

Minimal parameter implicit solvent model for *ab initio* electronic structure calculations

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We present an implicit solvent model for *ab initio* electronic structure calculations which is fully self-consistent and is based on direct solution of the nonhomogeneous Poisson equation. The solute cavity is naturally defined in terms of an isosurface of the electronic density according to the formula of Fattebert and Gygi (J. Comp. Chem. **23**, 6 (2002)). While this model depends on only two parameters, we demonstrate that by using appropriate boundary conditions and dispersion-repulsion contributions, solvation energies obtained for an extensive test set including neutral and charged molecules show dramatic improvement compared to existing models. Our approach is implemented in, but not restricted to, a linear-scaling density functional theory (DFT) framework, opening the path for self-consistent implicit solvent DFT calculations on systems of unprecedented size, which we demonstrate with calculations on a 2615-atom protein-ligand complex.

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The role of solvent is critical to a multitude of chemical, biological and physical processes. The accurate simulation of such processes, therefore, requires careful treatment of solvation effects. However, explicit inclusion of the solvent with full atomic detail is very costly due to the significant increase in the number of simulated atoms and the need for extensive averaging over the solvent degrees of freedom¹. Moreover, such explicit treatment may also be unnecessary, as it is often the long-range electrostatic effect of the solvent that is most significant, with only a small proportion of solvent molecules involved chemically. The implicit solvent approach addresses these issues by retaining only the atomic details of the solute, placed in a suitably defined cavity, and by representing the solvent environment by an unstructured dielectric continuum outside this cavity. The free energy of solvation is typically decomposed into two contributions – the electrostatic energy of interaction of the solvent with the polarized dielectric, and a nonpolar term accounting for the work required to create a cavity in the solvent (cavitation energy), and, in more complex models, for dispersion-repulsion interactions between the solute and solvent.

A multitude of implicit solvent models of differing sophistication have been proposed to date² for use in *ab initio* calculations. Many of these models are derived from the self-consistent reaction field (SCRF) formalism, where the effect of the electric field due to the dielectric (polarized by the solute) is included in the Hamiltonian in a self-consistent fashion. Two widely used classes of SCRF-type models are the polarizable continuum model (PCM) of Tomasi *et al.*³ and the conductor-like screening model (COSMO) of Klamt and Schüürmann⁴.

The shape of the cavity containing the solute varies between models – early models used spherical or elliptical cavities; in more recent models the cavity is usually constructed from overlapping atomic spheres of varying radii, which necessitates using a number of parameters. In contrast, the recent model proposed by Fattebert and Gygi^{5,6}, and later developed by Scherlis *et al.*⁷ (henceforth called the FGS model), utilizes a dielectric cavity constructed directly from the electronic density of the solute, which greatly reduces the number of parameters involved.

This “minimal-parameter” nature of the FGS model makes it attractive for *ab initio* calculations. However, this approach also has several shortcomings, which we address in this Letter. First, the original model did not include dispersion-repulsion effects, which, as the authors themselves note, is likely to impact its accuracy for larger neutral molecules. Second,

the model necessitates the use of an *a posteriori* correction to the energy in vacuum, obtained in periodic boundary conditions, to approximate open boundary conditions, whereas in the solvent the electrostatic energy is obtained subject to zero boundary conditions. Third, a severe numerical instability prevents this approach from being practical for large molecules.

This Letter describes how we have addressed these limitations, by including dispersion-repulsion interactions, employing open (Coulombic) boundary conditions, and identifying and circumventing the root cause of the abovementioned numerical instability. We then validate and evaluate the performance of the model on two sets of several tens of small molecules. Finally, by performing a calculation on a 2615-atom protein-ligand system, we demonstrate how the implemented model can be used to perform large-scale *ab initio* calculations in solution.

