 + \lambda_2 y_1, -\Delta^2 y_2). \quad (16)$$

We can write (16) as

$$F'_x(0,\lambda)y = A(y) + B(y),$$

where $A, B\colon X \to Y$ are given as follows:

$$A(y) = (\Delta^2 y_1, -\Delta^2 y_2), \quad B(y) = (2\lambda_1 \Delta y_1 + \lambda_2 y_1, 0).$$

It is known that $\Delta\colon C_0^{2,\mu}(\overline{D}) \to C^{0,\mu}(\overline{D})$ is an isomorphism. Moreover, it is a simple matter to check that B is compact, which finishes the proof.

Let $J_k\colon R \to R$, $k \in N_0$, denote the k-th Bessel function. It is well-known (see [8], [18]) that $\alpha \in R$ is an eigenvalue of $\Delta\colon C_0^{2,\mu}(\overline{D}) \to C^{0,\mu}(\overline{D})$ if and only if $\alpha < 0$ and there is $k \in N_0$ such that $J_k(\sqrt{-\alpha}) = 0$. Furthermore, if $J_0(\sqrt{-\alpha}) = 0$ then the eigenspace corresponding to α is one-dimensional. If $J_k(\sqrt{-\alpha}) = 0$ for a certain $k \in N$ then the corresponding eigenspace is two-dimensional.

For $\lambda = (\lambda_1, \lambda_2) \in R^2_+$, let $\delta = (\lambda_1)^2 - \lambda_2$, $a = -\lambda_1 - \sqrt{\delta}$ and $b = -\lambda_1 + \sqrt{\delta}$. Of course, a and b are determined on condition $\delta \geq 0$. Let $\Delta^2 + 2\lambda_1\Delta + \lambda_2 \mathrm{I}\colon C_{0,0}^{4,\mu}(\overline{D}) \to C^{0,\mu}(\overline{D})$ and $\Delta - a\mathrm{I}, \Delta - b\mathrm{I}\colon C_0^{2,\mu}(\overline{D}) \to C^{0,\mu}(\overline{D})$, where $\mathrm{I}(h) = h$ are natural embeddings of the appropriate Hölder spaces.

Lemma 4.5 (see Lemmas 4.1-4.3 of [12]). Under the above assumptions:
(i) If $\delta < 0$ then $\ker(\Delta^2 + 2\lambda_1\Delta + \lambda_2\mathrm{I}) = \{0\}$.
(ii) If $\delta = 0$ then $\ker(\Delta^2 + 2\lambda_1\Delta + \lambda_2\mathrm{I}) = \ker(\Delta + \lambda_1\mathrm{I})$.
(iii) If $\delta > 0$ then $\ker(\Delta^2 + 2\lambda_1\Delta + \lambda_2\mathrm{I}) = \ker(\Delta - a\mathrm{I}) \oplus \ker(\Delta - b\mathrm{I})$.

By (16), $N(\lambda) = \ker(\Delta^2 + 2\lambda_1\Delta + \lambda_2\mathrm{I}) \times \{0\}$. From this and Lemma 4.5 we obtain what follows.

Theorem 4.6. $\dim N(\lambda) = 1$ if and only if one of the below conditions is satisfied:
(I) $\delta = 0$ and $J_0(\sqrt{\lambda_1}) = 0$,
(II) $\delta > 0$, $J_0(\sqrt{-a}) = 0$ and $J_k(\sqrt{-b}) \neq 0$ for every $k \in N_0$,
(III) $\delta > 0$, $J_0(\sqrt{-b}) = 0$ and $J_k(\sqrt{-a}) \neq 0$ for every $k \in N_0$.

Suppose that $\lambda_0 = (\lambda_{01}, \lambda_{02})$ and $\dim N(\lambda_0) = 1$. Fix $e = (e_1, 0) \in N(\lambda_0)$ such that $\langle e, e\rangle = 1$. Set

$$c_0 = \begin{cases} a_0 & \text{if (I) or (II)}, \\ b_0 & \text{if (III)}, \end{cases}$$

where $a_0 = -\lambda_{01} - \sqrt{\delta_0}$, $b_0 = -\lambda_{01} + \sqrt{\delta_0}$ and $\delta_0 = (\lambda_{01})^2 - \lambda_{02}$. A trivial verification combining Theorem 4.1 with (14) shows that

$$E'''_{xx\lambda_1}(x,\lambda)(y,z,1) = \frac{2}{\pi}\iint_D (\Delta y_1) z_1 \, du dv,$$
$$E'''_{xx\lambda_2}(x,\lambda)(y,z,1) = \frac{1}{\pi}\iint_D y_1 z_1 \, du dv,$$

and

$$E'''_{xxx}(x,\lambda)(y,z,w) = -\frac{1}{\pi}\iint_D ([y_1,z_2]+[y_2,z_1]+6\gamma x_1 y_1 z_1) w_1\,dudv$$
$$-\frac{1}{\pi}\iint_D [y_1,z_1] w_2\,dudv,$$

where $x=(x_1,x_2)$, $y=(y_1,y_2)$, $z=(z_1,z_2)$, $w=(w_1,w_2)$. From this and Lemma 4.5 we receive

$$E'''_{xx\lambda_1}(0,\lambda_0)(e,e,1) = \frac{2}{\pi}\iint_D (\Delta e_1)e_1\,dudv = 2c_0\langle e,e\rangle = 2c_0,$$
$$E'''_{xx\lambda_2}(0,\lambda_0)(e,e,1) = \frac{1}{\pi}\iint_D e_1^2\,dudv = \langle e,e\rangle = 1,$$
$$E'''_{xxx}(0,\lambda_0)(e,e,e) = 0.$$

Applying Conclusion 3.10 we get the following theorem.

Theorem 4.7. Let $\lambda_0 \in R^2_+$ satisfy the above assumptions. Then the solution set of equation (13) in a certain neighbourhood of $(0,\lambda_0)\in X\times R^2_+$ is the union of two sets: Λ and Ξ. The set Ξ is given by

$$\Xi = \{(\hat{x}(\xi,\lambda_1),\lambda_1,f(\xi,\lambda_1))\colon |\xi|<r, |\lambda_1-\lambda_{01}|<r\},$$

where $\hat{x}$ and f are C^∞-smooth functions such that $\hat{x}(0,\lambda_{01}) = 0$, $f(0,\lambda_{01}) = \lambda_{02}$, $\hat{x}'_\xi(0,\lambda_{01}) = e$, $f'_\xi(0,\lambda_{01}) = 0$, $\hat{x}'_{\lambda_1}(0,\lambda_{01}) = 0$ and $f'_{\lambda_1}(0,\lambda_{01}) = -2c_0$.
Moreover, the intersection of Λ and Ξ in a sufficiently small neighbourhood of $(0,\lambda_0)$ can be parametrized as follows

$$I_{\Lambda,\Xi} = \{(0,\lambda_1,f(\hat{\xi}(\lambda_1),\lambda_1))\colon |\lambda_1-\lambda_{01}|<\varrho\},$$

where $0<\varrho\le r$ and $\hat{\xi}$ is a C^∞-smooth function such that $\hat{\xi}(\lambda_{01})=0$ and $\hat{\xi}'(\lambda_{01})=0$, which gives that $(0,\lambda_0)$ is a bifurcation point of (13).

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