In contrast to other SCRF models where the solute cavity has a discontinuous boundary, the FGS model defines a smooth transition of the relative permittivity according to:

$$\varepsilon(\mathbf{r}) = 1 + \frac{\varepsilon_\infty - 1}{2} \left(1 + \frac{1 - (\rho(\mathbf{r})/\rho_0)^{2\beta}}{1 + (\rho(\mathbf{r})/\rho_0)^{2\beta}} \right), \quad (1)$$

where $\rho(\mathbf{r})$ is the electronic density of the solute, ε_∞ is the bulk permittivity, the parameter β controls the smoothness of the transition of $\varepsilon(\mathbf{r})$ from 1 to ε_∞ , and ρ_0 is the density value for which the permittivity drops to $\varepsilon_\infty/2$. The cavitation contribution to the free energy is assumed to be proportional to the surface area, S , of the cavity (calculated at $\rho = \rho_0$), that is $\Delta G_{\text{cav}} = \gamma S(\rho_0)$, where γ is the solvent surface tension. Values for β and ρ_0 are found by a least-squares fit to the hydration energies of ammonia, nitrate and methylammonium (representative of neutral, anionic and cationic molecules, respectively)⁷. The total potential of the solute in the presence of the dielectric, $\phi(\mathbf{r})$ is obtained by solving the nonhomogeneous Poisson equation

$$\nabla \cdot (\varepsilon[\rho] \nabla \phi) = -4\pi \rho_{\text{tot}} \quad (2)$$

directly in real space subject to zero Dirichlet boundary conditions. The total charge density $\rho_{\text{tot}}(\mathbf{r})$ is a sum of the electronic density $\rho(\mathbf{r})$ and a Gaussian-smeared density of the cores, as proposed in ref. 7.

As outlined in ref. 7, the fact that the dielectric cavity responds self-consistently to changes in the electronic density means that the functional derivative of the electrostatic

energy, E_{es} , is no longer equal to the potential that is the solution of eq. (2), but rather:

$$\frac{\delta E_{\text{es}}}{\delta \rho}(\mathbf{r}) = \phi(\mathbf{r}) - \frac{1}{8\pi}(\nabla\phi(\mathbf{r}))^2 \frac{\delta \varepsilon}{\delta \rho}(\mathbf{r}). \quad (3)$$

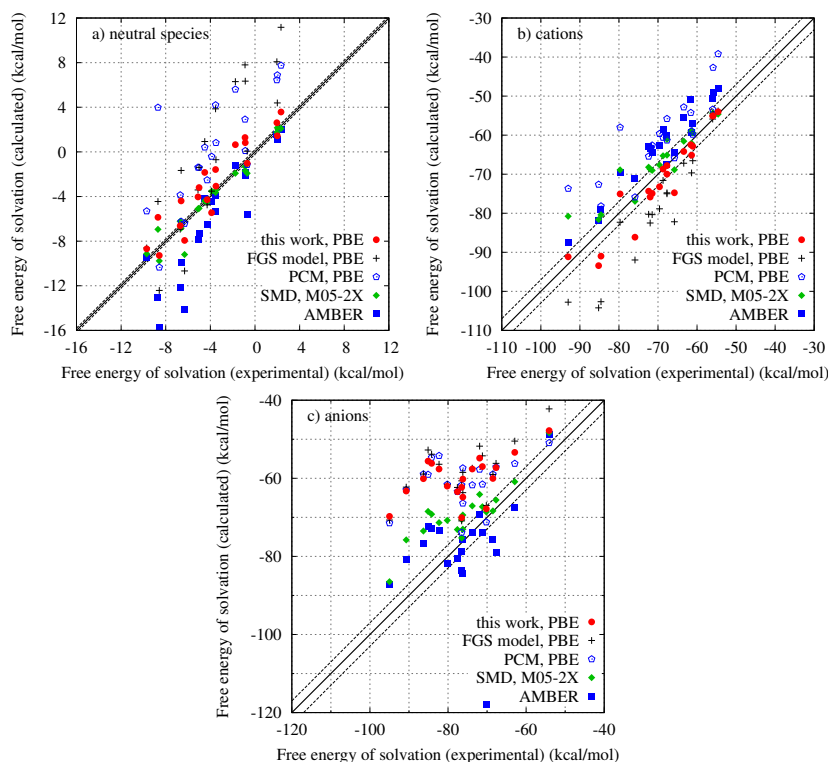


FIG. 1. Calculated free energies of solvation plotted against corresponding experimental values – a comparison between this work and other models. The solid diagonal line represents perfect agreement with experiment, the dashed black lines denote the estimated uncertainty of the experimental values.

The original FGS model does not set out to address dispersion-repulsion effects. This makes the results obtained for larger molecules dubious, especially for those that are neutral, as then the electrostatic contribution to solvation would be dwarfed by the nonpolar terms. As the authors duly note, this deficiency already becomes evident for the case of benzene where this model predicts a ΔG of 7.9 kcal/mol⁷ whereas the experimental value is -0.87 kcal/mol⁸. To appreciate the magnitude of the problem we refer to fig. 1 and the top of table I, where results obtained with the FGS model are shown for a representative selection of 20 neutral, 20 cationic and 20 anionic molecules chosen from ref. 8. The geometries of the molecules were not re-optimized in solvent, instead geometries optimized in the gas

phase readily available from ref. 8 were used. The FGS model underestimates the solvation effect for anions and overestimates it for cations. The predictions for neutral species are in moderate agreement with experiment. By examining the coefficient of correlation between the calculated and experimental values, we demonstrate that the obtained values do not correlate well with experiment (with the notable exception of cations), which makes the calculation of *relative* free energies of solvation, $\Delta\Delta G$, unreliable.

The second shortcoming of this approach is related to the boundary conditions used for the solution of eq. (2). References^{5,6} are concerned only with calculations in solution, where zero boundary conditions are used. Owing to the dielectric screening, this is a reasonable approximation, as long as the relative permittivity of the solvent is large. Ref.⁷ uses the same approach in solution, whereas for the reference vacuum calculation (needed to obtain free energies of solvation), where the Poisson equation becomes homogeneous, standard periodic plane-wave DFT calculations are performed. *In vacuo* energies thus obtained are subsequently corrected with the Makov-Payne formula⁹ to mimic the effect of open boundary conditions. This too is an approximation, since the correction cannot fully capture polarization effects⁹. Furthermore, only the energy is corrected, while the shape of the electronic density, and, in turn, the cavity generated in solution corresponds to periodic boundary conditions. As we demonstrate later, this subtly affects the free energies of solvation obtained for charged species, leading to a degree of cancellation of errors.

Further, we point out the root cause of the numerical instability inherent in the FGS model. The second term in the RHS of eq. (3) is extremely difficult to evaluate accurately, because $\frac{\delta\epsilon}{\delta\rho}$ is very close to zero everywhere, except on the boundary of the cavity, where, in turn, $(\nabla\phi(\mathbf{r}))^2$ is almost zero and thus difficult to distinguish from numerical noise. Because of this, the energy gradient calculated from eq. (3) is not numerically accurate and the method is found to converge only when high-order finite-differences and extremely fine grids (with a spacing of $0.15 a_0$ or finer) are used, as only then the gradient of the potential can be evaluated to sufficient accuracy. The memory requirements necessitated by such fine grids quickly make the technique impractical for larger molecules.

By addressing each of these limitations, we obtain a highly accurate and usable approach which retains the conceptual elegance of the FGS model.

We solve eq. (2) by means of a second-order multigrid^{11,12} approach, which is subsequently defect-corrected¹³ in an iterative fashion using 10-th order finite-difference stencils for the

TABLE I. Error (root mean square (rms) and maximum, in kcal/mol), with respect to experiment⁸, in the calculated free energies of solvation, and the corresponding coefficient of correlation, r , between the calculated and experimental values, for the 20 neutral, 20 cationic and 20 anionic species studied.

XC		neutral species			cations			anions		
Approach	functional	rms err.	max err.	r	rms err.	max err.	r	rms err.	max err.	r
FGS	PBE	5.0	8.8	0.87	9.7	19.0	0.95	19.5	32.4	0.55
this work ^a	PBE	1.6	2.8	0.93	4.4	10.2	0.95	18.1	29.5	0.53
this work ^b	PBE	5.0	8.9	0.87	10.4	19.0	0.95	21.2	35.1	0.54
this work ^c	PBE	1.8	3.1	0.93	3.9	8.3	0.94	18.1	29.4	0.54
PCM	PBE	4.9	12.7	0.75	10.5	21.7	0.83	17.8	29.5	0.36
PCM	B3LYP	4.7	12.0	0.78	10.4	21.8	0.83	17.0	28.4	0.41
PCM	M05-2X	4.4	11.1	0.79	10.2	21.7	0.81	15.7	26.8	0.46
SMD	M05-2X	0.9	2.9	0.97	4.6	12.2	0.95	8.5	16.6	0.86
AMBER ¹⁰	(classical)	3.3	7.84	0.64	6.9	10.8	0.96	12.8	47.8	0.32

^a With the cavity responding self-consistently to changes in density.

^b With the cavity responding self-consistently to changes in density, and without dispersion-repulsion.

^c With the cavity fixed.

first and second derivatives. We find that with a grid spacing of $0.125 a_0$ as few as 3-4 defect-correction iterations are sufficient to reduce the algebraic error in the obtained potential by four orders of magnitude with respect to the initial, uncorrected solution. The corresponding reduction in the magnitude of the residual is two orders of magnitude, due to the approximate nature of the calculated boundary conditions. With a grid spacing of $0.25 a_0$ ten iterations, on average, were necessary.

We have recast the solvation problem into open boundary conditions by computing the core-core and the local pseudopotential terms in real space and by using open (Coulombic) boundary conditions when solving eq. (2) – that is, we set up Dirichlet boundary conditions by evaluating the Coulombic potential due to the charge distribution ρ_{tot} . Since with a spatially localized density the calculation of the Coulombic integral for all the points on the boundary scales as $\mathcal{O}(L^2 N_{\text{at}})$ with the box length L and number of atoms N_{at} , charge

coarse-graining and interpolation were used to reduce the prefactor in this calculation by about three orders of magnitude. In so doing, we obtain *in vacuo* energies and densities that need not be corrected. In the solvated case, where the nonhomogeneity in ϵ prevents such an approach, we calculate the boundary conditions by approximating the dielectric as homogeneous, with the permittivity of the bulk solvent. Fig. 2, panels a) and b) and table I, rows 1 and 3 illustrate that for the simulation cell we used (a cube with an edge length of $L = 40.5 a_0$) this alone offers no improvement in accuracy. A difference in the free energy of solvation of 0.1 kcal/mol, 0.9 kcal/mol and -1.8 kcal/mol is observed on average for neutral, cationic and anionic species, respectively, compared to the predictions of the original FGS model (refer to fig. 3 for details). For charged species the application of consistent open boundary conditions leads to a slight increase in the error.

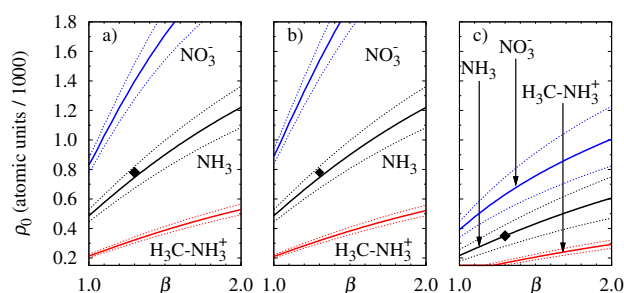


FIG. 2. Isolines of zero error with respect to experiment for the possible parametrizations of a) the FGS model, b) the FGS model with the boundary conditions we propose and c) the model proposed in this work, for three representative molecules. The dashed lines indicate an error of ± 1 kcal/mol. The black points indicate the final parametrization of the respective models.

A parameter sweep for the three molecules used to parametrize the FGS model demonstrates (cf. fig. 2) that there exists no parametrization that would result in even moderate agreement with experiment for the three species simultaneously. The model would consistently either underestimate free energies of solvation for anions or overestimate them for cationic species.

We attribute this failure to a combination of factors – the poor performance of the Perdew-Burke-Ernzerhof (PBE) exchange-correlation (XC) functional; the fact that *any* isodensity formulation will use larger cavities for anions than for corresponding cations, whereas the charge asymmetry in solvation effects is in fact opposite¹⁴; and, finally, to the lack of inclusion of dispersion-repulsion effects, which leads to an overestimation of the nonpolar

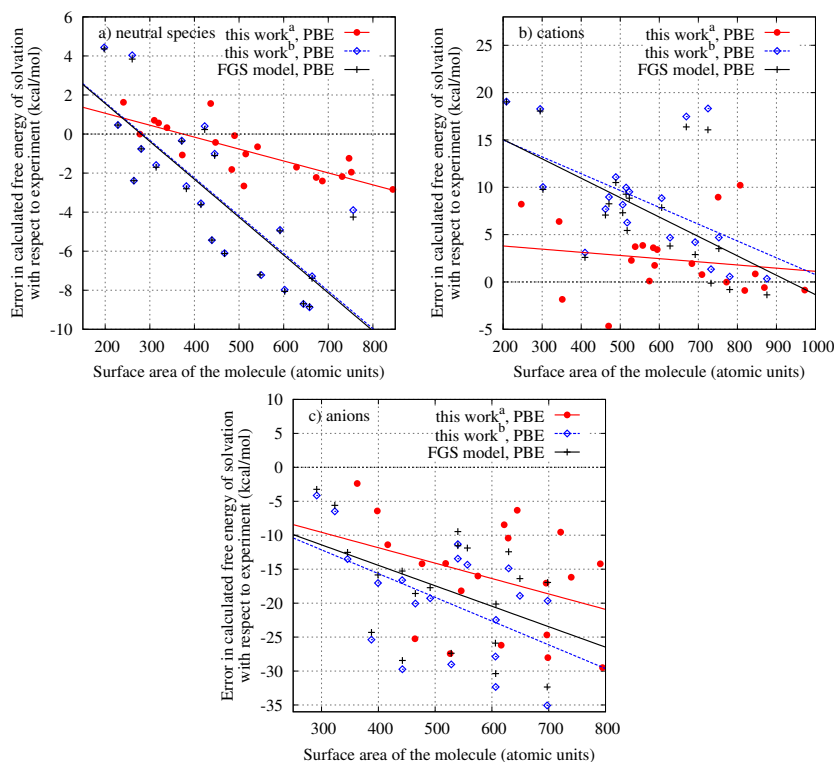


FIG. 3. Absolute error in the calculated free energies of solvation with respect to experimental values, plotted against the surface area of the molecule. The surface areas differ between models because the parametrization is different. The solid and dashed lines represent a linear fit. The horizontal dotted line corresponds to perfect agreement with experiment.

component of solvation. The middle rows of table I show, on the example of PCM, how using a hybrid functional such as B3LYP or a hybrid meta-GGA functional such as M05-2X¹⁵ addresses the first problem, by reducing the self-interaction error, which otherwise leads to excessive delocalization of the electrons, but does not address the other two problems.

The increase in the magnitude of the difference between the calculated and experimentally obtained free energies of solvation with the size of the molecule, especially in the case of neutral molecules, demonstrated in fig. 3 indicates that the neglect of dispersion-repulsion effects is detrimental to the predictive quality of the FGS model. We propose including dispersion-repulsion effects in the free energy of solvation, $\Delta G_{\text{dis,rep}}$, using an approximate relation derived by Floris *et al.*¹⁶. Since this relation is linear, it amounts to a simple rescaling of the surface tension of the solvent, including the approximate $\Delta G_{\text{dis,rep}}$ in the cavitation term. From the slope of the linear relation plotted in fig. 1 of ref.¹⁶ it follows that the surface tension should be rescaled by a factor of 0.281. Even this crude method for taking dispersion-

repulsion into account dramatically improves the accuracy of the model, as evidenced by figs. 1 and 2 and table I, from which it is apparent that the resulting approach is in much better agreement with experiment than both PCM and the force-field Poisson-Boltzmann (PB) approach of AMBER¹⁰, offering comparable quality to the much more complex SMD¹⁷¹⁸ model. The improvement offered by the inclusion of dispersion-repulsion effects is evidenced by fig. 3 and can be quantified by comparing rows denoted with *a* and *b* in table I. The results corresponding to the row denoted with *b* were obtained by turning off the dispersion-repulsion contribution whilst using the parameters proposed in ref.⁷, denoted with a point in fig. 2, panels a) and b).

The numerical instability caused by the second term in the RHS of eq. (3) can be circumvented without loss of accuracy. We first note that this term disappears when, instead of responding to changes in the electronic density, the dielectric cavity is fixed. We propose constructing the cavity by the application of eq. (1) to the converged electronic density of the solute obtained in the vacuum calculation and keeping the cavity fixed throughout the calculation in solvent. We show (cf. tables I, II) that the associated reduction in accuracy is insignificant, while both the wall time and the memory requirements of the computation are reduced by about an order of magnitude, as convergence is readily achieved with a more reasonable real-space grid spacing of $0.25 a_0$. We should point out that a similar attempt to fix the cavity in the FGS model would probably lead to larger errors due to the fact that the fixed cavity would come from the *periodic* density of the vacuum calculation – as the Makov-Payne correction⁹ only corrects the energy. We note that this simplified approach is still suitable for geometry optimization in solution, provided the additional contribution to the forces due to the cavity variation with atomic positions is included. Sá *et al.*¹⁹ also note the abovementioned instability and propose a somewhat different way of circumventing it.

We further validate our model on 71 neutral molecules taken from the blind tests of refs. 20 and 21, for which the experimental energies of solvation are reported in ref.⁸. Again, the geometries were not re-optimized in solution, but rather the gas-phase geometries from ref.⁸ were used. The results, shown in fig. 4 and table II, again show that our approach is consistently more accurate than both PCM and the force-field PB approach of AMBER¹⁰ and that our model offers a level of agreement with experiment which is comparable to the SMD¹⁸ model, even when the cavity is fixed.

Conventional *ab initio* calculations are typically limited to only a few hundred atoms at

TABLE II. Error (root mean square (rms) and maximum, in kcal/mol), with respect to experiment, in the calculated free energies of solvation, and the corresponding coefficient of correlation, r , between calculated and experimental values for the 71 molecules studied^{20,21}.

Approach	XC functional	rms max		r
		error	error	
this work ^a	PBE	3.8	8.3	0.83
this work ^b	PBE	4.1	9.1	0.83
PCM	PBE	10.9	23.3	0.53
SMD	M05-2X	3.4	14.5	0.87
AMBER	(classical)	5.1	19.9	0.77

^a With cavity responding self-consistently to changes in density.

^b With cavity fixed.

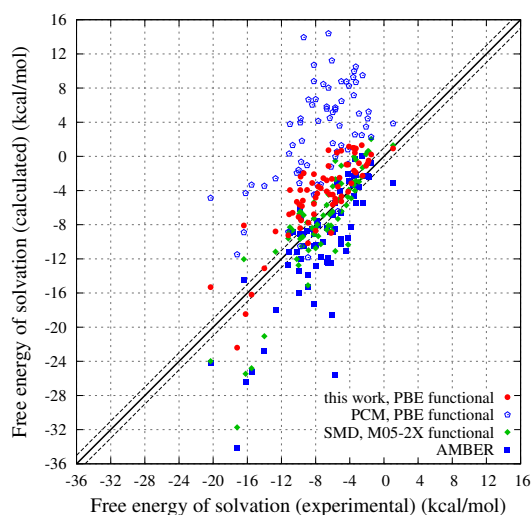


FIG. 4. Calculated free energies of solvation plotted against corresponding experimental values for the 71 neutral molecules studied^{20,21} – a comparison between models. The solid diagonal line represents perfect agreement with experiment, the dashed black lines denote the estimated uncertainty in the experimental values.

most. However, with recent advances in linear-scaling density functional theory (LS-DFT) approaches²³ a number of codes^{24–26} have been developed which are capable of performing calculations with many thousands of atoms. The combination of LS-DFT with implicit solvent models would enable highly realistic simulations of important phenomena such as

TABLE III. Free energies of solvation (in kcal/mol) of L99A/M102Q T4 lysozyme (ΔG_{host}), its complex with catechol (ΔG_{cplx}), catechol (ΔG_{lig}), desolvation energy of the ligand ($\Delta G_{\text{d}} = \Delta G_{\text{cplx}} - \Delta G_{\text{host}} - \Delta G_{\text{lig}}$), binding energy *in vacuo* (ΔE_{gas}) and in solvent (ΔE_{sol}), as predicted by our model (with PBE and a fixed cavity) and AMBER.

Approach	ΔG_{cplx}	ΔG_{host}	ΔG_{lig}	ΔG_{d}	ΔE_{gas}	ΔE_{sol}
this work	-2423.0	-2421.3	-7.5	5.8	-28.6	-22.8
AMBER	-2428.3	-2433.0	-17.6	22.4	-27.7	-5.3
expt. ²²	-	-	-9.3	-	-	-

biomolecular processes or the chemical modification and self-assembly of nanostructures.

As a demonstration of the potential applications of this approach in large-scale DFT calculations, we have implemented our solvent model in the LS-DFT code ONETEP²⁴ and used it to calculate the free energy of solvation for a 2615-atom system composed of the catechol ligand bound to a L99A/M102Q mutant of the T4 lysozyme. The results of this calculation are shown in table III and the overview of the system in question, along with an outline of the dielectric cavity (at $\rho = \rho_0$) is shown in fig. 5. AMBER greatly overestimates the solvation energy of catechol, and consequently the binding energy in solvent differs significantly between the two models. The need for an *ab initio* model, where the density is polarized by the solvent is demonstrated by the different behavior of ΔG_{host} (as compared to ΔG_{cplx}) between the proposed model and AMBER.

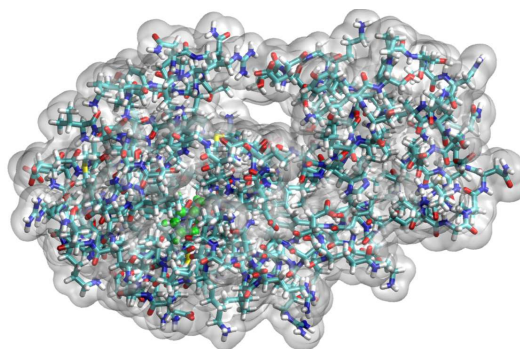


FIG. 5. An overview of the lysozyme-catechol complex, with the dielectric cavity indicated in grey and catechol in green.

In summary, we have outlined and validated an implicit solvent model for *ab initio* calculations, which, despite using only two parameters, offers a substantial improvement over

existing models, as measured by the agreement of absolute and relative free energies of solvation with experiment (compared to PCM and AMBER) or the number of parameters needed to achieve similar agreement (compared to SMD). We have shown how the implementation of the proposed model in the LS-DFT code ONETEP paves the way for first-principles implicit-solvent calculations for molecules with thousands of atoms.

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